Fluctuations in Fluid Dynamics

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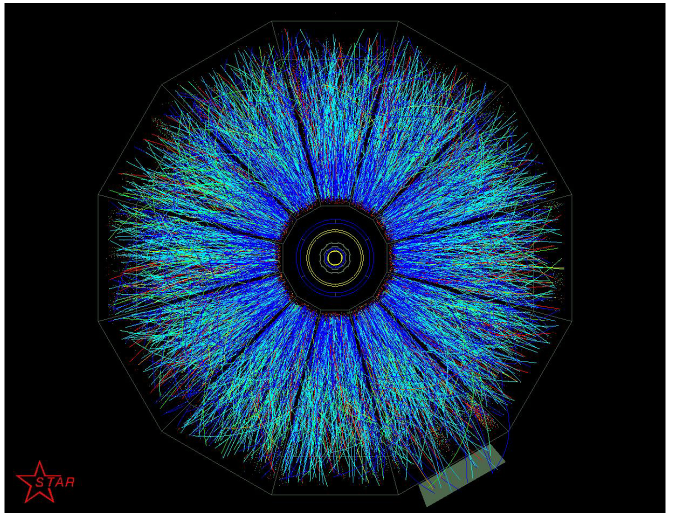
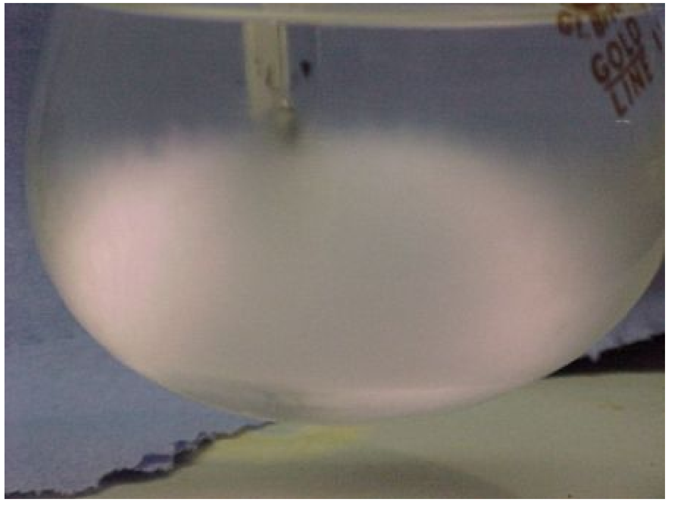
BEST COLLABORATION





Why consider fluctuations?

For consistency: Satisfy fluctuation-dissipation relations.

Fluid dynamics as an EFT: Fluctuations determine non-analyticities in (ω, k), and encode the resolution dependence of low energy parameters (such as transport coefficients).

Role of fluctuations enhanced in nearly perfect fluids ($\eta/s \lesssim 1$).

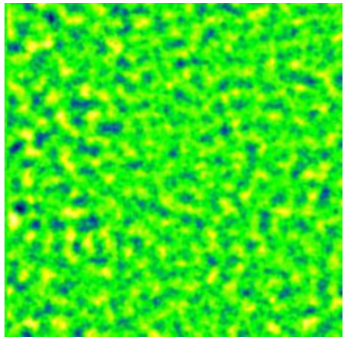
Fluctuations are dominant near critical points.

Part I: Non-critical Fluids

Main application: Ultracold Fermi Gases

The unitary Fermi gas is a scale invariant, strongly interacting, non-relativistic fluid ($(\eta/s)_{min} < \hbar/k_B$). Can detune from unitarity to study scale breaking, and tune temperature to study classical to quantum transition, including transition to superfluid.

Beyond gradients: Hydrodynamic fluctuations



Hydrodynamic variables fluctuate

$$\langle \delta v_i(x,t)\delta v_j(x',t)\rangle = \frac{T}{\rho}\delta_{ij}\delta(x-x')$$

Linearized hydrodynamics propagates fluctuations as shear or sound

$$\langle \delta v_i^T \delta v_j^T \rangle_{\omega,k} = \frac{2T}{\rho}(\delta_{ij} - \hat{k}_i\hat{k}_j)\frac{\nu k^2}{\omega^2 + (\nu k^2)^2} \qquad shear$$

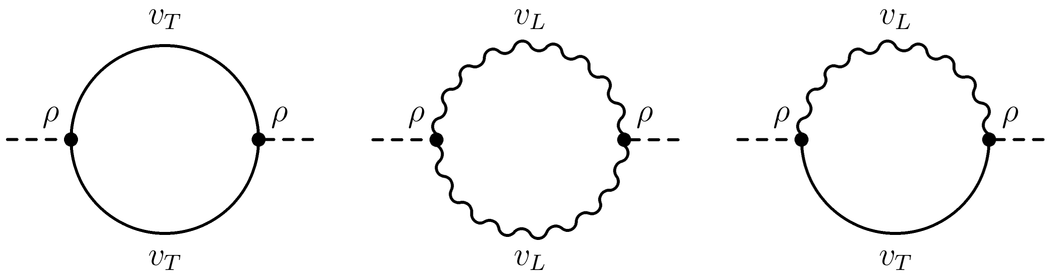
$$\langle \delta v_i^L \delta v_j^L \rangle_{\omega,k} = \frac{2T}{\rho}\hat{k}_i\hat{k}_j\frac{\omega k^2\Gamma}{(\omega^2 - c_s^2k^2)^2 + (\omega k^2\Gamma)^2} \qquad sound$$

$v = v_T + v_L$: $\nabla \cdot v_T = 0$, $\nabla \times v_L = 0$ $\qquad \nu = \eta/\rho$, $\Gamma = \frac{4}{3}\nu + \ldots$

Hydro Loops: "Breakdown" of second order hydro

Correlation function in hydrodynamics

$$G_S^{xyxy} = \langle \{ \Pi^{xy}, \Pi^{xy} \} \rangle_{\omega,k} \simeq \rho_0^2 \langle \{ v_x v_y, v_x v_y \} \rangle_{\omega,k}$$



Match to response function in $\omega \to 0$ (Kubo) limit

$$G_R^{xyxy} = P + \delta P - i\omega[\eta + \delta\eta] + \omega^2 \left[\eta\tau_\pi + \delta(\eta\tau_\pi) \right]$$

with

$$\delta P \sim T\Lambda^3 \qquad \delta\eta \sim \frac{T\rho\Lambda}{\eta} \qquad \delta(\eta\tau_\pi) \sim \frac{1}{\sqrt{\omega}} \frac{T\rho^{3/2}}{\eta^{3/2}}$$

Hydro Loops: RG and "breakdown" of 2nd order hydro

Cutoff dependence can be absorbed into bare parameters. Non-analytic terms are cutoff independent.

