Instantons, the η' Mass, and the Large N_c Limit

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Outline

- Introduction: Why Instantons?

  $U(1)_A$ puzzle, topology, and instantons

- Instantons, the η' meson, and the Witten-Veneziano relation: large baryon density

  $\rho \ll \Lambda_{QCD}^{-1}$: semi-classical approximation under control

- Instantons and the large N_c limit

  smooth large N_c limit? Witten-Veneziano relation?

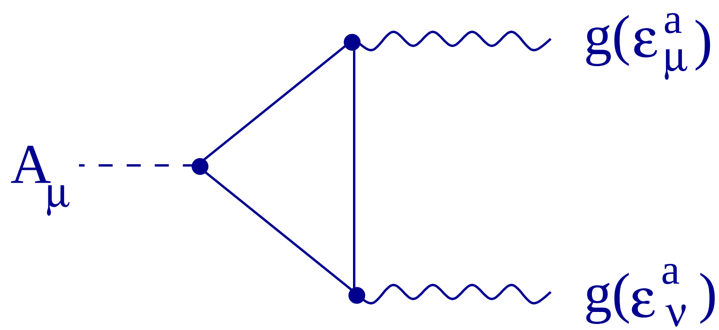
$U(1)_A$ Puzzle

- QCD has a $U(1)_A$ symmetry $\psi_L \to e^{i\phi}\psi_L, \;\; \psi_R \to e^{-i\phi}\psi_R$
  which is spontaneously broken $\langle \bar{\psi}_L \psi_R \rangle = \Sigma \neq 0$

- Goldstone Theorem: massless Goldstone boson $m_{\eta'} \to 0$ $(m_q \to 0)$

  But: η' is heavy, $m_{\eta'}^2 \gg m_q \Lambda_{QCD}$

- Resolution: axial anomaly



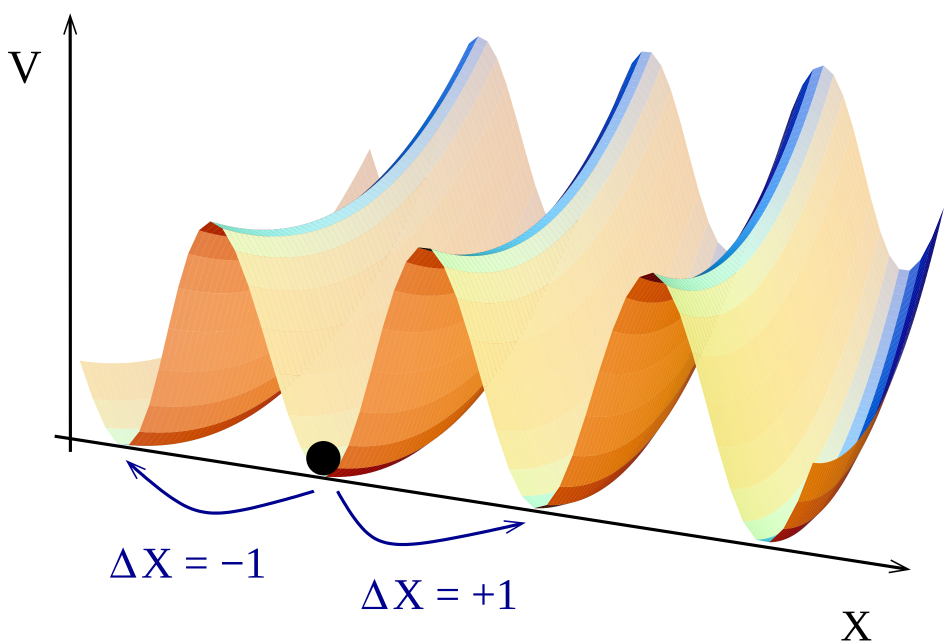
$$\partial_\mu A^\mu = \frac{N_f}{16\pi^2} G^a_{\mu\nu} \tilde{G}^a_{\mu\nu}$$

But: RHS is a total divergence $G\tilde{G} \sim \partial^\mu K_\mu$

$\Rightarrow$ Topology is important

Topology in QCD

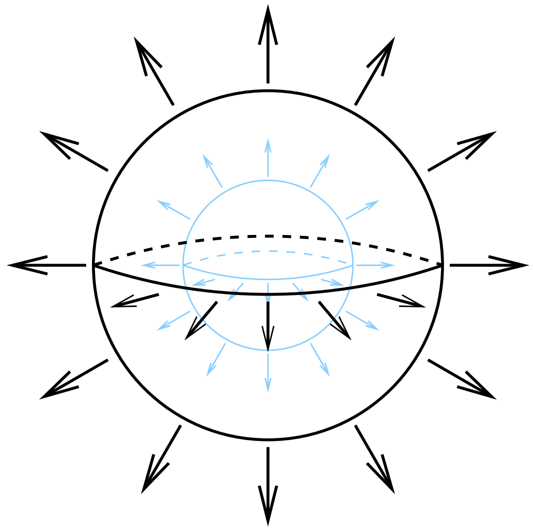
- classical potential is periodic in variable X



