

Homework 10, due 11-18

In homework assignment 1 we introduced the Pauli spin matrices $\vec{\sigma}$, studied their commutation relations $[\sigma_i, \sigma_j]$, and computed the matrix $\exp(i\vec{\phi}\cdot\vec{\sigma})$.

1. Show that $S_i = \hbar\sigma_i/2$ satisfy the same commutation relations as the angular momentum operators L_i . Check that the vectors $|\uparrow\rangle = (1, 0)$ and $|\downarrow\rangle = (0, 1)$ are eigenstates of \vec{S}^2, S_z . What is the value of the angular momentum?
2. Construct the 2×2 matrix $R(\hat{n}, \phi)$ corresponding to a finite rotation around the \hat{n} axis by an angle ϕ . What is $R(\hat{n}, 2\pi)$?
3. There are many ways to describe a general rotation in three dimensions. Euler suggested a method where we imagine that there is a separate coordinate system attached to the body that is being rotated (called the “body-fixed coordinate system” as opposed to the “space-fixed system”). We can now write a general rotation as a rotation around the z -axis by an angle α , followed by a rotation around the body fixed y -axis y' by β , followed by a rotation around the body fixed z -axis z' by γ . (α, β, γ) are known as Euler angles.

This requires some thought, because we have to relate the body-fixed rotations to space-fixed rotations. One can show that

$$R(\alpha, \beta, \gamma) = R(\hat{z}', \gamma)R(\hat{y}', \beta)R(\hat{z}, \alpha) = R(\hat{z}, \alpha)R(\hat{y}, \beta)R(\hat{z}, \gamma).$$

Using this result, and the result of part 2, find the 2×2 matrix $R(\alpha, \beta, \gamma)$.

This matrix is called the $j = 1/2$ irreducible representation of the rotation operator or the $j = 1/2$ Wigner \mathcal{D} -function $\mathcal{D}_{mm'}^{1/2}(\alpha, \beta, \gamma)$.

4. Find an explicit expression for the eigenstates of the operator $\vec{S} \cdot \hat{n}$

$$\vec{S} \cdot \hat{n} |(\vec{S} \cdot \hat{n}), \pm\rangle = \pm \left(\frac{\hbar}{2}\right) |(\vec{S} \cdot \hat{n}), \pm\rangle,$$

in terms of the eigenstates $|\uparrow\rangle, |\downarrow\rangle$ of the operator S_z .