Fluid dynamics is a "renormalizable" effective theory.

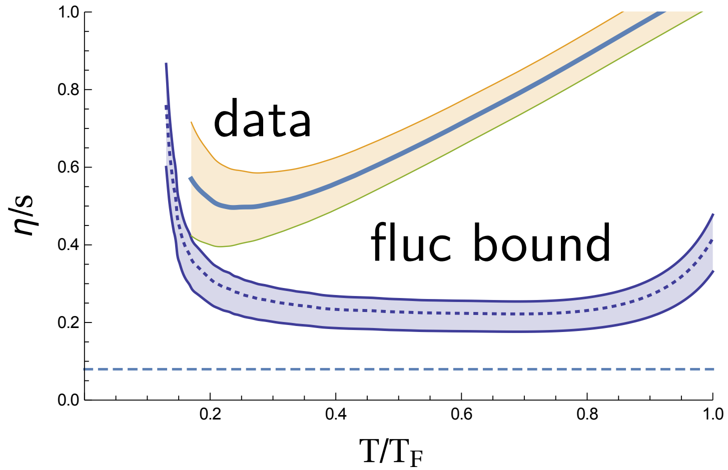
Small η enhances fluctuation corrections: $\delta\eta \sim T\left(\frac{\rho}{\eta}\right)^2\left(\frac{P}{\rho}\right)^{1/2}$

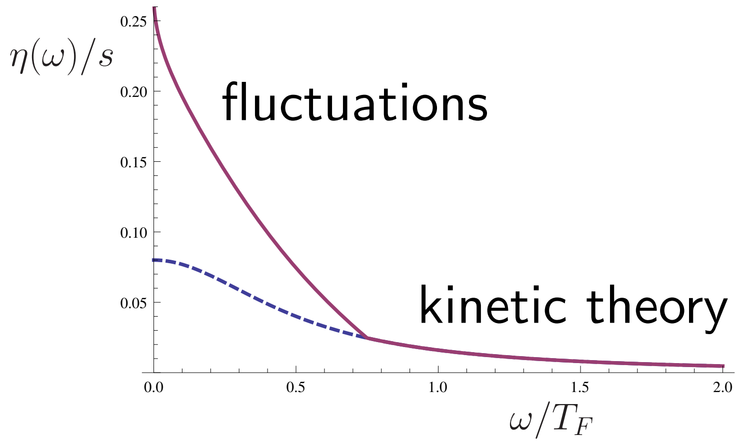
Small η leads to large $\delta\eta$: There must be a bound on η/n.

Relaxation time diverges: $\delta(\eta\tau_\pi) \sim \frac{1}{\sqrt{\omega}}\left(\frac{\rho}{\eta}\right)^{3/2}$

2nd order hydro without fluctuations inconsistent.

Fluctuation induced bound on η/s





$(\eta/s)_{min} \simeq 0.2$

spectral function

non-analytic $\sqrt{\omega}$ term

Schaefer, Chafin (2012), see also Kovtun, Moore, Romatschke (2011)

Fluctuation induced bulk stresses

Kubo relation for bulk viscosity

$$\zeta = -\lim_{\omega\to 0} \mathrm{Im}\frac{1}{9\omega}\int dt d^3x\, e^{-i\omega t}\,\langle[\Pi_{ii}(t,x),\Pi_{jj}(0)]\Theta(t)\rangle$$

Scale invariance not manifest

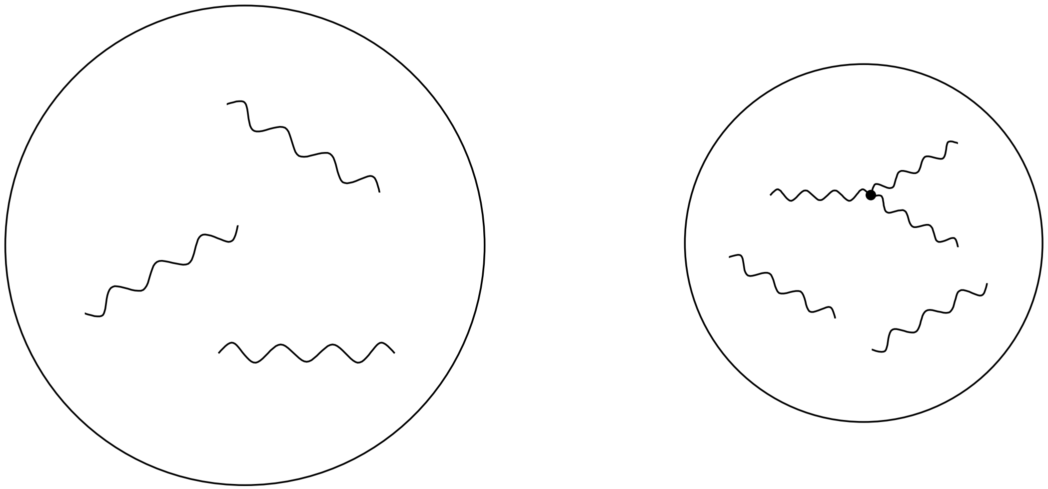
May use conservation of energy $\partial_t\mathcal{E} + \vec{\nabla}\cdot\vec{\jmath}^{\epsilon} = 0$ to rewrite Kubo formula

$$\zeta = -\lim_{\omega\to 0} \mathrm{Im}\frac{1}{\omega}\langle[\mathcal{O}(t,x),\mathcal{O}(0)]\rangle_{\omega k} \qquad \mathcal{O} = \frac{1}{3}\Pi_{ii} - \frac{2}{3}\mathcal{E}$$

and consider coupling to fluctuations of ρ and T

$$\mathcal{O} = \mathcal{O}_0 + a_{\rho\rho}(\Delta\rho)^2 + a_{\rho T}\Delta\rho\Delta T + a_{TT}(\Delta T)^2 + \ldots .$$

Fluctuation induced bulk stresses



Fluctuation contribution to bulk spectral function ($A_i \sim (P - \frac{2}{3}\mathcal{E})^2$):

$$\zeta(\omega) = \zeta(0) - \left(\frac{A_T}{(2D_T)^{3/2}} + \frac{A_\Gamma}{\Gamma^{3/2}} \right) \frac{\sqrt{\omega}}{36\sqrt{2}\pi} .$$