$$X = \int d^3x\, K_0(x,t)$$

$$\partial^\mu K_\mu = \frac{1}{32\pi^2} G^a_{\mu\nu} \tilde{G}^a_{\mu\nu}$$

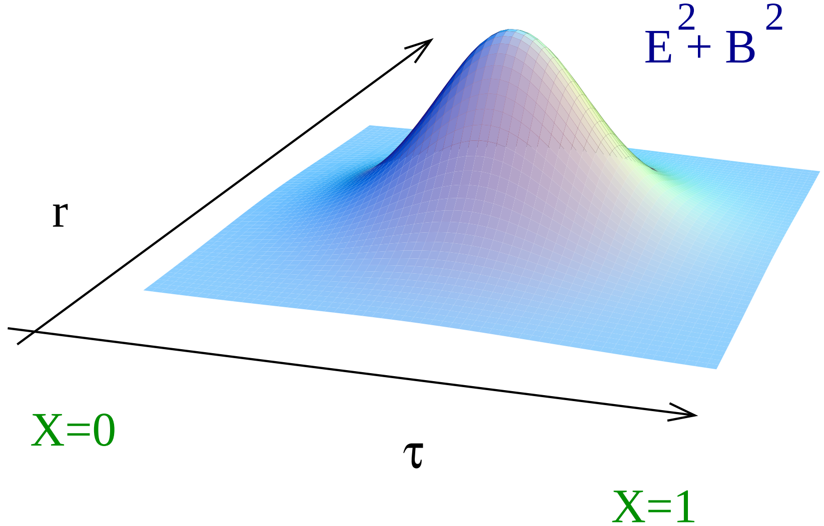
- classical minima correspond to pure gauge configurations



$$A_i(x) = iU^\dagger(x)\partial_i U(x)$$

$$E^2 = B^2 = 0$$

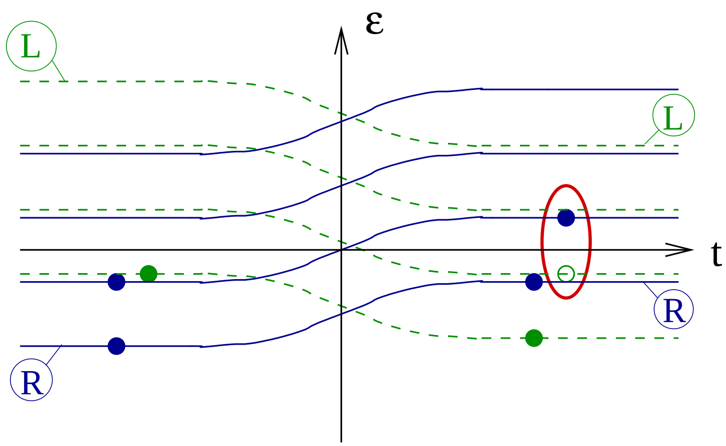
- classical tunneling paths: Instantons

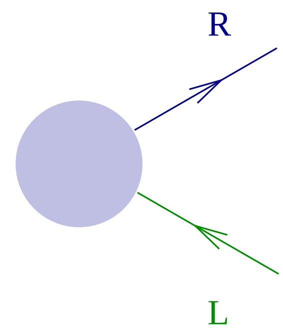


$$A^a_\mu(x) = 2\frac{\eta_{a\mu\nu}x_\nu}{x^2+\rho^2},$$

$$G^a_{\mu\nu}\tilde{G}^a_{\mu\nu} = \frac{192\rho^4}{(x^2+\rho^2)^4}.$$

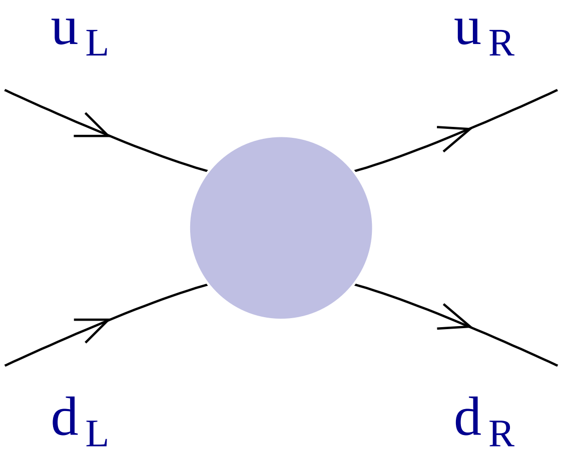
• Dirac spectrum in the field of an instanton





