Stochastic Fluid Dynamics and the QCD Critical Point

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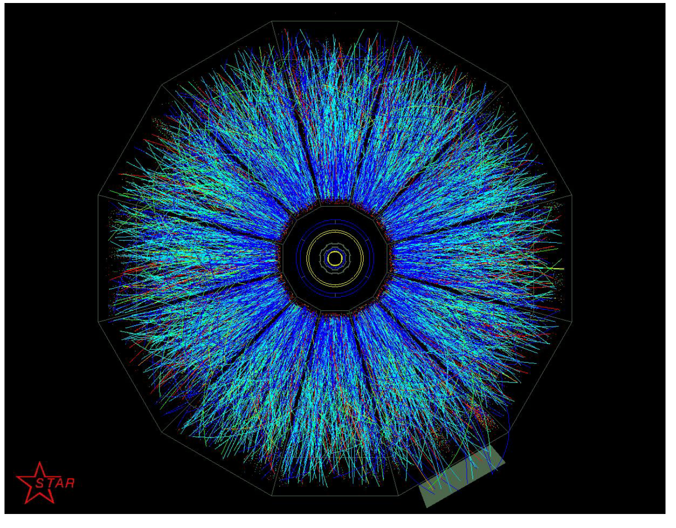
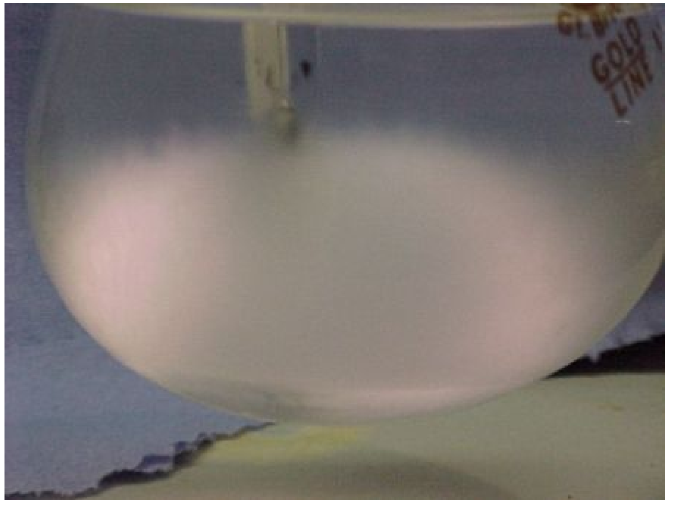
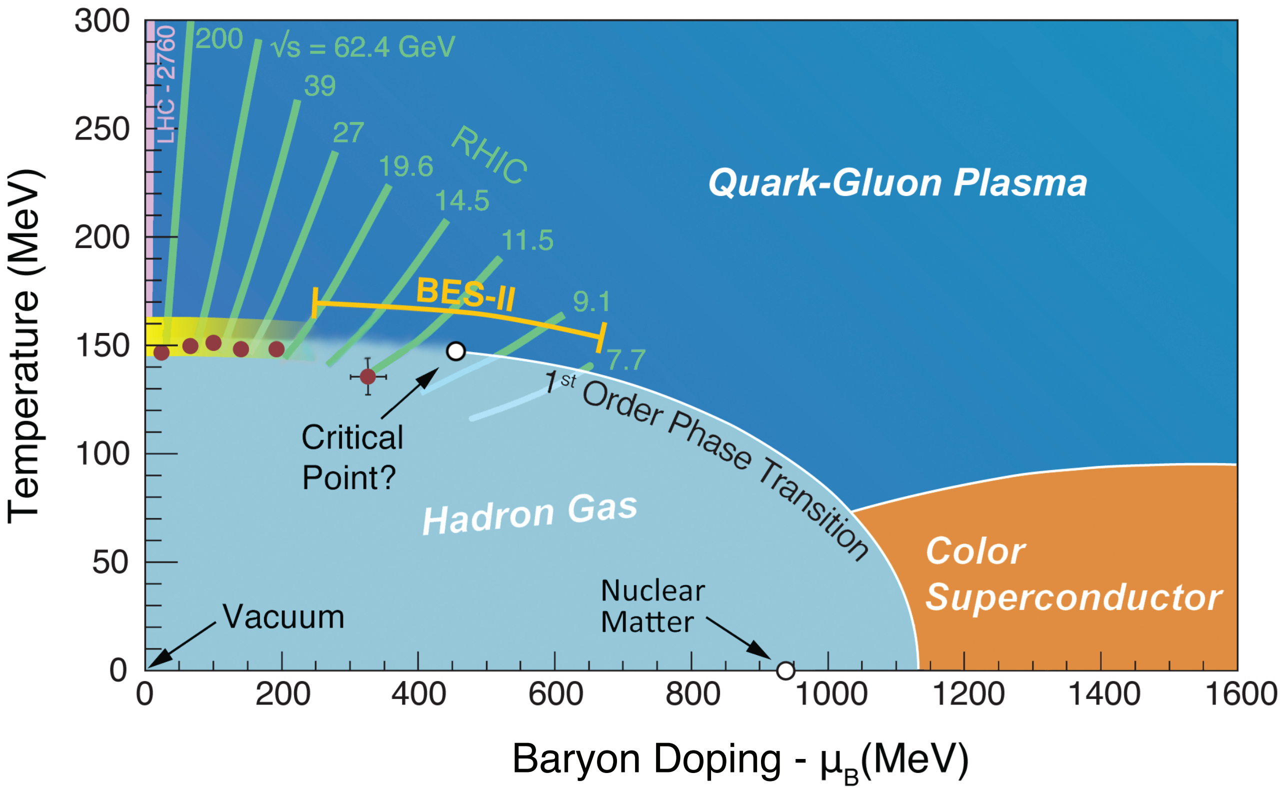
BEST COLLABORATION