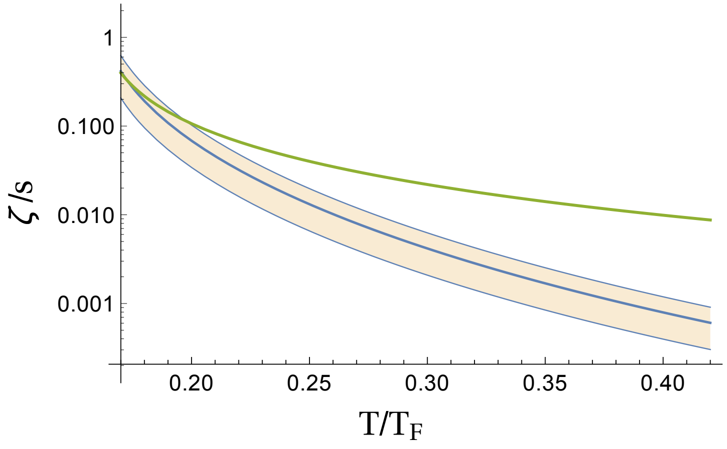
Fluctuation bound

$$\zeta_{min} = \left(\frac{A_T}{2D_T^2} + \frac{\sqrt{5}A_\Gamma}{\sqrt{3}\Gamma^2} \right) \sqrt{\frac{T}{m}} .$$

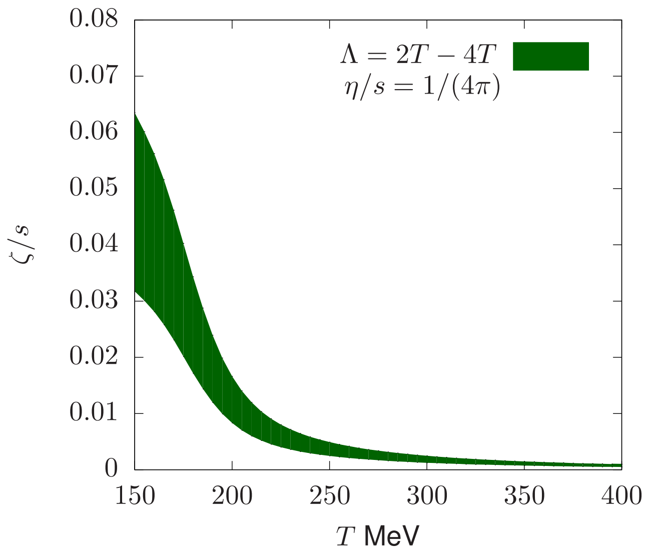
Consider $\lambda/a \sim 1$. Get $\zeta/s \gtrsim 0.1$

Fluctuation induced bound on ζ/s

(Detuned) Unitary Fermi Gas



Quark Gluon Plasma



Non-relativistic fluid, M. Martinez, T. S. (2017); relativistic fluid, Akamatsu et al. (2018)

See also Kovtun, Yaffe (2003)

Part II: Critical Fluids

Main application: QGP at RHIC

Expectation: If there is a critical endpoint in the QCD phase diagram, then the dynamical universality class is that of model H (liquid-gas).
Consider simplifications: Purely diffusive dynamics of order parameter mode (model B), or coupled dynamics truncated at second order moments.

Digression: Diffusion

Consider a Brownian particle

$$\dot{p}(t) = -\gamma_D p(t) + \zeta(t) \qquad \langle \zeta(t)\zeta(t') \rangle = \kappa \delta(t - t')$$

drag (dissipation) white noise (fluctuations)

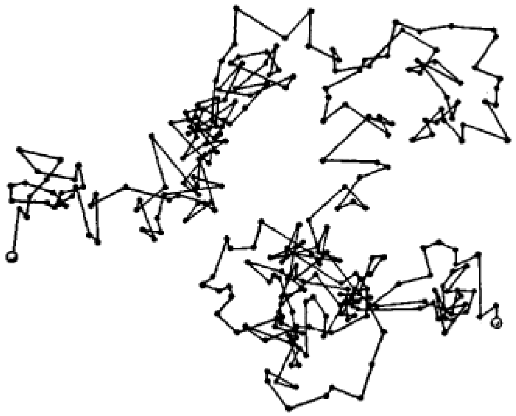
For the particle to eventually thermalize

$$\langle p^2 \rangle = 2mT$$

drag and noise must be related

$$\kappa = \frac{mT}{\gamma_D}$$

Einstein (Fluctuation-Dissipation)



Hydrodynamic equation for critical mode

Equation of motion for critical mode ϕ ("model H")

$$\frac{\partial \phi}{\partial t} = D\nabla^2 \frac{\delta \mathcal{F}}{\delta \phi} - g\vec{\nabla}\phi \cdot \frac{\delta \mathcal{F}_T}{\delta \vec{\pi}} + \zeta_\phi$$

Diffusive Reactive White Noise

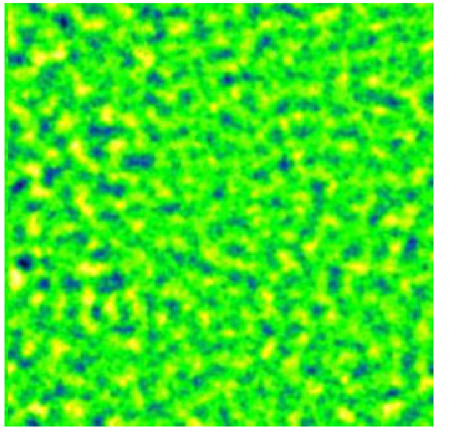
Free energy functional: Order parameter ϕ, momentum density $\vec{\pi} = \rho\vec{v}$

$$\mathcal{F} = \int d^d x \left[\frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{r}{2}\phi^2 + \lambda\phi^4 + \frac{1}{2}\vec{\pi}^2 \right]$$

Fluctuation-Dissipation relation

$$\langle \zeta_\phi(x,t)\zeta_\phi(x',t') \rangle = 2DT\delta(x-x')\delta(t-t')$$

ensures $P[\phi] \sim \exp(-\mathcal{F}[\phi]/T)$

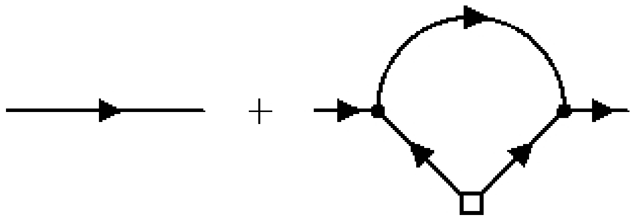


Linearized analysis (non-critical fluid)

Navier-Stokes equation: $\partial_0 \vec{v} + \nu \nabla^2 \vec{v} = mode\ couplings + noise$

Linearized propagator: $\langle \delta v_i^T \delta v_j^T \rangle_{\omega,k} = \frac{1}{\rho} \frac{-\nu k^2 P_{ij}^T}{-i\omega + \nu k^2}$ $\qquad \nu = \frac{\eta}{\rho}$

