Homework 2, due 11-3

In this homework we consider the Lorentz group. The defining representation is four dimensional. The Lorentz group is the set of real matrices Λ that leave the Minkowski space metric invariant

$$\Lambda^T \eta \Lambda = \eta, \qquad \eta = \mathrm{diag}(1,-1,-1,-1), \qquad \det(\Lambda) = 1.$$

With this definition the Lorentz group preserves the inner product of 4-vectors. This means that $v_\mu w^\mu = \eta_{\nu\mu} v^\nu v^\mu = v^T \eta w$ is invariant under $v \to \Lambda v$, $w \to \Lambda w$.

1. If we write $\Lambda = \exp(i\alpha_a X_a)$ show that the generators X_a must satisfy

   $$X_a^T \eta + \eta X_a = 0.$$

   Determine from this condition and the reality of Λ the number of generators.

2. Show that the six matrices

   $$(J_i)_{\mu\nu} = -i\epsilon_{0i\mu\nu}, \qquad (K_i)_{\mu\nu} = -i(\delta_{\mu 0}\delta_{\nu i} - \delta_{\mu i}\delta_{\nu 0}),$$

   with $i = 1,2,3$ and $\mu,\nu = 1,\ldots,4$ form a complete basis of the generators X_a. Here, $\epsilon_{\mu\nu\alpha\beta}$ is the completely anti-symmetric four index tensor with $\epsilon_{0123} = +1$. The J_i and K_i are called rotations and boosts, respectively. Are J_i, K_i hermitean?

3. Compute the commutators $[J_i, J_j]$, $[K_i, K_j]$, $[J_i, K_j]$.

4. Define $A_i = (J_i + iK_i)/2$ and $B_i = (J_i - iK_i)/2$. Compute $[A_i, A_j]$, $[B_i, B_j]$, $[A_i, B_j]$. Your result shows that the algebra of the Lorentz group is isomorphic to that of $SU(2) \times SU(2)$. This implies that representations of the Lorentz group are labeled by pairs of half-integers (j_A, j_B).

5. Using

   $$\alpha_a X_a = \theta_i J_i + \omega_i K_i = (\theta_i - i\omega_i)A_i + (\theta_i - i\omega_i)B_i$$

   you can find the explicit form of a Lorentz transformations in the (j_A, j_B) representation. As an example, consider a two-spinor $\psi = (\alpha, \beta)^T$ in the $(1/2, 0)$ representation. How does ψ transform under rotations around the z-axis by an angle θ_3 or boosts along the z-axis by a boost parameter ω_3?