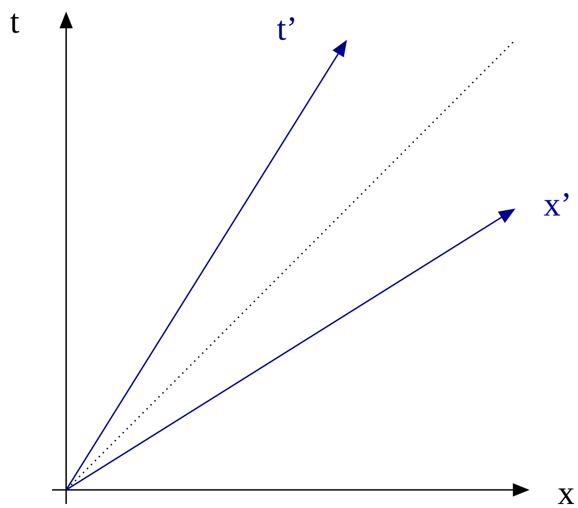
The Twin Paradox

We would like to clarify the twin paradox using space time diagrams. This is also a nice exercise in drawing space time diagrams. We consider two observers: S and S', where S' is moving with velocity $0.625c$ relative to S.
We will use Lorentz transformations to help us draw the diagrams correctly. We start with the S coordinate system (x, t). The transformation from S to S' is

$$x' = \gamma(x - \beta ct), \quad (1)$$
$$ct' = \gamma(ct - \beta x), \quad (2)$$

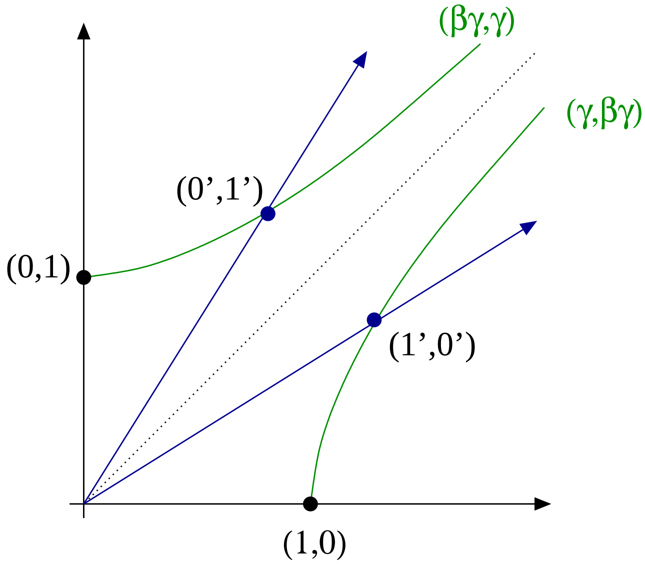
where $\beta = v/c$ and $\gamma = 1/(1-\beta^2)^{1/2}$. The t' axis corresponds to $x' = 0$, so $ct = x/\beta$. The x' axis satisfies $t' = 0$ and $ct = \beta x$. The two slopes are inversely proportional. We have



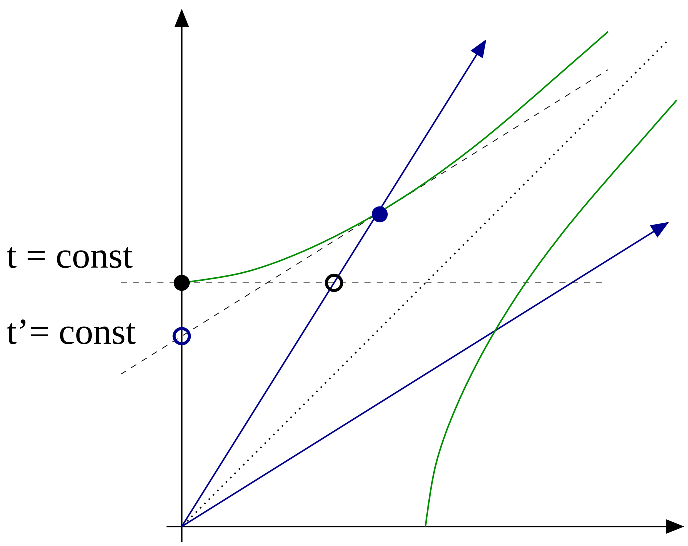
In order to find the units of S' in the S-system we use the inverse Lorentz transformation

$$x = \gamma(x' + \beta ct'), \quad (3)$$
$$ct = \gamma(ct' + \beta x'). \quad (4)$$