Fluctuation correction:



Renormalized viscosity:

$$\eta = \eta_0 + c_\eta \frac{T\rho\Lambda}{\eta_0} - c_\tau \sqrt{\omega} \frac{T\rho^{3/2}}{\eta_0^{3/2}}$$

Hydro is a renormalizable stochastic field theory

Linearized analysis (critical fluid)

Consider order parameter mode

$$\partial_0 \phi = -D\nabla^2 \frac{\delta \mathcal{F}}{\delta \phi} + mode\ couplings + \zeta_\phi$$

$$\mathcal{F} = \int d^3x \left\{ \frac{1}{2}(\nabla\phi)^2 + \frac{r}{2}\phi^2 + \lambda\phi^4 + \frac{1}{2}\vec{\pi}^2 \right\}$$

Dispersion relation $i\omega = Dq^2(r+q^2) + \ldots$

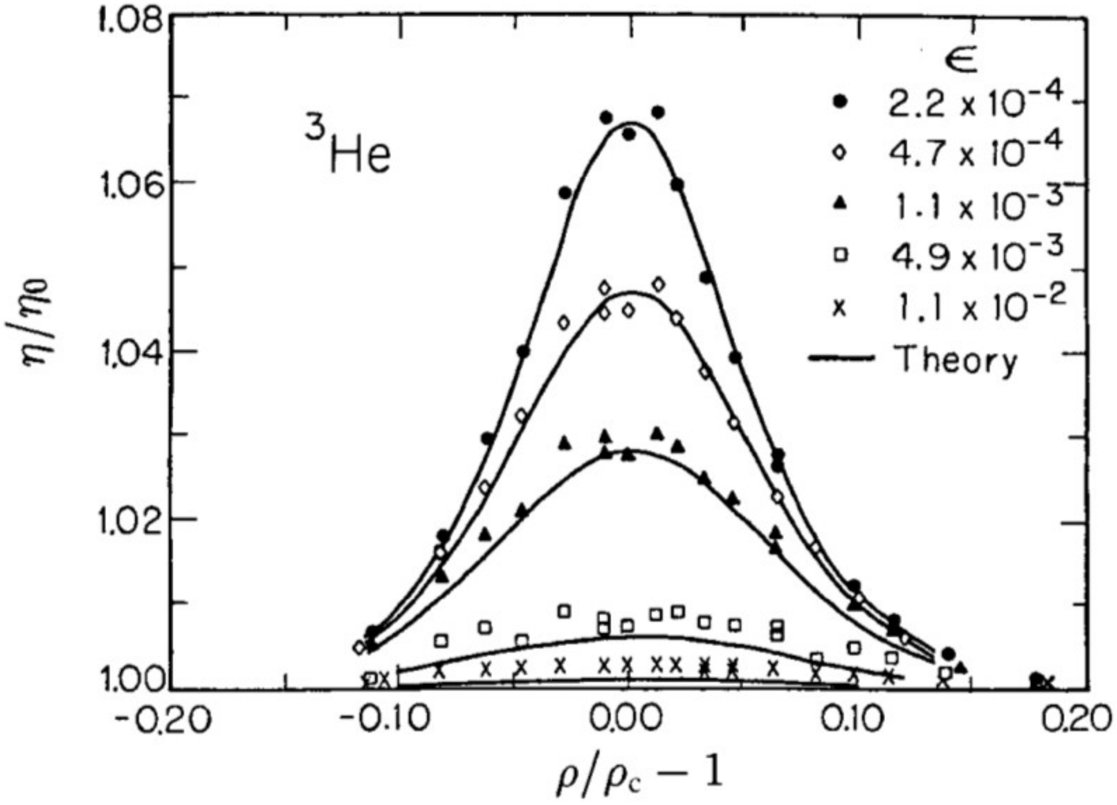
Use $r \sim \xi^{-2}$. Relaxation time for modes $q \sim \xi^{-1}$:

$$\tau \sim \xi^z \quad (z = 4) \qquad \text{"Critical slowing down"}$$

A more sophisticated analysis gives $z \simeq 3$ and

$$\eta \sim \xi^{0.05} \qquad \kappa \sim \xi^{0.9} \qquad \zeta \sim \xi^{2.8}$$

Critical transport (helium)



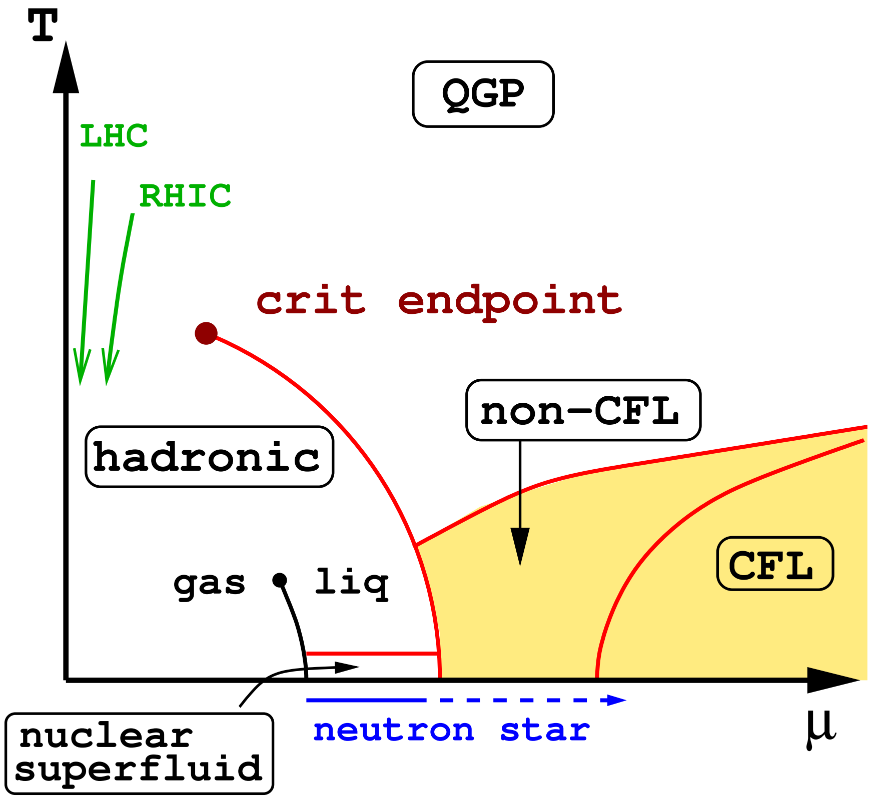
Agosta, Wang, Meyer (1987)

More dramatic enhancement in thermal conductivity and bulk viscosity (sound attenuation)

Critical endpoint in QCD?