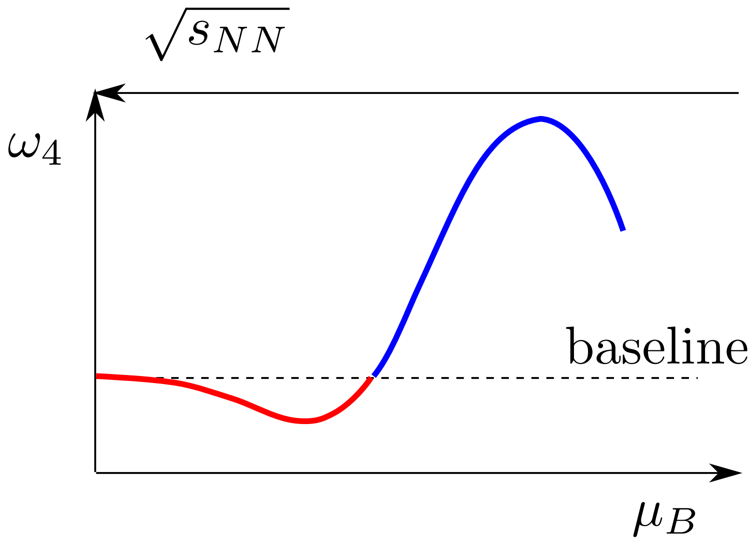


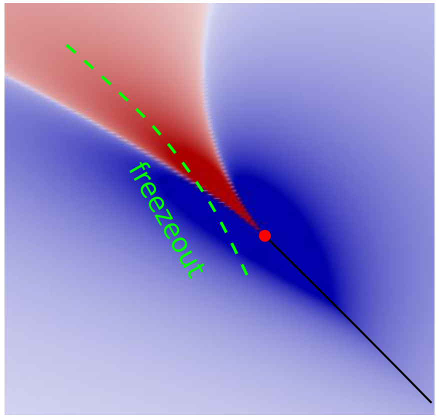
RHIC beam energy scan

Can we experimentally locate the QCD phase transition, either by detecting a critical point, or by identifying a first order transition?



Basic discovery idea: Study fluctuation observables. Expect non-monotonic variation of 4th order gallant near Ising critical point.





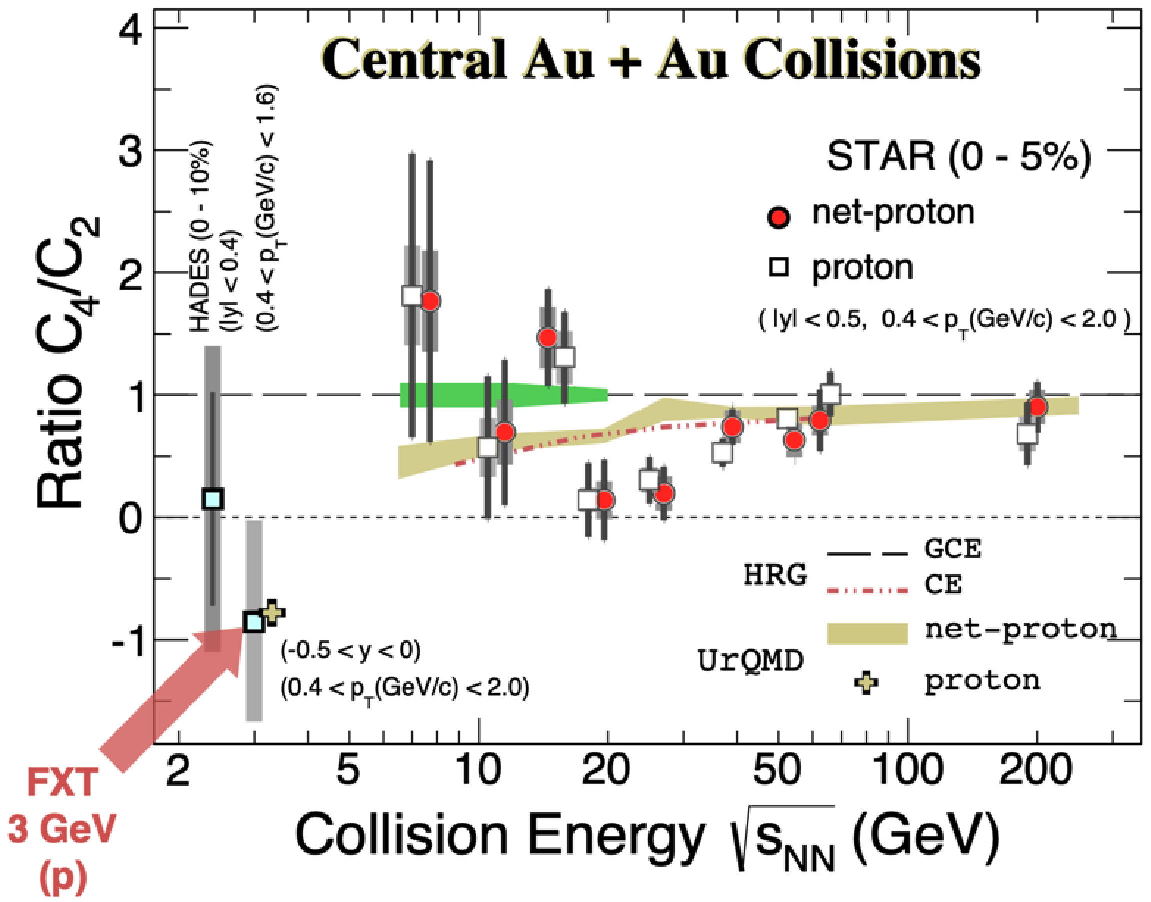
Real world may well be more complicated:

