Homework 8, due 11-4

1. Consider a particle governed by the Hamiltonian

$$H = \frac{p^2}{2m} + V(x).$$

Consider $[[H, x], x]$ in order to show that

$$\sum_m (E_m - E_n)|\langle n|x|m\rangle|^2 = \frac{\hbar^2}{2m},$$

where $|n\rangle$ are energy eigenstates satisfying $H|n\rangle = |n\rangle E_n$. This relation is known as the dipole (or Thomas-Reiche-Kuhn) sum rule. Check the dipole sum rule in the case of the 1-dimensional harmonic oscillator.

2. Consider the 1-dimensional harmonic oscillator with Hamiltonian $H = p^2/(2m) + (m/2)\omega^2 x^2$. Show that the classical action is given by

$$S_{cl} = \frac{m\omega}{2\sin(\omega(t_b - t_a))}\left\{\left(x_a^2 + x_b^2\right)\cos(\omega(t_b - t_a)) - 2x_a x_b\right\},$$

where $x_{cl}(t_a) = x_a$ and $x_{cl}(t_b) = x_b$. What is the meaning of the singularity when $\omega(t_b - t_a) = n\pi$?

3. In class we defined the infinitesimal translation operator through its action on coordinate eigenstates, $T(\epsilon)|x\rangle = |x + \epsilon\rangle$. Compute

$$T^\dagger(\epsilon) X T(\epsilon), \qquad T^\dagger(\epsilon) P T(\epsilon),,$$

where X, P are the coordinate and momentum operators.

4. Consider the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2a^2} m\omega^2 (x^2 - a^2)^2.$$

Construct approximate wave functions for the ground state and the first excited state in the limit $a \to \infty$. Explain your reasoning using parity symmetry.