Homework 10, due 11-25

1. We define an infinitesimal rotation around the x-axis by

$$R_x(\delta\phi_x)|x,y,z\rangle = \left(1 - \frac{i\delta\phi_x}{\hbar}L_x\right)|x,y,z\rangle = |x, y - z\delta\phi_x, z + y\delta\phi_x\rangle.$$

   In class we showed that

$$\langle x,y,z|L_x|\psi\rangle = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\langle x,y,z|\psi\rangle.$$

   Introduce spherical coordinate eigenstates $|r,\theta,\phi\rangle$ and determine

$$\langle r,\theta,\phi|L_x|\psi\rangle.$$

   Repeat the procedure for L_y and $L_\pm$. Remember that $(x,y,z) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$.

2. The orbital angular momentum operator $\vec{L}$ is given by $L_i = \epsilon_{ijk}x_j p_k$. Use the canonical commutation relations $[p_i, x_j] = -i\hbar\delta_{ij}$ in order to show that

$$\vec{L}^2 = \vec{x}^2\vec{p}^2 - (\vec{x}\cdot\vec{p})^2 + i\hbar\vec{x}\cdot\vec{p}.$$

   Use this result in order to show that

$$\langle r,\theta,\phi|\frac{\vec{p}^2}{2m}|\psi\rangle = -\left(\frac{\hbar^2}{2m}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)\langle r,\theta,\phi|\psi\rangle + \frac{1}{2mr^2}\langle r,\theta,\phi|\vec{L}^2|\psi\rangle.$$

3. (Only if you have too much energy) Use the result of problem 1 and $\vec{L}^2 = L_z^2 + (L_+L_- + L_-L_+)/2$ in order to show that

$$\langle r,\theta,\phi|\vec{L}^2|\psi\rangle = -\hbar^2\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right]\langle r,\theta,\phi|\psi\rangle.$$