- Finite size and finite expansion rate effects.
- Non-equilibrium effects (memory, critical slowing).
- Freezeout, resonances, global charge conservation, etc.

Motivates dynamical studies.

Figure from Bzdak et al. [1906.00936]

RHIC beam energy scan, BESI



BESII data have been taken, and are being analyzed.

Dynamical Theory

What is the dynamical theory near the critical point?

The basic logic of fluid dynamics still applies. Important modifications:

- Critical equation of state.
- Stochastic fluxes, fluctuation-dissipation relations.
- Possible Goldstone modes (chiral field in QCD?)

Outline:

1. Stochastic field theories
2. Functional methods: The nPI action
3. Numerical approaches to stochastic diffusion
4. Hydrokinetics in an expanding background

1. Stochastic diffusion

Consider diffusion of a conserved charge

$$\partial_0 \psi + \vec{\nabla} \cdot \vec{j} = 0 \qquad \vec{j} = -D \nabla \psi + \ldots$$

Introduce noise and non-linear interactions

$$\partial_0 \psi = \kappa \nabla^2 \frac{\delta \mathcal{F}}{\delta \psi} + \xi$$

$$\mathcal{F} = \int d^d x \left[\frac{\gamma}{2} (\vec{\nabla} \psi)^2 + \frac{m^2}{2} \psi^2 + \frac{\lambda}{3} \psi^3 + \frac{u}{4} \psi^4 \right]$$

$$\langle \xi(x,t) \xi(x',t') \rangle = D T \nabla^2 \delta(x - x') \delta(t - t') \qquad D = \kappa m^2$$

Equilibrium distribution

$$P[\psi] \sim \exp\left(- \frac{\mathcal{F}[\psi]}{k_B T} \right)$$

Stochastic Field Theory

Noise average (noise kernel $L_0 = DT\nabla^2$)

$$\langle\psi\psi\ldots\rangle = \frac{1}{Z}\int D\xi\, e^{-\int \xi L_0^{-1}\xi}\int D\psi\,\delta\left(\partial_0\psi - \kappa\nabla^2\frac{\delta\mathcal{F}}{\delta\psi} - \xi\right)\psi\psi\ldots$$

Auxiliary field $\tilde{\psi}$

$$\langle\psi\psi\ldots\rangle = \frac{1}{Z}\int D\tilde{\psi}\,D\psi\,D\xi\, e^{-\int \xi L_0\xi} e^{-\int \tilde{\psi}(\partial_0\psi+\ldots)}\,\psi\psi\ldots$$

Integrate out noise, include sources for ψ and $\tilde{\psi}$

$$Z[\jmath,\tilde{\jmath}] = \int D\tilde{\psi}\,D\psi\,\exp(-S[\psi,\tilde{\psi},\jmath,\tilde{\jmath}])$$

Stochastic Field Theory

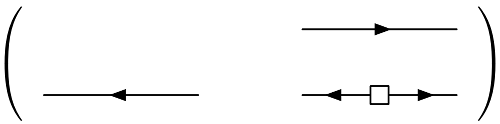
Stochastic effective lagrangian

$$\mathcal{L} = \tilde{\psi}\left(\partial_0 - D\nabla^2\right)\psi + \tilde{\psi}DT\nabla^2\tilde{\psi} + \tilde{\psi}D\lambda\nabla^2\psi^2 + \ldots$$

Diffusion Noise Interactions

Matrix propagator

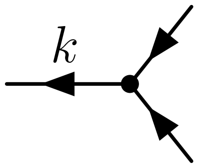
$$\begin{pmatrix} \langle\tilde{\psi}\tilde{\psi}\rangle & \langle\tilde{\psi}\psi\rangle \\ \langle\psi\tilde{\psi}\rangle & \langle\psi\psi\rangle \end{pmatrix} = \begin{pmatrix} 0 & G_R \\ G_A & G_S \end{pmatrix} =$$



Analytic structure of the Schwinger-Keldysh propagator

Interaction vertex

$D\lambda k^2$



What are the rules for constructing more general vertices?

