Homework 1, due 9-17

Suggestion: Read chapter 1 of Shankar or Sakurai.

1. The Pauli matrices are defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   (a) Are the Pauli matrices hermitean? Are they unitary? Determine the commutation relations $[\sigma_i, \sigma_j]$.

   (b) Consider a 2×2 matrix $X = a_0 + \vec{a} \cdot \vec{\sigma}$. Determine the coefficients a_i in terms of X_{ij} and also in terms of $\mathrm{tr}[X]$ and $\mathrm{tr}[\vec{\sigma} X]$.

2. We can define $\exp(X)$ for a matrix X through the power series expansion $\exp(X) = \sum_n X^n/n!$ (provided the series is convergent).

   (a) Compute $\exp(X)$ for

$$X = \begin{pmatrix} 41/25 & 12/25 \\ 12/25 & 34/25 \end{pmatrix}.$$

   (Hint: Diagonalize X).

   (b) Compute $\exp(i\vec{\phi} \cdot \vec{\sigma})$ where ϕ is a real vector and $\vec{\sigma}$ are the Pauli matrices introduced above. Is $\exp(i\vec{\phi} \cdot \vec{\sigma})$ hermitean/unitary?

   (c) If A and B are matrices find a sufficient condition for $\exp(A+B) = \exp(A)\exp(B)$.

3. The anti-commutator of two linear operators is defined by $\{A, B\} = AB+BA$. Under what condition can two operators that satisfy $\{A, B\} = 0$ have a simultaneous eigenvector?

4. $|a\rangle$ and $|b\rangle$ are eigenvectors of a hermitean operator A. Under what condition can $|a\rangle + |b\rangle$ be an eigenvector?