What happens for $\mu \neq 0$? Lattice calculations cannot tell (the QCD sign problem). Two options: The transition weakens, or it strengthens.

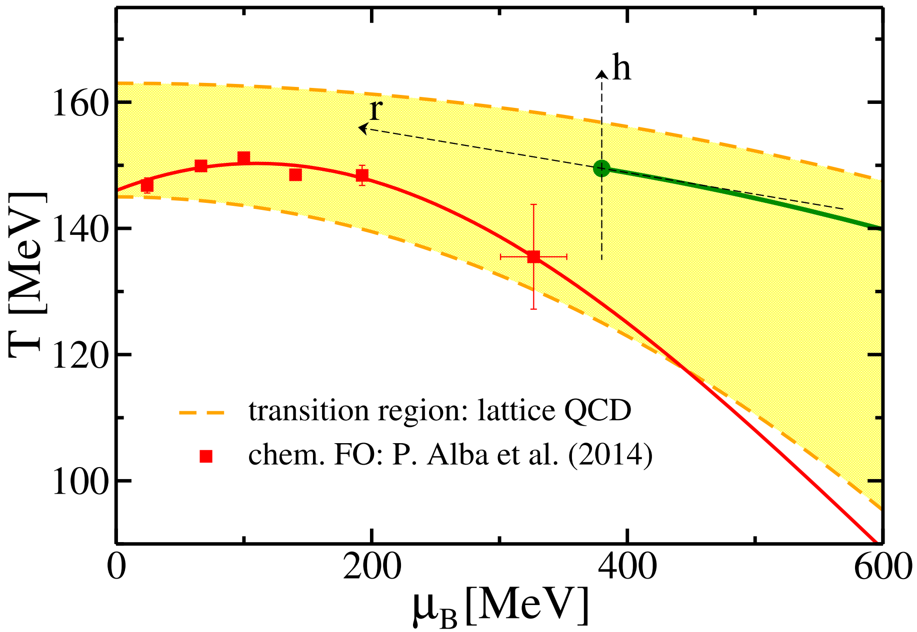
If the transition strengthens for $\mu > 0$ then there is a critical endpoint.



Critical endpoint in QCD?

Several possible order parameters: $\langle\bar\psi\psi\rangle - \Sigma_0$, $\rho - \rho_0$, $s - s_0$.

All of them mix, obtain one critical mode. Free energy in $d = 3$ Ising universality class.



$$\mathcal{F} = \kappa(\nabla\phi)^2 + r\phi^2 + \phi h + \lambda\phi^4$$

External field h.

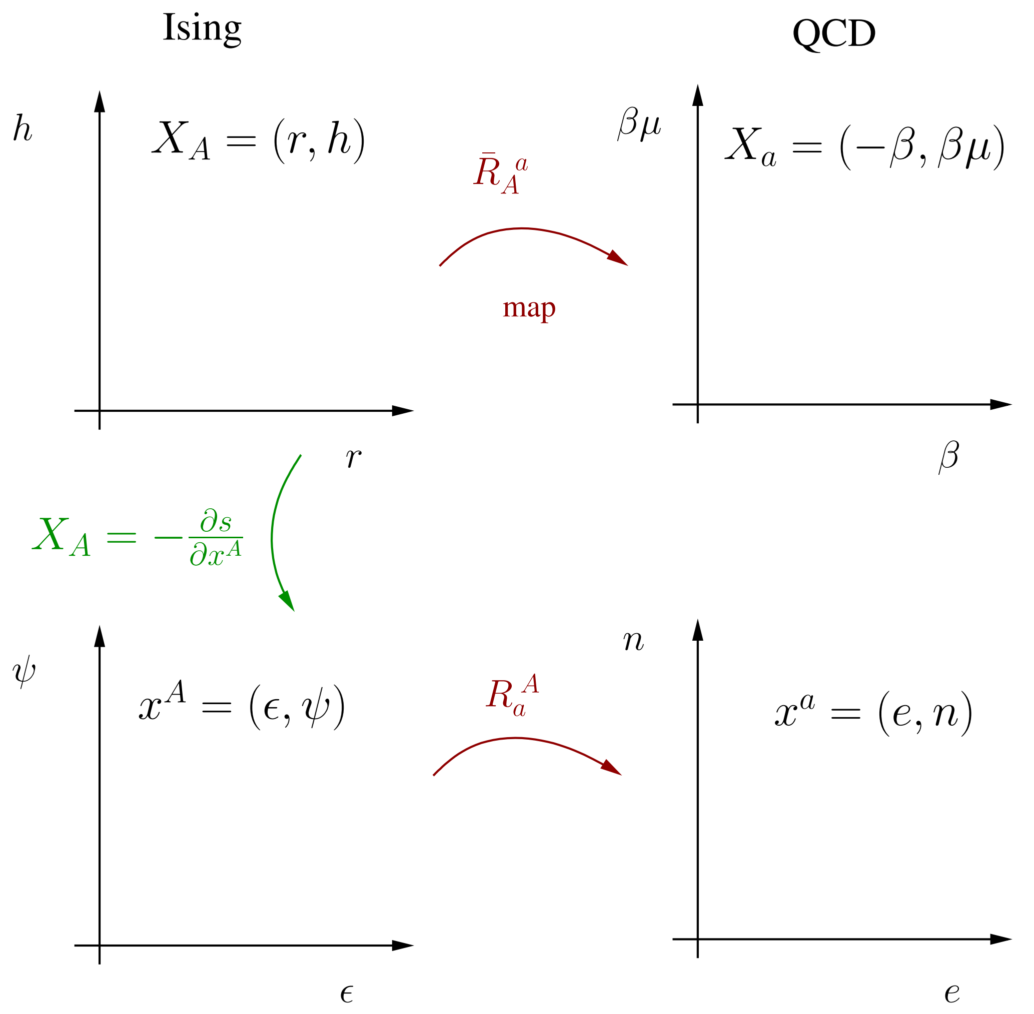
Reduced temperature r.

$$\xi \sim r^{-\nu}$$

Freezout curve (exp). Transition regime (lattice). Critical line (model).