axial charge violation: $\Delta Q_A = 2$

• Instanton induced quark interaction $(N_f = 2)$



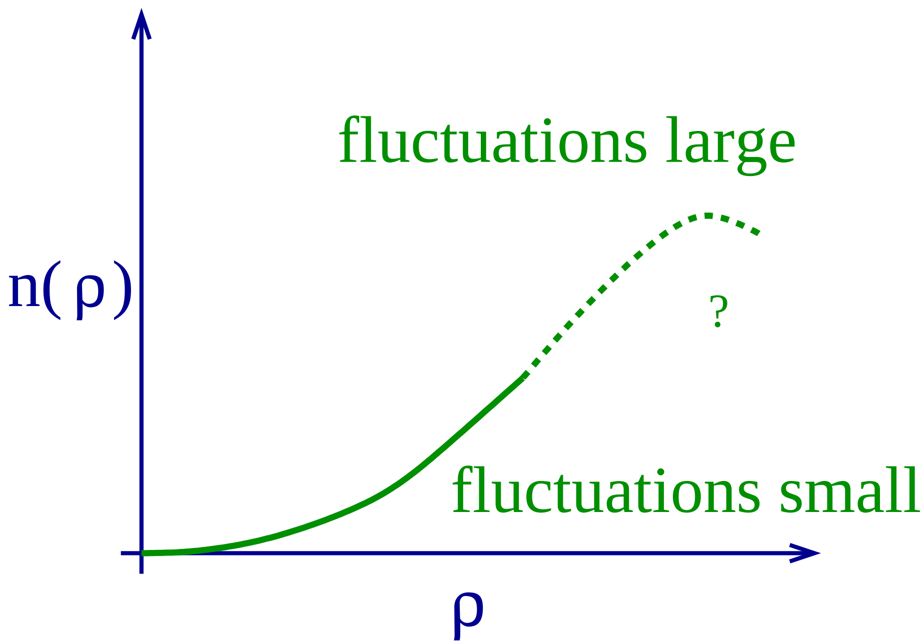
$$\mathcal{L} = G \det_f(\bar{\psi}_{L,f}\psi_{R,g})$$

$$G = \int d\rho\, n(\rho)$$

violates $U(1)_A$ but preserves $SU(2)_{L,R}$

. . . and contributes to the η' mass

- Tunneling rate (barrier penetration factor)



$$n(\rho) \sim \exp\left[-\frac{8\pi^2}{g^2(\rho)}\right] \sim \rho^{b-5}$$

QCD at large N_c

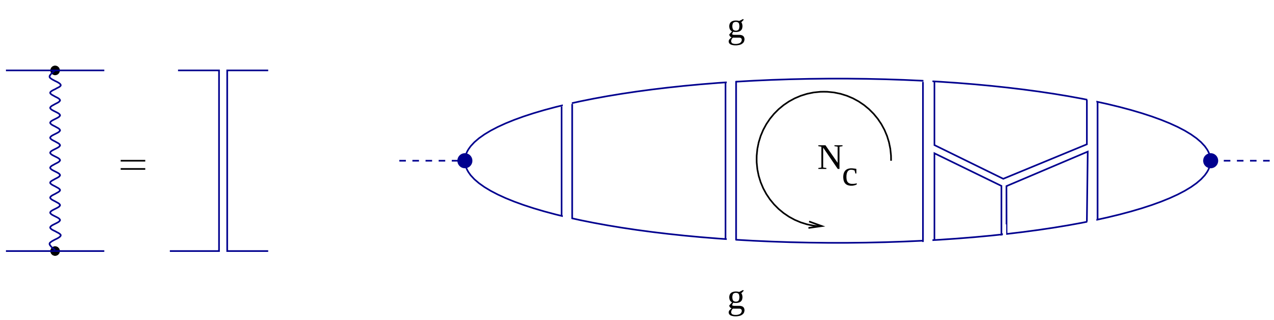
- QCD $(m = 0)$ is a parameter free theory. Very beautiful.

  But: No expansion parameter

- 't Hooft: Consider $N_c \to \infty$ and use $1/N_c$ as a small parameter

  $N_c \to \infty \quad \Rightarrow \quad$ classical master field

- keep Λ_{QCD} fixed $\Rightarrow g^2 N_c = const$



- Could the master field be a multi-instanton?

  Witten : No! $\qquad dn \sim \exp\left(-\frac{1}{g^2}\right) \sim \exp\left(-N_c\right)$