Time reversal invariance

Stochastic theory must describe detailed balance

$$\frac{P(\psi_1 \to \psi_2)}{P(\psi_2 \to \psi_1)} = \exp\left(-\frac{\Delta\mathcal{F}}{k_B T}\right)$$

Related to T-reversal symmetry

$$\begin{aligned} \psi(t) &\to \psi(-t) \\ \tilde{\psi}(t) &\to -\left[\tilde{\psi}(-t) + \frac{\delta\mathcal{F}}{\delta\psi}\right] \end{aligned}$$

$$\mathcal{L} \to \mathcal{L} + \frac{d\mathcal{F}}{dt}$$

Ward identities: Fluctuation-Dissipation relations

$$2\kappa\,\mathrm{Im}\left\{k^2\langle\psi(\omega,k)\tilde{\psi}(-\omega,-k)\rangle\right\} = \omega\langle\psi(\omega,k)\psi(-\omega,-k)\rangle$$

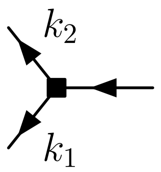
New and non-classical interactions

At this order (Ψ^3, ∇^2) there is one more interaction

$$\kappa \to \kappa(\psi) = \kappa_0 \left[1 + \lambda_D \psi\right] \qquad \mathcal{L} \sim D\lambda_D \tilde{\psi} \nabla^2 \psi^2$$

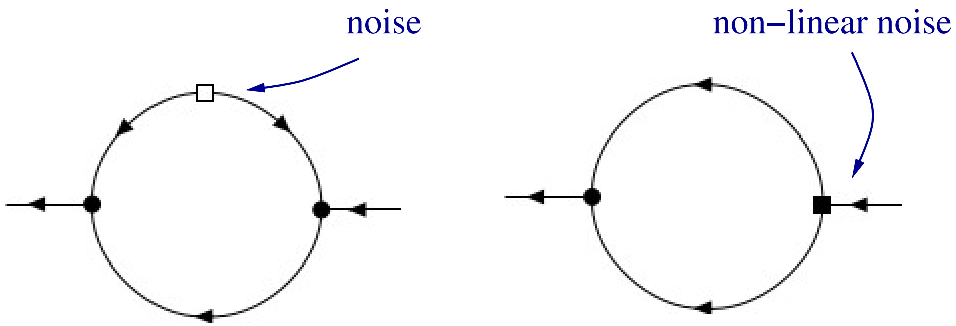
T-invariance fixes noise: $\mathcal{L} \sim D\lambda_D \psi (\vec{\nabla}\tilde{\psi})^2$

$D\lambda_D k_1 \cdot k_2$



Non-linear noise vertex

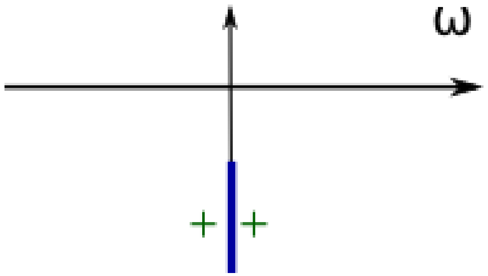
Retarded self energy



Contribute to (non-critical) order parameter relaxation

$$\Sigma(\omega, k) = \frac{\lambda'}{32\pi}\left(i\lambda'\omega k^2 + \lambda_D\left[i\omega - Dk^2\right]k^2\right)\sqrt{k^2 - \frac{2i\omega}{D}}$$

Analytical structure



Diffusive cut dominates over (split) diffusive pole.

Even higher order: Non-linear noise with no contribution to constitutive equations.

Chao, T.S. [2008.01269], see also Chen-Lin et al. [1811.12540] and Jain & Kovtun [2009.01356]

KMS effective actions

Modern ideas: Use "GB" field φ: $\dot{\varphi} \sim \mu \sim (\delta\mathcal{F})/(\delta\psi)$.

Define field on Keldysh contour: $\varphi_{1,2}$

$$\varphi_r = \frac{1}{2}(\varphi_1 + \varphi_2) \qquad \varphi_a = \varphi_2 - \varphi_1$$

Impose KMS condition and microscopic T-reversal

$$\varphi_1 \to \varphi_1(-t + i\theta) \qquad \varphi_2 \to \varphi_2(-t - i(\beta - \theta))$$

Semi-classical limit: $\tilde{\varphi}_r = \varphi_r$, $\tilde{\varphi}_a = \hbar\varphi_a$ ($\hbar \ll 1$)

$$\varphi_r \to \varphi_t(-t) \qquad \varphi_a \to -\varphi_a(-t) + i\dot{\varphi}_r$$

"KMS" symmetry

Glorioso & Liu [1805.09331]

KMS effective actions

Construct KMS invariants: Consider gauge invariant building blocks

$B^{\mu}_{r,a} = \partial^{\mu}\varphi_{r,a} + A^{\mu}_{r,a}$

$$\mathcal{L} = P'(\mu)B^t_a + iTDB^i_a\left(B^i_a + \partial_t B^i_r\right) + \ldots$$

Use $P'(\mu) = \psi$ and $\mu = (\delta\mathcal{F})/(\delta\psi)$

$$\mathcal{L} = \varphi_a\left(\partial_t + D\nabla^2\right)\psi + \ldots$$

Agrees with MSR effective action

More powerful tool for going to higher orders.