Mapping the Isisng EOS to QCD

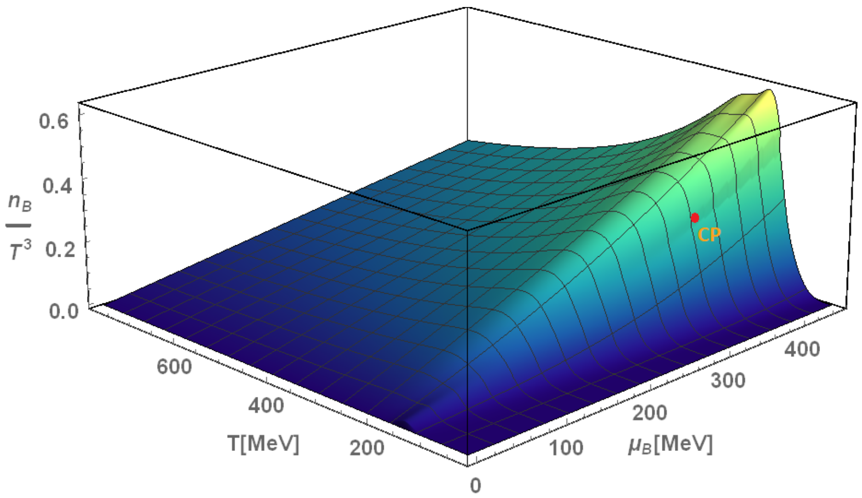
Map QCD variables on Ising equation of state $F(h,t)$

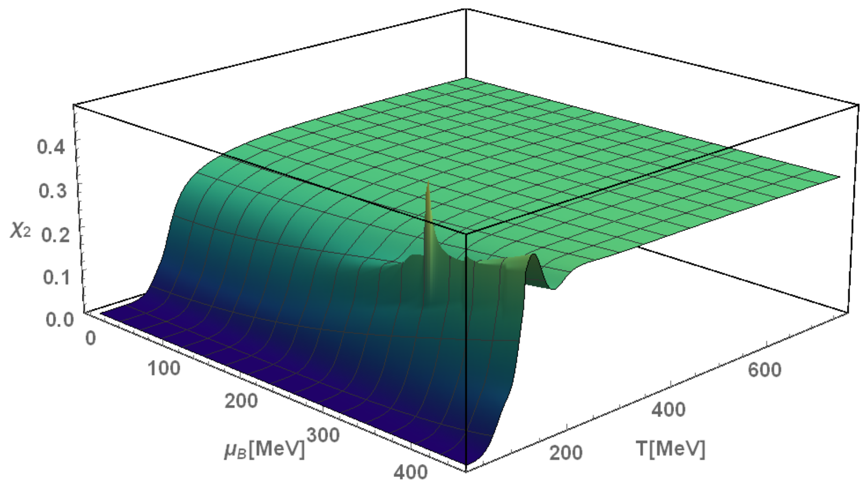


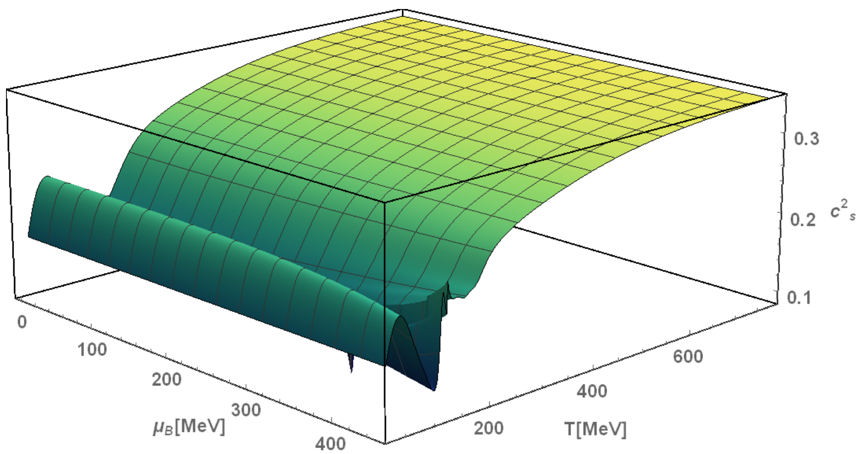
Zinn-Justin parmeterization: $G(\psi,t) = R^{2-\alpha} g(\theta)$

$$\{\psi = R^\beta \theta, t = R(1-\theta^2)\}$$

Critical equation of state for QCD







Add non-critical QCD part (using lattice Taylor exapnsion)

Parotto et al. (2018)

Critical contribution to bulk viscsosity

Consider critical entropy functional

$$S[\psi,\epsilon] = -\int d^3x \left\{\kappa(\nabla\psi)^2 + \frac{r}{2}\psi^2 + \frac{u}{4}\psi^4 + \gamma\epsilon\psi^2 + \frac{1}{2C_0}\epsilon^2\right\} + S_0 + \frac{E_0}{T}$$

Can be related back to critical Gibbs free energy

$$\beta G[\psi,t] = \int d^3x \left\{\kappa(\nabla\psi)^2 + \frac{\tilde{r}}{2}\psi^2 + \frac{\tilde{u}}{4}\psi^4\right\}$$

where $\tilde{r} = r - 2\gamma C_0 t/T_0$

Fluctuations of QCD pressure

$$\delta P = \frac{e+P}{\beta}\frac{\partial\psi}{\partial(\delta e)}\frac{\partial s^{Is}}{\partial\psi} + \frac{n}{\beta}\frac{\partial\epsilon}{\partial(\delta n)}\frac{\partial s^{Is}}{\partial\epsilon}$$

$$\delta P = nTAa_{n\epsilon}\gamma\psi^2$$

Critical contribution to bulk viscsosity

Kubo relation for bulk viscosity

$$\zeta(\omega) = (nAa_{\epsilon n}\gamma)^2 T \int_0^\infty dt d^3x e^{-i\omega t} \langle \psi^2(0)\psi^2(t,x)\rangle$$