$U(1)_A$ anomaly at large N_c

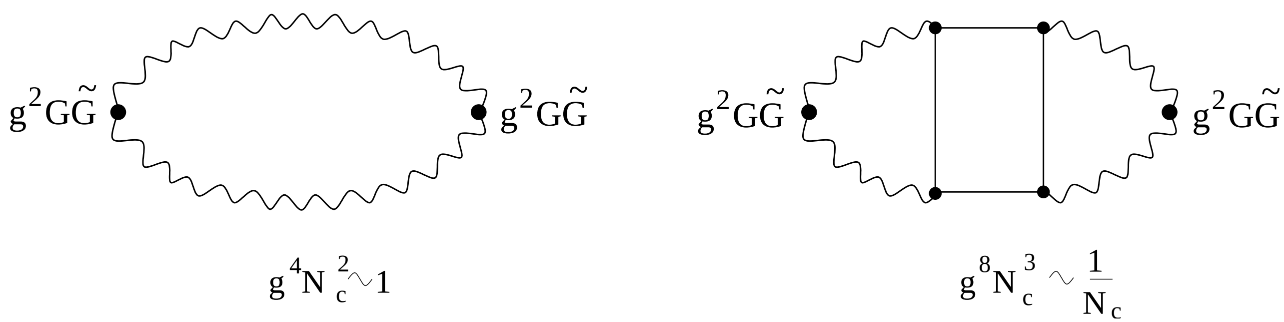
- consider θ term $\qquad \mathcal{L} = \frac{ig^2\theta}{32\pi^2} G^a_{\mu\nu}\tilde{G}^a_{\mu\nu}$

- no θ dependence in perturbation theory.

  Witten: *non-perturbative θ dependence* $\qquad \chi_{top} = \left.\frac{d^2E}{d\theta^2}\right|_{\theta=0} \sim O(1)$

- massless quarks: topological charge screening $\qquad \lim_{m\to 0}\chi_{top} = 0$

- How can that happen? Fermion loops are suppressed!



Witten: $\qquad f_\pi^2 m_{\eta'}^2 = 2N_f\,\chi_{top}(no\ quarks)$

η' has to become light $\qquad \Rightarrow \qquad m_{\eta'}^2 = O(1/N_c)$

Witten-Veneziano relation "works"

$$\chi_{top} \simeq (200\,\mathrm{MeV})^4 \quad (\textit{quenched lattice}) \;\Rightarrow m_{\eta'} \simeq 900\,\mathrm{MeV}$$

Where does $\chi_{top} \neq 0$ come from?

Instantons

- large N_c limit?
- Witten-Veneziano relation?
- ...

???

- topology?
- semi-classical limit?
- charge screening?

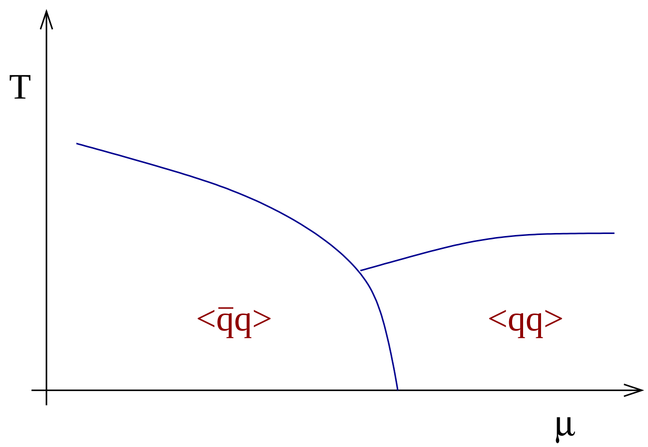
Digression:

Instantons and the mass of the η':

Large Baryon Density

QCD at large density ($N_f = 2$)

- schematic phase diagram

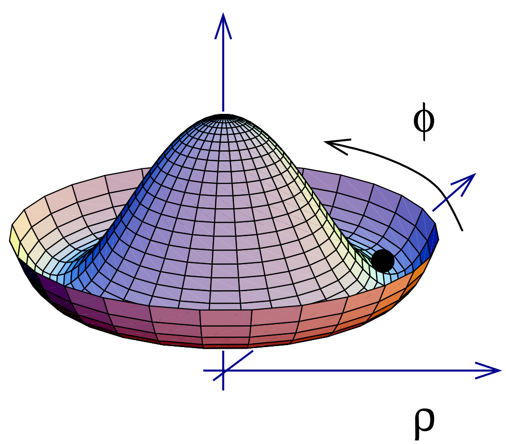


diquark condensate breaks

$U(1)_B$ and $U(1)_A$

$$\langle q_L q_L \rangle = \rho\, e^{i(\chi+\phi)/2}$$
$$\langle q_R q_R \rangle = \rho\, e^{i(\chi-\phi)/2}$$

- effective lagrangian for $U(1)_A$ Goldstone boson

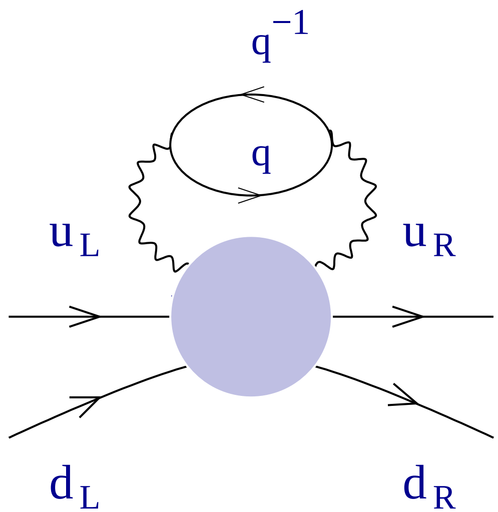


$$\mathcal{L} = \frac{f^2}{2}\left[(\partial_0\phi)^2 - v^2(\partial_i\phi)^2\right] - V(\phi+\theta) + \mathcal{L}(\rho,\chi)$$

