Homework 12, due 12-9

1. The Lagrange function for a point particle with mass m and charge $-e$ in an electromagnetic field is
$$L = \frac{m}{2}\vec{v}^2 + e\phi - \frac{e}{c}\vec{v}\cdot\vec{A},$$
where $\vec{v} = \dot{\vec{x}}$ is the velocity, ϕ is the scalar potential and $\vec{A}$ is the vector potential. Show that the Hamilton function $\mathcal{H}$ is given by
$$\mathcal{H} = \frac{1}{2m}\left(\vec{p} + \frac{e}{c}\vec{A}\right)^2 - e\phi.$$

2. Consider a particle of charge $-e$ in a vector potential
$$\vec{A} = \frac{B}{2}(-y\hat{e}_x + x\hat{e}_y),$$
where $\hat{e}_x$ and $\hat{e}_y$ are unit vectors in the x, y direction. Show that the magnetic field is $\vec{B} = B\hat{e}_z$. Also show that a classical particle in this potential moves in circles with an angular frequency $\omega_0 = eB/(mc)$.

3. Consider the corresponding quantum Hamiltonian
$$H = \frac{1}{2m}\left\{\left(p_x - \frac{e}{2c}yB\right)^2 + \left(p_y + \frac{e}{2c}xB\right)^2 + p_z^2\right\}.$$
Find the spectrum of H. Hint: Introduce the operators
$$Q = -\frac{1}{eB}\left(cp_x - \frac{e}{2}yB\right), \quad P = \left(p_y + \frac{e}{2c}xB\right).$$
Compute $[P, Q]$ and express H in terms of P, Q.

4. Compute the ground state wave function. Hint: Follow the strategy we employed in the case of the harmonic oscillator. Introduce the complex variables $u = x + iy$ and $u^* = x - iy$. Use the ansatz
$$\psi_0 = f(z)g(u, u^*)\exp(-\alpha|u|^2),$$
with a suitably chosen α. Is the solution for g unique? Explain!