Coupling to conserved momentum density

Couple to momentum density. New ingredient: Poisson brackets

$$\partial_t \psi = \{\mathcal{H}, \psi = \int dx \, \frac{\delta \mathcal{H}}{\delta \pi} \{\pi, \psi\} = \frac{1}{w} \pi \cdot \nabla \psi + \ldots$$

Effective action

$$\mathcal{L} = \tilde{\pi}_T \left(\partial_t - \gamma \nabla^2\right) \pi_T + \frac{1}{w} \left[\tilde{\psi} \pi \cdot \nabla \psi + \tilde{\pi}_k \pi \cdot \nabla \pi_k\right] + \tilde{\pi}_k (\nabla^2 \psi) \nabla_k \psi + \ldots$$

Diffusion of shear waves, $\gamma = \eta / w$. Advection of ψ, π by π.

PB couplings do not get renormalized.

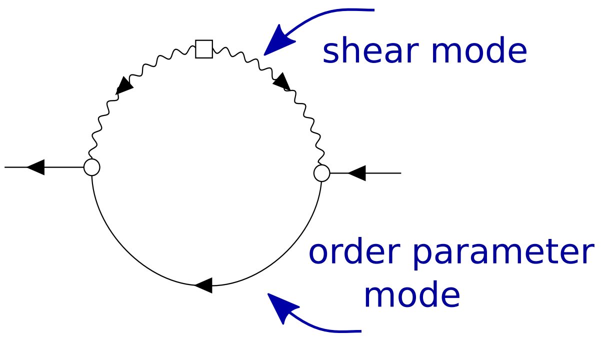
Model H: Critical Dynamics

Non-critical fluids: Gradient expansion $k\xi \ll 1$.

Critical fluids: RG analysis, study possible fixed points.

"Mode Coupling" approximation: Use bare shear viscosity, and static susceptibility χ_k

$$G^{-1}(\omega, k) = i\omega - Dk^2 - \delta\Gamma_k$$



Order parameter relaxation rate ("Kawasaki function").

$$\Gamma_k = \frac{T\xi^{-3}}{6\pi\eta_0} K(k\xi) \qquad K(x) = \frac{3}{4}\left[1 + x^2 + \left(x^3 + x^{-1}\right)\arctan(x)\right] .$$

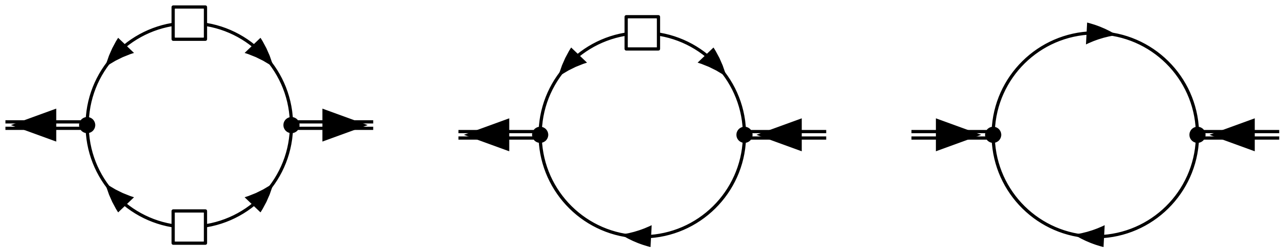
Dynamic critical exponent: $\Gamma_{\xi^{-1}} \sim \xi^{-z}$ with $z = 3$

2. 1PI effective action

Consider 1PI effective action

$$\Gamma[\Psi, \tilde{\Psi}] = W[J, \tilde{J}] - \int dt\, d^3x \left(J\Psi + \tilde{J}\tilde{\Psi} \right) \qquad \frac{\delta W}{\delta J} = \langle \psi \rangle = \Psi \,,$$

Loop expansion



"Classical" equation of motion

$$(\partial_t - D\nabla^2)\Psi - \frac{D\lambda^2}{2}\nabla^2\Psi^2 + \int d^3x\, dt\, \Psi(x', t')\Sigma(x, t; x', t') = 0$$

2PI effective action

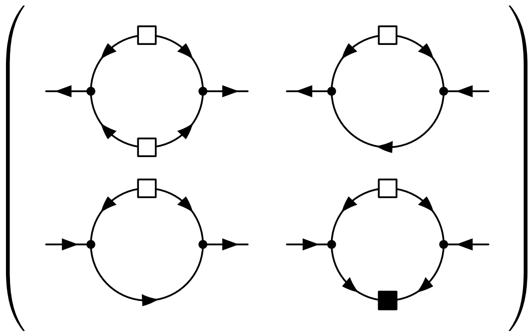
Consider 2PI effective action

$$\Gamma[\Psi_a, G_{ab}] = W[J_a, K_{ab}] - J_A \Psi_A - \frac{1}{2} K_{AB} \left[\Psi_A \Psi_B + G_{AB} \right]$$

Matrix propagator G_{ab}, Bilocal source K_{ab}

Equation of motion for Ψ_a unchanged, but Σ_{ab} satifies Dyson-Schwinger equation

$$\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} =$$



3. Stochastic diffusion

Stochastic relaxation equation ("model A")

$$\partial_t \psi = -\Gamma \frac{\delta \mathcal{F}}{\delta \psi} + \zeta \qquad \langle \zeta(x,t)\zeta(x',t')\rangle = DT\delta(x-x')\delta(t-t')$$