$V(\phi+\theta)$ vanishes in perturbation theory

η' mass at large baryon density $(N_c = N_f = 2)$

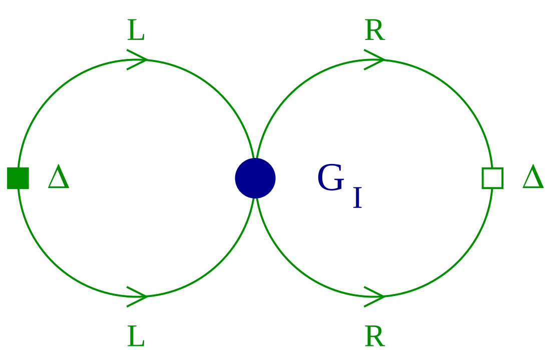
- instanton induced effective interaction for quarks with $p \sim p_F$



$$n(\rho,\mu) = n(\rho,0)\exp\left[-N_f\rho^2\mu^2\right]$$

$$\rho \sim \mu^{-1} \ll \Lambda_{QCD}^{-1}$$

- instanton contribution to vacuum energy

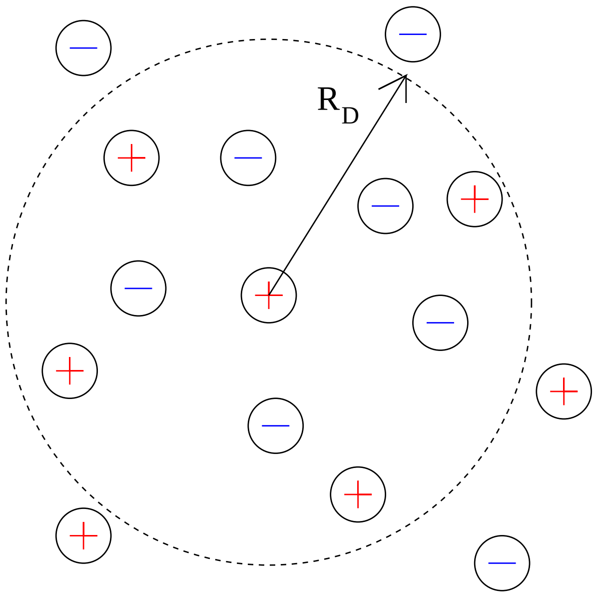


$$\langle \mathcal{L} \rangle = A\cos(\phi+\theta)$$

$$A = C_N \Phi^2 \left[\log\left(\frac{\mu}{\Lambda}\right)\right]^4 \left(\frac{\Lambda}{\mu}\right)^8 \Lambda^{-2}$$

