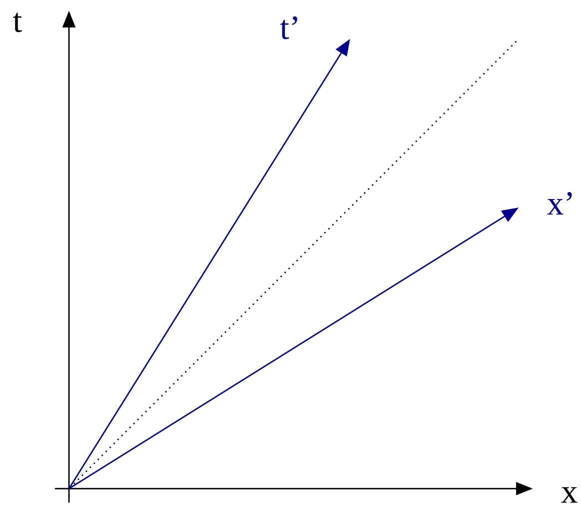
The Twin Paradox

We would like to clarify the twin paradox using space time diagrams. This is also a nice exercise in drawing space time diagrams. We consider two observers: S and S', where S' is moving with velocity $0.625c$ relative to S. We will use Lorentz transformations to help us draw the diagrams correctly. We start with the S coordinate system (x, t). The transformation from S to S' is

$$x' = \gamma(x - \beta ct), \quad (1)$$
$$ct' = \gamma(ct - \beta x), \quad (2)$$

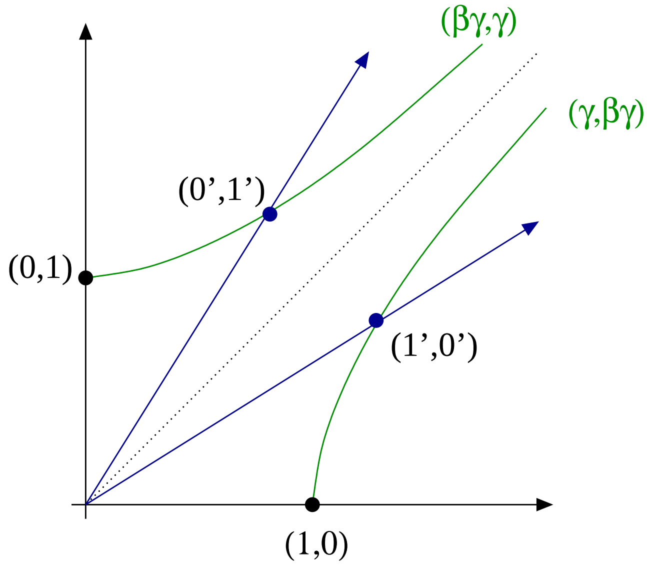
where $\beta = v/c$ and $\gamma = 1/(1 - \beta^2)^{1/2}$. The t' axis corresponds to $x' = 0$, so $ct = x/\beta$. The x' axis satisfies $t' = 0$ and $ct = \beta x$. The two slopes are inversely proportional. We have



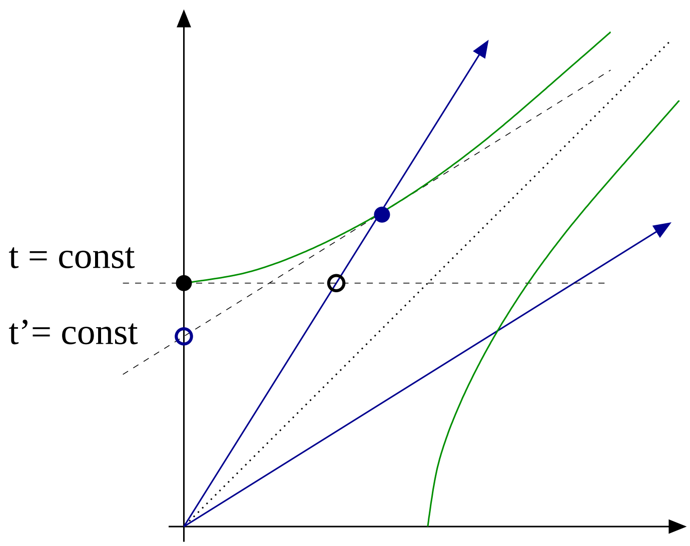
In order to find the units of S' in the S-system we use the inverse Lorentz transformation

$$x = \gamma(x' + \beta ct'), \quad (3)$$
$$ct = \gamma(ct' + \beta x'). \quad (4)$$