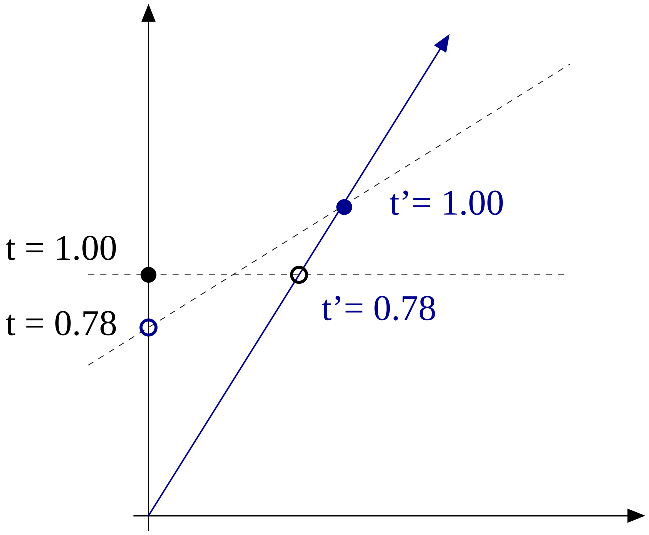
The unit on the t' axis is $(x', ct') = (0, 1)$. This corresponds to $(x, ct) = (\gamma\beta, \gamma)$. Analogously, $(x', ct') = (1, 0)$ is $(x, ct) = (\gamma, \gamma\beta)$. For arbitrary $\beta = v/c$ these curves are hyperbolas. The intersection of the hyperbolas with the S and S' axes fixes the units.



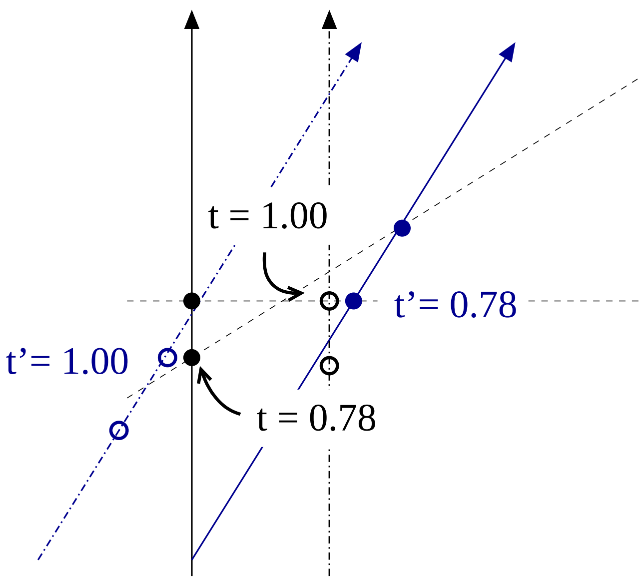
Now we can draw an equal time line in S through the point $(x, ct) = (0, 1)$. This is a line parallel to the x-axis. We can also draw an equal time line in S' through the point $(x', ct') = (0, 1)$. This line is parallel to the x'-axis.



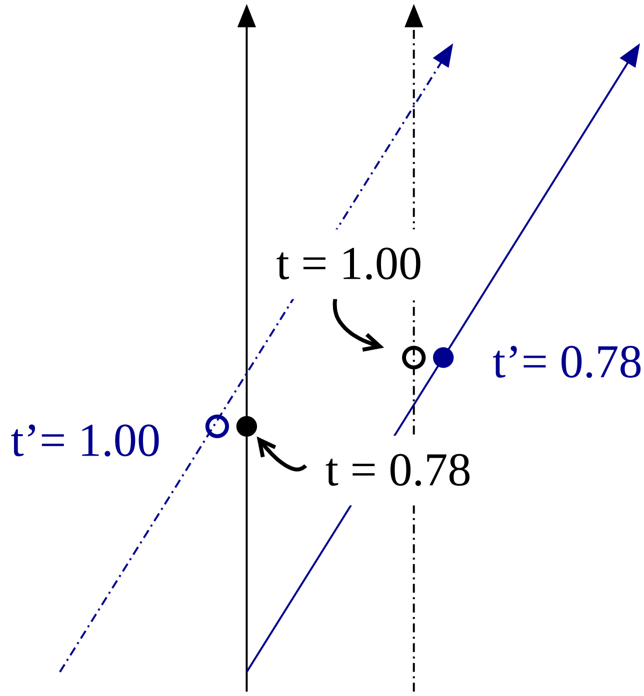
We can see that in S, the point $(x, ct) = (0, 1)$ is at equal time to $(x', ct') = (0, 1/\gamma)$. Analogously, the point $(x', ct') = (0, 1)$ in S' is at equal time (in S') to $(x, ct) = (0, 1/\gamma)$. For $\beta = 0.625$ we have $\gamma = 1.28$ and $1/\gamma = 0.78$.