- η' mass satisfies "Witten-Veneziano" relation $\boxed{f^2 m_\phi^2 = A}$

- very dilute instanton gas



$$\rho \ll r_{IA} \ll R_D$$

$$\rho \sim \mu^{-1}$$

$$r_{IA} = A^{1/4}$$

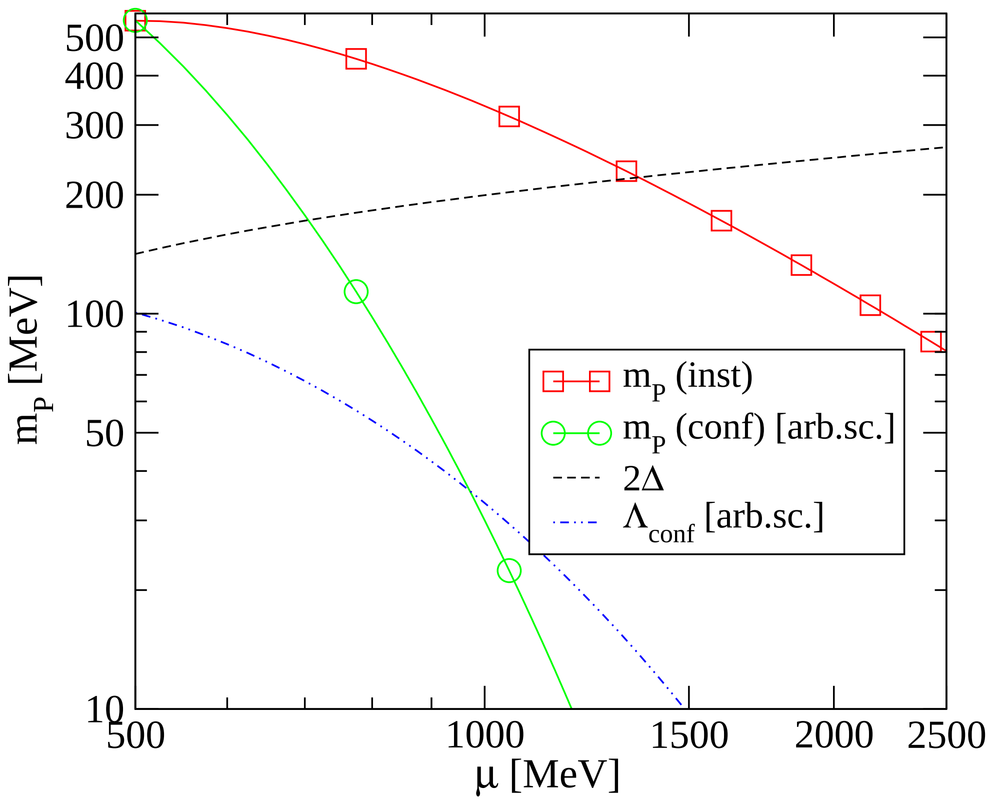
$$R_D = m_\phi^{-1}$$

- A is the local topological susceptibility

$$A = \chi_{top}(V) = \frac{\langle Q_{top}^2 \rangle_V}{V} \qquad r_{IA}^4 \ll V \ll R_D^4$$

- Global topological susceptibility vanishes

$$\chi_{top} = \lim_{V \to \infty} \frac{\langle Q_{top}^2 \rangle_V}{V} = 0 \qquad (m = 0)$$



- extrapolate to zero density

$$\left(\frac{N}{V}\right) \sim 1\,\mathrm{fm}^{-4} \qquad\qquad m'_\eta \sim 800\,\mathrm{MeV}$$

- Instantons predict density dependence of $m_{\eta'}$

can be checked on the lattice

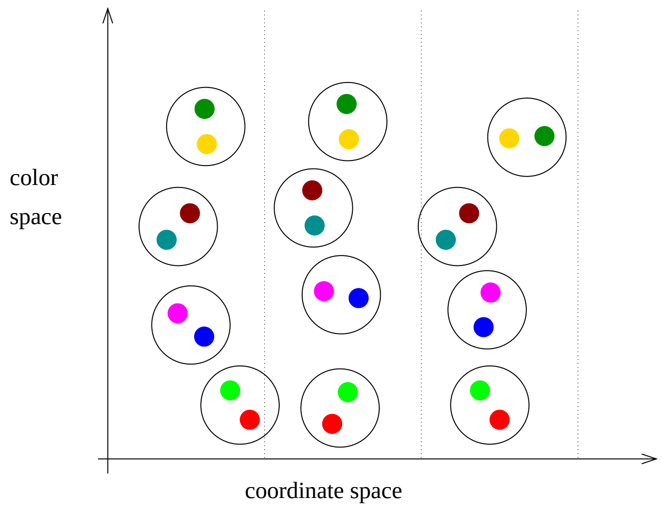
Main Course

Instantons and the mass of the η':

Large Number of Colors

Instantons at large N_c

- semi-classical ensemble of instantons at large N_c



instantons are $N_c = 2$ configurations

$$\left(\frac{N}{V}\right) = O(N_c)$$

$$\Rightarrow \epsilon_{vac} = O(bN_c) = O(N_c^2)$$

- instantons are semi-classical

$$\rho \simeq \rho^* = O(1) \qquad S_{inst} = O(N_c)$$

- density $dn \sim \exp(-S_{inst}) = O(\exp(-N_c))$?

$$NO! \qquad \textit{large entropy} \qquad dn \sim \exp(+N_c)$$

- topological susceptibility $\chi_{top} \simeq (N/V) = O(N_c)$?

$$NO! \qquad \textit{fluctuations suppressed} \qquad \chi_{top} = O(1)$$

Digression: $\mathcal{N}=4$ SUSY Yang Mills

- string/field theory duality (Maldacena)

$$\mathcal{N}=4 \text{ SUSY YM} \quad \Leftrightarrow \quad \text{IIB strings on } AdS_5\times S_5$$
$$\lambda = g^2 N \to \infty \quad \Leftrightarrow \quad (l_s/R)^4 \to 0$$
$$(g^2 \to 0) \qquad\qquad (g_s \to 0)$$

- string theory contains D-instantons characterized by location on $AdS_5\times S_5 \Leftrightarrow$ field theory instantons

$$\int d^4x \int \frac{d\rho}{\rho^5}\ \int d\Lambda^{ab}$$
$$AdS_5\times S_5$$

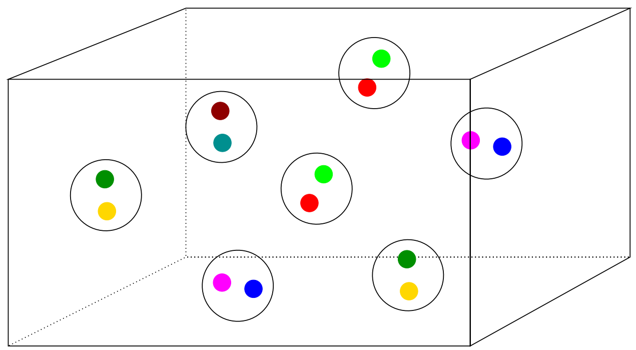
- k instanton amplitudes

$$(AdS_5\times S_5)^k \quad \to \quad (AdS_5\times S_5)$$
$$\text{k instantons in commuting } SU(2)'s$$
$$\text{(bound by fermions)}$$