Naive discretization

$$\psi(t+\Delta t) = \psi(t) + (\Delta t)\left[-D\frac{\delta \mathcal{F}}{\delta \psi} + \sqrt{\frac{DT}{(\Delta t)a^3}}\theta\right] \qquad \langle \theta^2 \rangle = 1$$

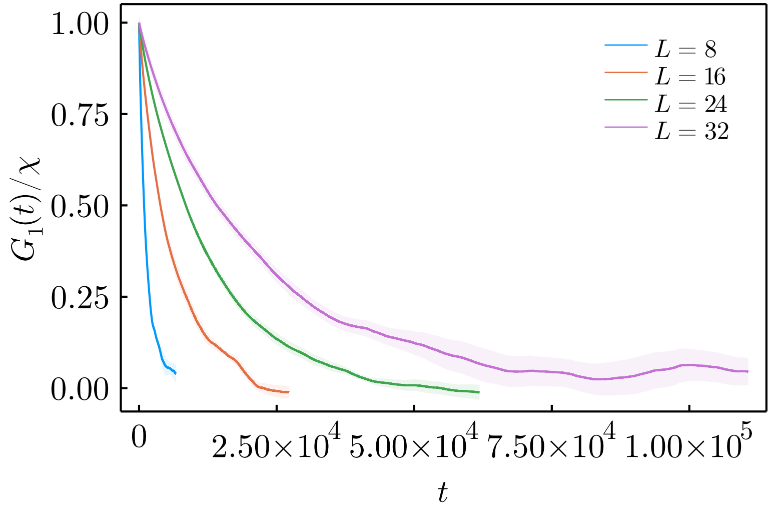
Noise dominates as $\Delta t \to 0$, leads to discretization ambiguities in the equilibrium distribution.

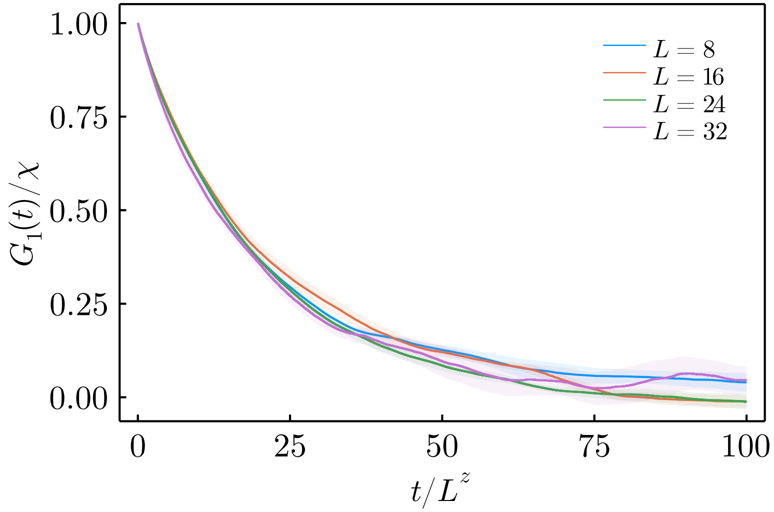
Idea: Add Metropolis step

$$\psi(t+\Delta t) = \psi(t) + \sqrt{2\Gamma(\Delta t)}\theta \qquad p = min(1, e^{-\beta\Delta\mathcal{F}})$$

Dynamic scaling (model A)