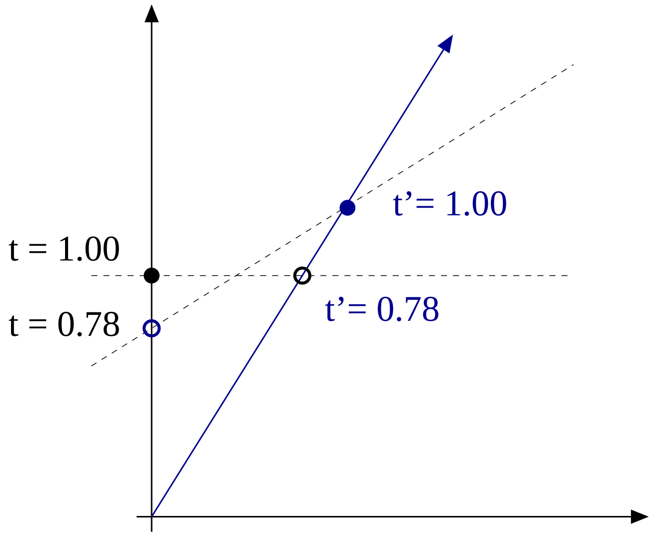
The unit on the t' axis is $(x', ct') = (0, 1)$. This corresponds to $(x, ct) = (\gamma\beta, \gamma)$. Analogously, $(x', ct') = (1, 0)$ is $(x, ct) = (\gamma, \gamma\beta)$. For arbitrary $\beta = v/c$ these curves are hyperbolas. The intersection of the hyperbolas with the S and S' axes fixes the units.



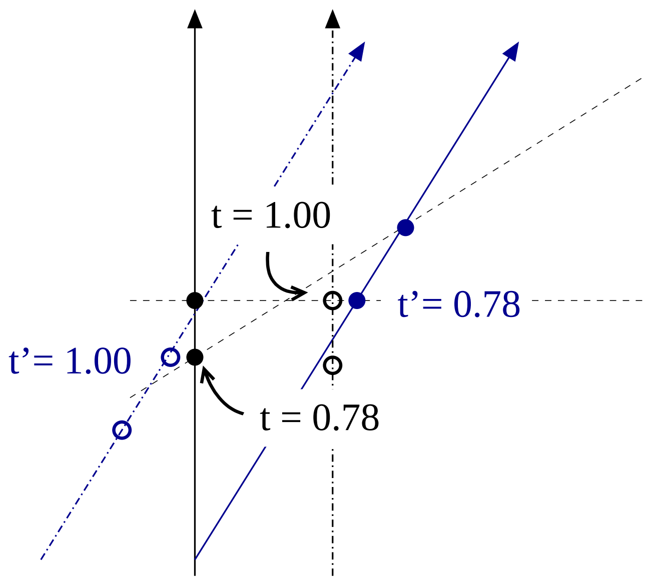
Now we can draw an equal time line in S through the point $(x, ct) = (0, 1)$. This is a line parallel to the x-axis. We can also draw an equal time line in S' through the point $(x', ct') = (0, 1)$. This line is parallel to the x'-axis.



We can see that in S, the point $(x, ct) = (0, 1)$ is at equal time to $(x', ct') = (0, 1/\gamma)$. Analogously, the point $(x', ct') = (0, 1)$ in S' is at equal time (in S') to $(x, ct) = (0, 1/\gamma)$. For $\beta = 0.625$ we have $\gamma = 1.28$ and $1/\gamma = 0.78$.



The situation is completely symmetric. Both observers conclude that it is the other clock that is running slow (the other clock is showing $ct = 0.78$ when their own clock shows $ct = 1$). There is no paradox, because there are (effectively) more than two clocks involved. Both S and S' have sets of synchronized clocks, and comparisons between S and S' involve clocks that are at the same space time point. The worldlines of clocks are shown as lines with arrows.



Finally, we focus on the two clock comparisons that cause both S and S' to conclude that the other clock is running slow.