Instanton ensemble

instanton ensemble

described by partition function



$$Z = \frac{1}{N_I! N_A!} \prod_I^{N_I+N_A} \int [d\Omega_I \, n(\rho_I)] \times \exp(-S_{int})$$

$$n(\rho) = C_{N_c} \left(\frac{8\pi^2}{g^2}\right)^{2N_c} \rho^{-5} \exp\left[-\frac{8\pi^2}{g(\rho)^2}\right]$$

$$C_{N_c} = \frac{0.47\exp(-1.68 N_c)}{(N_c-1)!(N_c-2)!} \qquad \frac{8\pi^2}{g^2(\rho)} = -b\log(\rho\Lambda), \qquad b = \frac{11}{3} N_c$$

$$S_{int} = -\frac{32\pi^2}{g^2}|u|^2 \left\{\frac{\rho_I^2\rho_A^2}{R_{IA}^4}\left(1 - 4\cos^2\theta\right) + S_{core}\right\}$$

- complicated ensemble, size distribution

$$n(\rho) \sim \begin{cases} \exp(-N_c) & \rho < \rho^* \\ const & \rho \sim \rho^* \end{cases} \qquad \rho^* \sim O(1)$$

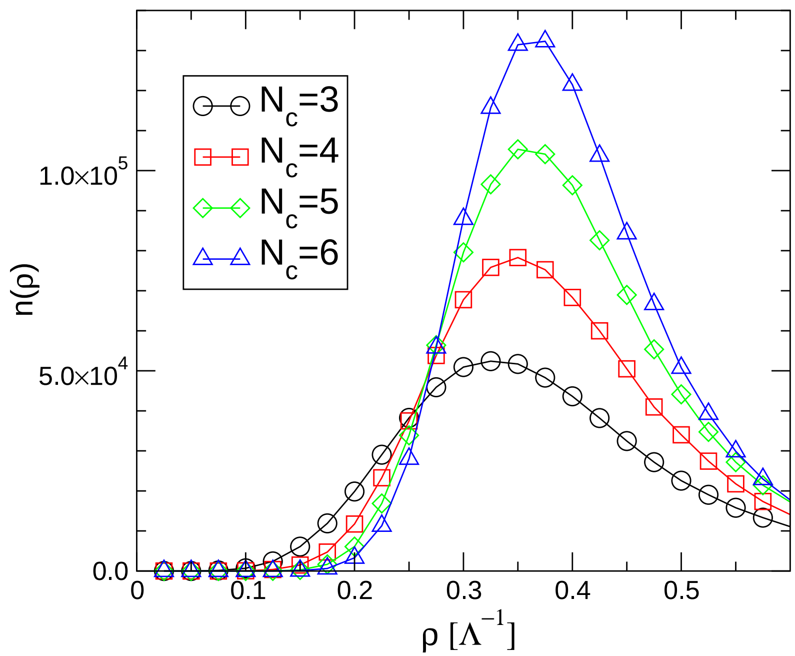
- total density determined by interactions

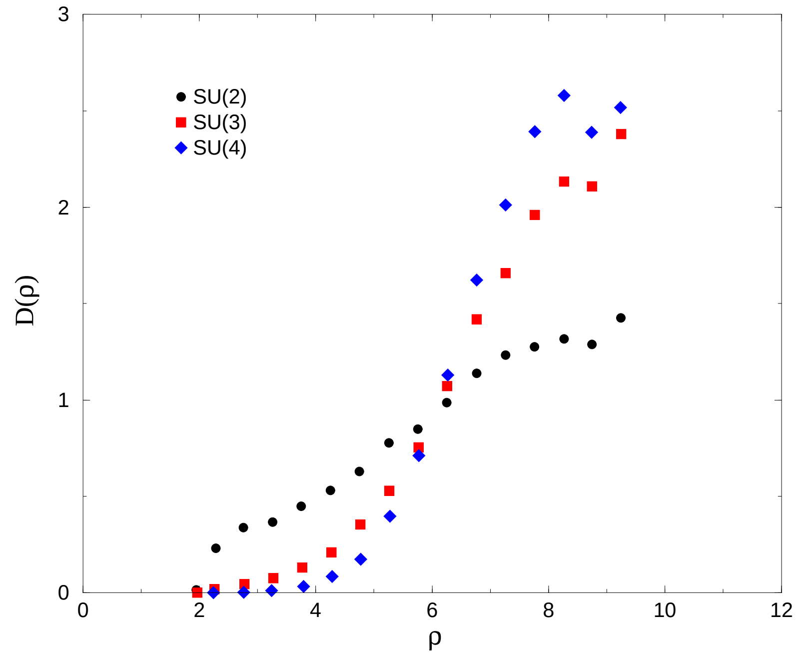
$$\begin{array}{ccccccc} S(1-body) & \sim & & & S(2-body) & & \\ N_c & \sim & N_c & \times & \frac{1}{N_c} & \times & \left(\frac{N}{V}\right) \\ classical & \sim & classical & \times & color\ overlap & \times & density \end{array}$$