Order parameter relaxation rate

Final result

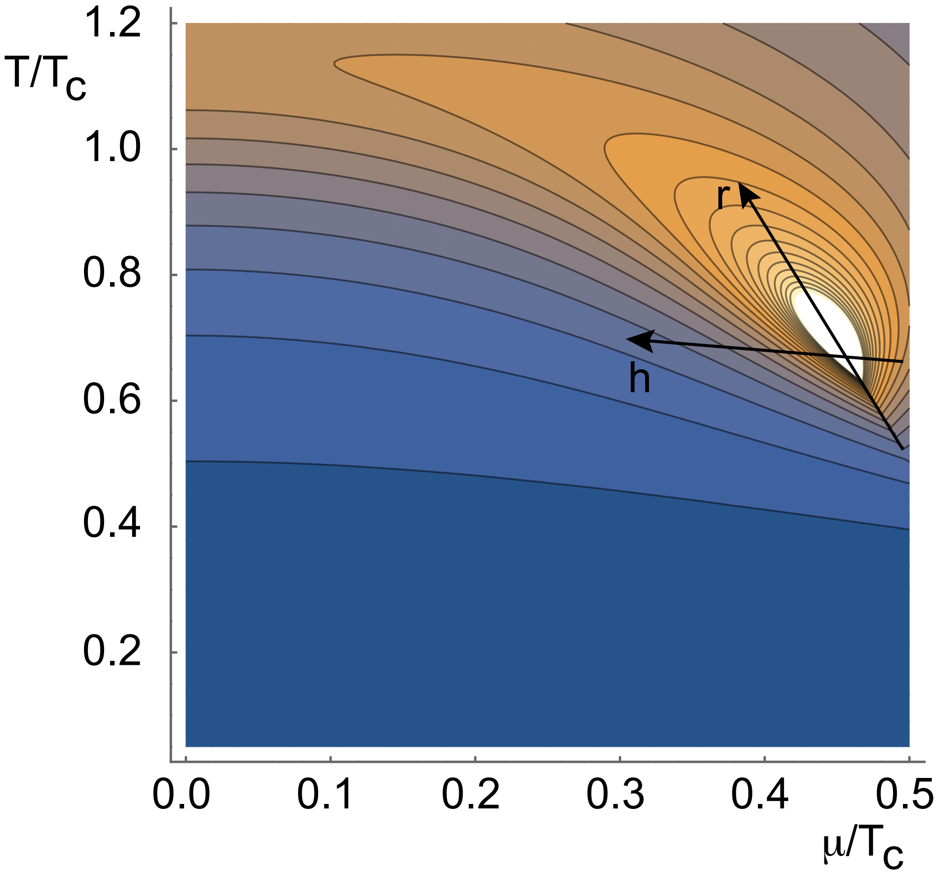
$$\frac{\zeta(0)}{s} = \left(\frac{n}{s}\right)^2 (\gamma_\pm a_{\epsilon n})^2 (Tt_0) \frac{1}{4\pi^2} \left(\frac{4\pi}{s/\eta}\right) \left(\frac{\xi}{\xi_0}\right)^{z-\alpha/\nu}$$

$(n/s)^2 \ll 1$ related to orientation of Ising axes.

Amplitude ratio $(\gamma_-/\gamma_+)^2 \simeq 10.$

First order regime: $\zeta(0) \simeq 3 \cdot 10^{-4} (\xi/\xi_0)^3$

The role of the Ising Map



Phase diagram in random matrix model, tuned to reproduce T_χ/T_{pc}

$$\frac{\zeta}{s} = \sin^2(\alpha_1)\left(\frac{4\pi}{s/\eta}\right)\left(\frac{\xi}{\xi_0}\right)^3 \begin{cases} 3.4 \cdot 10^{-2} & r > 0 \\ 2.2 \cdot 10^{-1} & r < 0 \end{cases}$$

$$\sin^2(\alpha_1) \simeq 1/4$$

Summary

Non-critical tails: Hard to observe in both relativistic and non-relativistic fluids.

Viscosity and diffusion bounds potentially relevant to nearly perfect fluids.

Critical bulk viscosity in QCD: Suppressed for "standard" orientation of Ising axes, large amplitude ratio.

Numerical Simulation: Stochastic Diffusion

Stochastic diffusion equation

$$\partial_t n_B(x,t) = \Gamma \nabla^2 \left(\frac{\delta \mathcal{F}}{\delta n_B} \right) + \nabla \cdot J(x,t)$$

$$\vec{J}(x,t) = \sqrt{2T\Gamma}\,\vec{\zeta}(x,t) \qquad \langle \zeta_i(x,t)\zeta_j(x',t') \rangle = \delta(x-x')\delta(t-t')\delta_{ij}$$

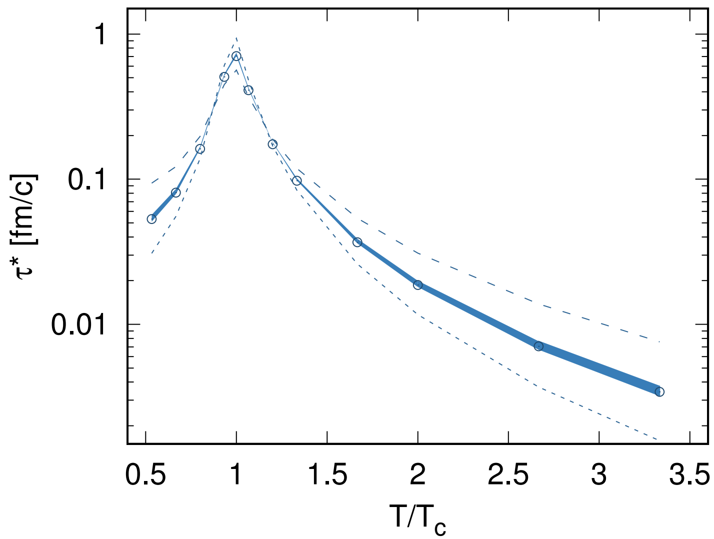
Free energy functional

$$\mathcal{F}[n_B] = T \int d^3x \left(\frac{m^2}{2n_c^2} (\Delta n_B)^2 + \frac{K}{2n_c^2} (\nabla n_B)^2 \right.$$

$$\left. + \frac{\lambda_3}{3n_c^3} (\Delta n_B)^3 + \frac{\lambda_4}{4n_c^4} (\Delta n_B)^4 + \frac{\lambda_6}{6n_c^6} (\Delta n_B)^6 \right)$$

Scale $m^2 \sim \xi^{-2}$, $\lambda_3 \sim \xi^{-3/2}$ etc., parameterize $\xi(t)$ with $t = \frac{T-T_c}{T_c}$.

Numerical results (diffusion in expanding critical fluid)