Correlation functions at T_c, $V = L^3$, $L = 8, 16, 24, 32$



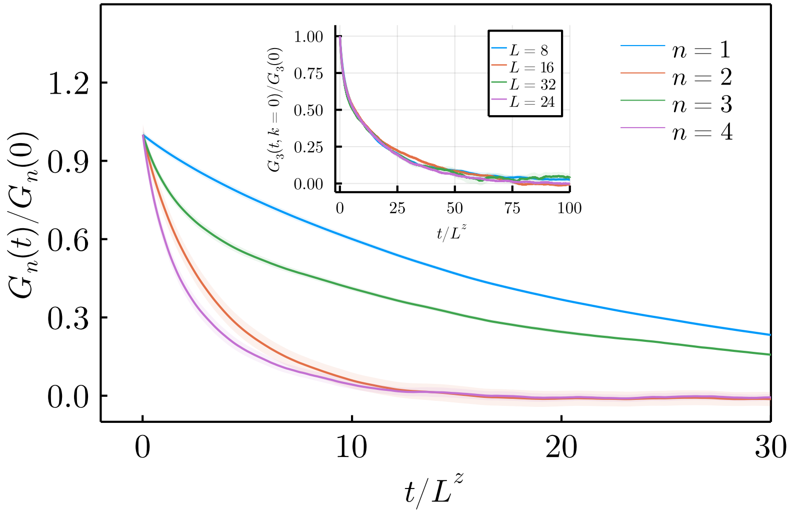


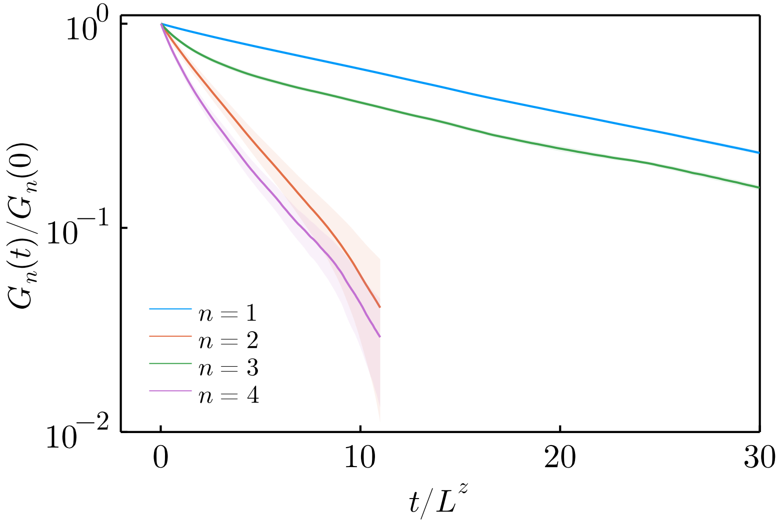
$$G_1(t) = \langle M(0)M(t) \rangle \qquad M(t) = \int d^3x\, \psi(x)$$

Dynamic critical exponent $z = 2.026(56)$.

Correlation functions of higher moments

Correlation functions at T_c



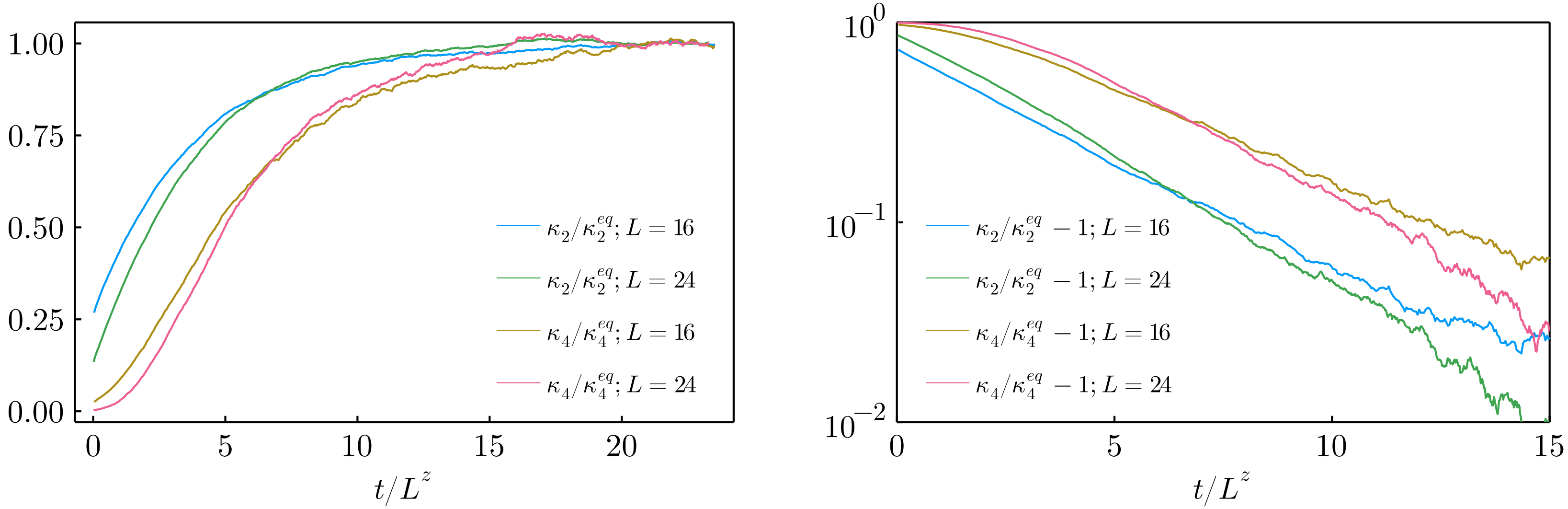


$$G_n(t) = \langle M^n(0) M^n(t) \rangle \qquad M(t) = \int d^3x\, \psi(x)$$

Inset: Dynamic scaling of $G_3(t)$ with $z = 2.026(56)$.

Relaxation after a quench

Thermalize at $T > T_c$. Study evolution at T_c



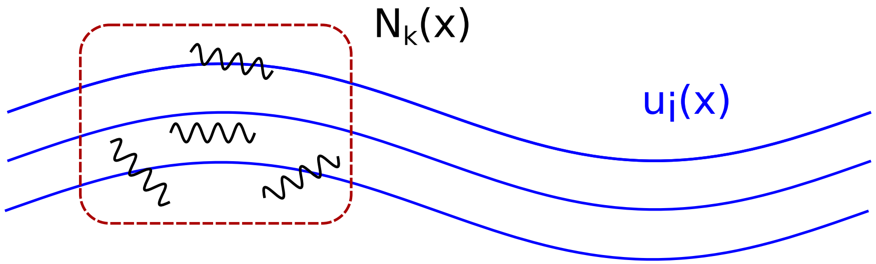
$$C_n(t) = \langle\langle M^n(t)\rangle\rangle_{M(0)} \qquad M(t) = \int d^3x\, \psi(x)$$

4. Fluctuations in an expanding fluid

Consider linearized stochastic dynamics about a fluid background.

Determine eigenmodes: two sound $\phi_\pm$, three diffusive modes $\phi_\psi, \phi_{\vec{\pi}_T}$.

Noise average: Consider equal time 2-point fct $W_{ab} = \langle \phi_a(\tau, x)\phi_b(\tau, x')\rangle$.