The situation is completely symmetric. Both observers conclude that it is the other clock that is running slow (the other clock is showing $ct = 0.78$ when their own clock shows $ct = 1$). There is no paradox, because there are (effectively) more than two clocks involved. Both S and S' have sets of synchronized clocks, and comparisons between S and S' involve clocks that are at the same space time point. The worldlines of clocks are shown as lines with arrows.



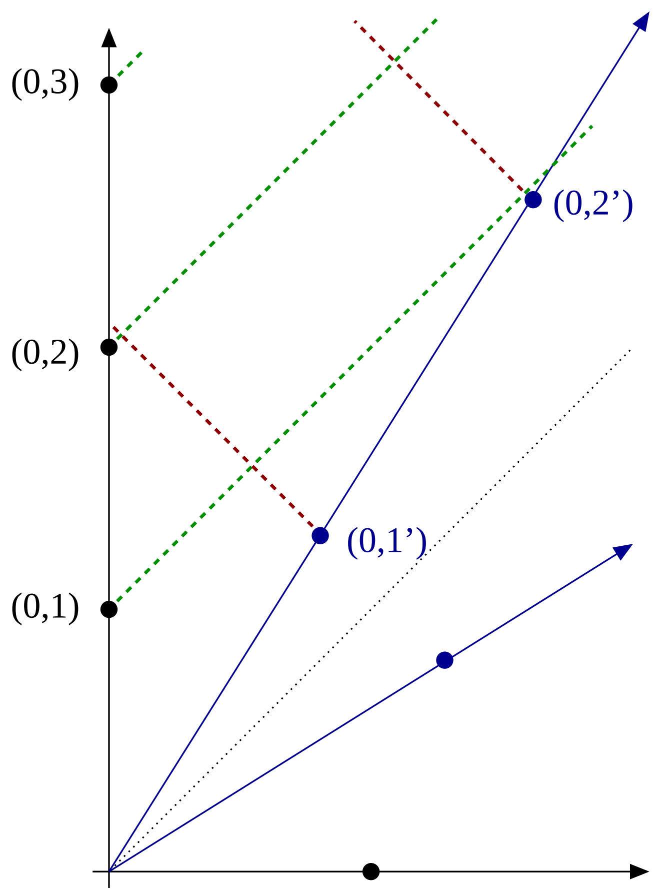
Finally, we focus on the two clock comparisons that cause both S and S' to conclude that the other clock is running slow.



The Twin Paradox, revisited

The original twin paradox is resolved by the observation that there are more than two clocks that are being compared. As a result there is no sense in which the clocks in S or S' are "really" slower. S and S' are completely symmetric

Now imagine that we try to reduce the twin paradox to the comparison of just two clocks. For this purpose imagine that both S and S' have only one clock, located as $x = 0$ and $x' = 0$, respectively. Both observers send out a light signal (dark red and dark green dashed lines) for every tick of their clock. The figure shows that the situation is still symmetric: both observers conclude that the other clock is slow (the light signal corresponding to tick number n of the other clock arrives after the local clock has sent out the n'th signal). Note that this effect is more than just time dilation – it also involves the extra delay due to the travel time of the light signal. The combination of these two effects is called the Doppler effect.



Now imagine that S' starts to decelerate at some point, and then reverses direction until he reaches the velocity $-0.625c$. Then both S and S' will start to pick the light signals of the other clock at an increased rate. When the word lines of the two clocks intersect there is no time delay anymore. In the figure we can see that S' has collected three "ticks" sent by S, whereas S barely receives two ticks. The situation is no longer symmetric, the S' clock is "really" slower. This is possible because S and S' are no longer equivalent. S is an inertial observer, whereas S' undergoes acceleration and is not an inertial observer.