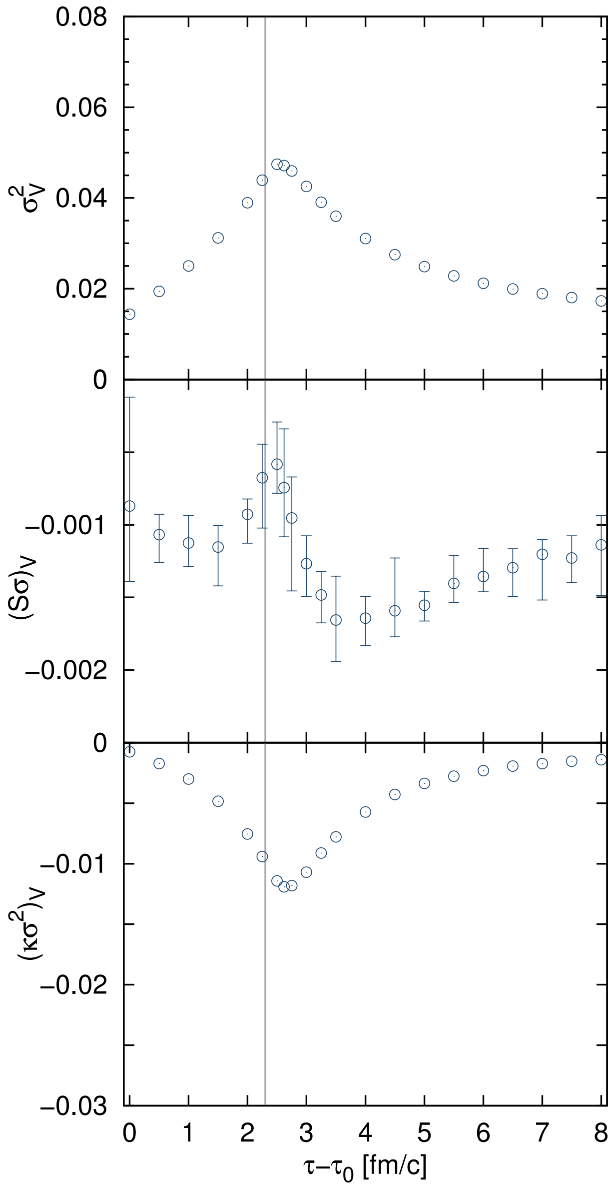
Dynamical scaling: Consider correlation function

$$C_2(t) = \langle \Delta n_B(k,0) \Delta n_B(-k,t) \rangle \ \text{ for } \ k = k^* \sim \xi^{-1}$$

Determine decay rate $C_2(t) \sim \exp(-t/\tau^*)$.

Blue line: Expectation for $z = 4$.

M. Nahrgang et al. (2018)



M. Nahrgang et al. (2018)

Numerical results
(diffusion in expanding critical fluid)

Variance

Skewness

Kurtosis

Analytic study: Hydro tails in Bjorken geometry

Consider linearized stochastic dynamics about some fluid backround (Bj)

Determine eigenmodes: two sound $\phi_\pm$, three diffusive modes ϕ_d, ϕ_{T_i}.

Noise average: Deterministic equ for 2-point fct $C_{ab} = \langle \phi_a(\tau, x)\phi_b(\tau, x')\rangle$.

$$\partial_0 C + [\mathcal{A}, C] + \{\mathcal{D}, C\} = \mathcal{P}C + C\mathcal{P}^\dagger + \mathcal{N}$$

evolution+reactive + diffusive = sources + noise-correlator

Mixed representation: $C_{ab}(\tau, \vec{k})$. Local quantities after momentum integration.

Contain divergences, can be renormalized by subtraction in homogeneous system.

Akamatsu et al. (2016)

Homogeneous System

Coupled equation for two-pont function of hydro modes

$$\partial_0 C_{\pm\pm} + \frac{4}{3}\gamma\, k^2\, C_{\pm\pm} = \mathcal{P}_{\pm\pm} C_{\pm\pm} + \mathcal{N}_{\pm\pm}$$

$$\partial_0 C_{T_l T_l} + 2\,\gamma\, k^2\, C_{T_l T_l} = \mathcal{P}_{T_l T_l} C_{T_l T_l} + \mathcal{N}_{T_l T_l}$$

$$\partial_0 C_{dd} + 2D\, k^2\, C_{dd} = \mathcal{N}_{dd}$$

$$\partial_0 C_{dT_l} + (\gamma + D)\, k^2\, C_{dT_l} = \mathcal{P}_{dT_l} C_{T_l T_l} + \mathcal{P}_{T_l d} C_{dd} + \mathcal{P}_{T_l T_l} C_{dT_l}$$

Off-diagonal couplings important for diffusive tails

$$G_R(\omega) = -i\omega[\sigma + \delta\sigma] - (1+i)\omega^{3/2}\,\frac{\chi T}{(D+\gamma)^{3/2}}$$

M. Martinez, T. S. (2018)

Expanding System

Coupled equations in Bj geometry

$$\partial_\tau C_{\pm\pm} + \frac{4}{3}\gamma\, k^2\, C_{\pm\pm} = -\frac{2 + c_s^2 + \cos^2\theta_K}{\tau} C_{\pm\pm} \mp \frac{\hat{K}\cdot E}{\bar{w}} \frac{1+c_s^2}{c_s^2} C_{\pm\pm} + \mathcal{N}_{\pm\pm}$$

$$\partial_\tau C_{T_1T_1} + 2\gamma\, k^2\, C_{T_1T_1} = -\frac{2}{\tau} C_{T_1T_1} + \mathcal{N}_{T_1T_1}$$

$$\partial_\tau C_{dd} + 2D\, k^2\, C_{dd} = -\frac{2}{\tau} C_{dd} + \mathcal{N}_{dd}$$

$$\partial_\tau C_{dT_1} + (\gamma + D)\, k^2\, C_{dT_1} = -\frac{2}{\tau} C_{T_1T_1} + \frac{1}{\bar{w}} \hat{e}_{T_1} \cdot E(C_{dd} - c_s C_{T_1T_1}$$

Asymptotic solution (as $\gamma k^2 \tau \gg 1$)

$$C_{dT_1}^{as} = \frac{\chi}{\bar{w}} \frac{1}{(D+\gamma)K^2\tau} \hat{e}_{T_1} \cdot E$$

renormalizes $\sigma \to \sigma + \delta\sigma$. $C - C^{as}$ generates hydro tails in expanding fluid.

Expanding System

Proper time evolution of longitudinal pressure

$$\langle \tau^2 T^{\eta\eta} \rangle = (e+P)\left\{ \frac{p}{e+P} - \frac{4}{3}\frac{\gamma_\eta}{\tau} + \frac{1.083}{s(4\pi\gamma_\eta)^{3/2}} + \frac{8(\lambda_1 - \eta\tau_\pi)}{9(e+P)\tau^2} + \ldots \right\}$$

fractional power corrections

$$\langle \tau^2 T^{\eta\eta} \rangle = \frac{e+P}{4}\left\{ 1 - 0.092\left(\frac{4.5}{\tau T}\right) + 0.034\left(\frac{4.5}{\tau T}\right)^{3/2} - 0.008\left(\frac{4.5}{\tau T}\right)^{2} + \ldots \right\}$$

Evolution of diffusive current response

$$\frac{\langle J^x \rangle}{\mathcal{E}^x} = \frac{T^3 (s/\eta)^2}{4\pi^2 w_0^2}\left\{ 1 - 0.14\left(\frac{4\pi}{s/\eta}\right)\left(\frac{4.5}{\tau T}\right)^{1/2} + \ldots \right\}$$

Fractional powers "in between" first and second order hydro