Wigner function representation: $W_{ab}(\tau, x, k)$. Diagonal component $N_{a,k}(\tau, x)$ is a phase space density of hydro fluctuations.

Akamatsu et al. (2016), Martinez, T.S. (2017).

Critical mode in expanding system

Study transit of critical point: Consider $\hat{s} = s/n$ and follow "mode coupling" philosophy. Use static susceptibility and critical relaxation rate $\Gamma_{\hat{s}}$.

$$\partial_t N_{\hat{s}}(t,k) = -2\Gamma_{\hat{s}}(t,k)\left[N_{\hat{s}}(t,k) - N^0_{\hat{s}}(t,k)\right] + \ldots,$$

$$\Gamma_{\hat{s}}(t,k) = \frac{\lambda_T}{C_p\xi^2}(k\xi)^2(1+(k\xi)^{2-\eta}), \qquad N^0_{\hat{s}}(t,k) = \frac{C_p(t)}{(1+(k\xi)^{2-\eta})},$$

Correlation length $\xi(t) = \xi(n(t), e(t)) = \xi_0 f_\xi(r(t), h(t))$

$$\text{hydro}: \frac{\partial_t n}{n} \sim \frac{\partial_t e}{e} \sim \frac{1}{\tau_{exp}} \qquad \text{Ising map}: (e,n) \to (r,h)$$

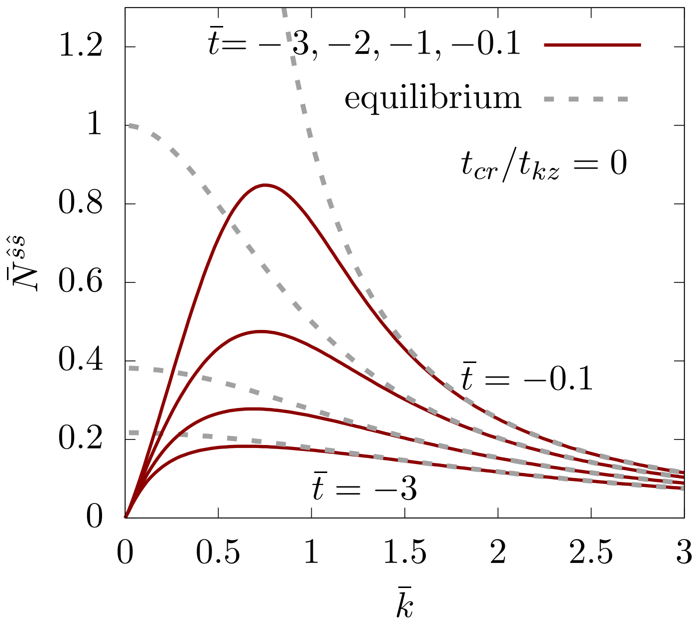
Emergent time scale t_{KZ}: Expansion rate matches relaxation time for modes with $k^* \sim \xi^{-1}$ (modes fall out of equilibrium).

Emergent length scale l_{KZ}: $l_{KZ} = \xi(t_{KZ})$. $l_{KZ} \sim 1.6$ fm

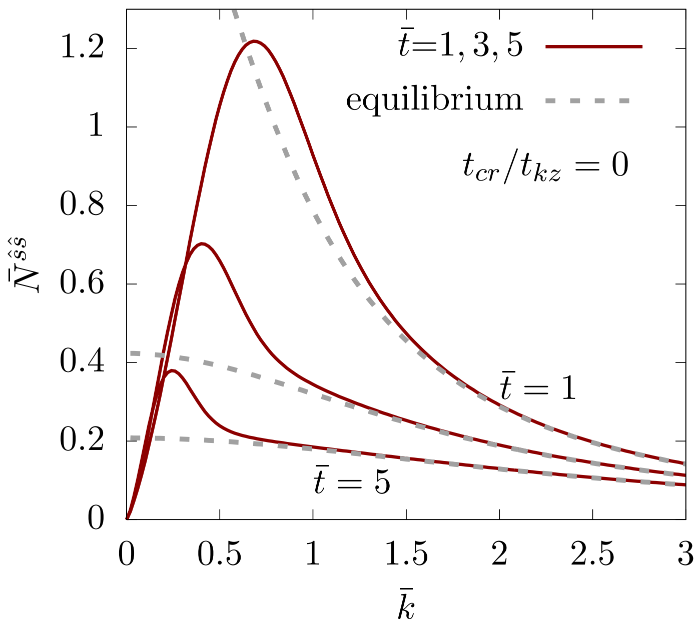
Akamatsu, Teaney, Yan, Yi [1811.05081], Martinez, Schaefer, Skokov [1906.11306], see also Berdnikov, Rajagopal [hep-ph/9912274]

Expanding System: Numerical Results

$$\bar{k} = kl_{KZ},\ \bar{t} = t/t_{KZ}$$



before CP



after CP

Akamatsu, Teaney, Yan, Yi [1811.05081]

Summary

Dynamical evolution of fluctuations is important.

Old and new ideas about effective actions on the Keldysh contour. In principle allows systematic derivation of hydro equations for n-point functions.

Alternative approach: Direct simulation of stochastic fluid dynamics. New idea: Ignore backreaction, and use Metropolis (or heat bath?) algorithm.

Not discussed: From conserved charges to particles.