Quantum Mechanics

1. The state of the system is described by a vector $|\psi(t)\rangle$, called the wave function.

2. Observables correspond to hermitean operators R. The possible outcomes of a measurement of R are the eigenvalues r, $R|r\rangle = r|r\rangle$. The probability of observing r is

$$P_r = |\langle r|\psi(t)\rangle|^2 \, .$$

   Important special cases are the probability to observe a given position, $P_x = |\langle x|\psi\rangle|^2$, or a given momentum p, $P_p = |\langle p|\psi\rangle|^2$. The quantity $\psi(x,t) = \langle x|\psi(t)\rangle$ is called the coordinate space wave function, and $\psi(p,t) = \langle p|\psi(t)\rangle$ is the momentum space wave function. Note that $\psi(p,t)$ is the (suitably normalized) Fourier transform of $\psi(x,t)$.

   Immediately after a measurement in which the value of R was observed to be r the state of the system changes from $|\psi(t_{bef})\rangle$ to $|\psi(t_{aft})\rangle = |r\rangle$. ("The collapse of the wave function.")

3. The state of the system evolves according to the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$$

   where H is the Hamiltonian. The Hamiltonian of a single particle moving in a potential $V(x)$ is

$$H = \frac{p^2}{2m} + V(x) \, .$$

   Note that $\langle x|p|\psi\rangle = -i\hbar\nabla_x\langle x|\psi\rangle$, so that we can write

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left\{-\frac{\hbar^2\nabla_x^2}{2m} + V(x)\right\}\psi(x,t) \, .$$

4. If the eigenvalues and eigenfunctions of H are known,

$$H|n\rangle = E_n|n\rangle \,,$$

then the solution of the Schrödinger equation is given by

$$|\psi(t)\rangle = \sum_n |n\rangle c_n e^{-iE_n t/\hbar} \,,$$

where $c_n = \langle n|\psi(0)\rangle$. The eigenvalue equation for H in the coordinate representation,

$$\left\{-\frac{\hbar^2\nabla_x^2}{2m} + V(x)\right\}\psi_n(x) = E_n\psi_n(x)$$

where $\psi_n(x) = \langle x|n\rangle$, is sometimes called the time-independent Schrödinger equation.

5. If two hermitean operators commute, $[R, S] = 0$, then we can construct a wave function that is an eigenstate of both R and S,

$$R|r,s\rangle = r|r,s\rangle, \qquad S|r,s\rangle = s|r,s\rangle \,.$$

In such a state, measurements of both R and S can be predicted with absolute certainty. If R and S do not commute this is not true. In particular, we cannot find a state for which both R and S are certain. The most important example is position x and momentum p. We have

$$[p, x] = -i\hbar$$

and

$$(\Delta p)^2(\Delta x)^2 \geq \frac{\hbar^2}{4}$$

where $(\Delta p)^2 = \langle\psi|(p-\bar{p})^2|\psi\rangle$ is the uncertainty in p, $\bar{p} = \langle\psi|p|\psi\rangle$ is the average p, and the uncertainty and average in x are defined analogously. This is known as the uncertainty relation.