- conclude

$$\boxed{\left(\frac{N}{V}\right) = O(N_c)}$$

instanton size distribution





B. Lucini, M. Teper

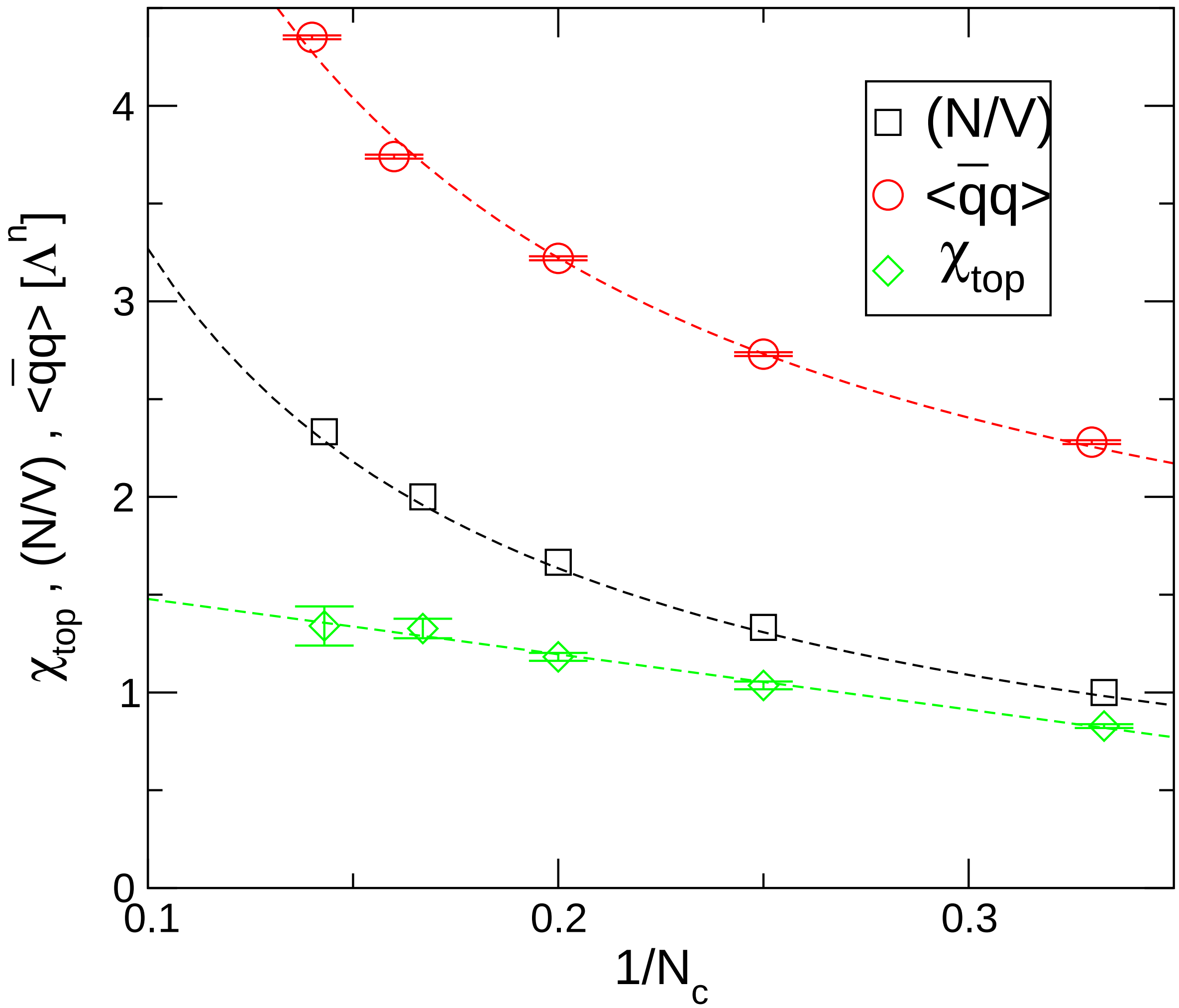
fluctuations in N are $1/N_c$ suppressed

$$\langle N^2\rangle - \langle N\rangle^2 = \frac{4}{b}\langle N\rangle \sim O(1) \qquad\qquad (\text{not } O(N_c)!)$$

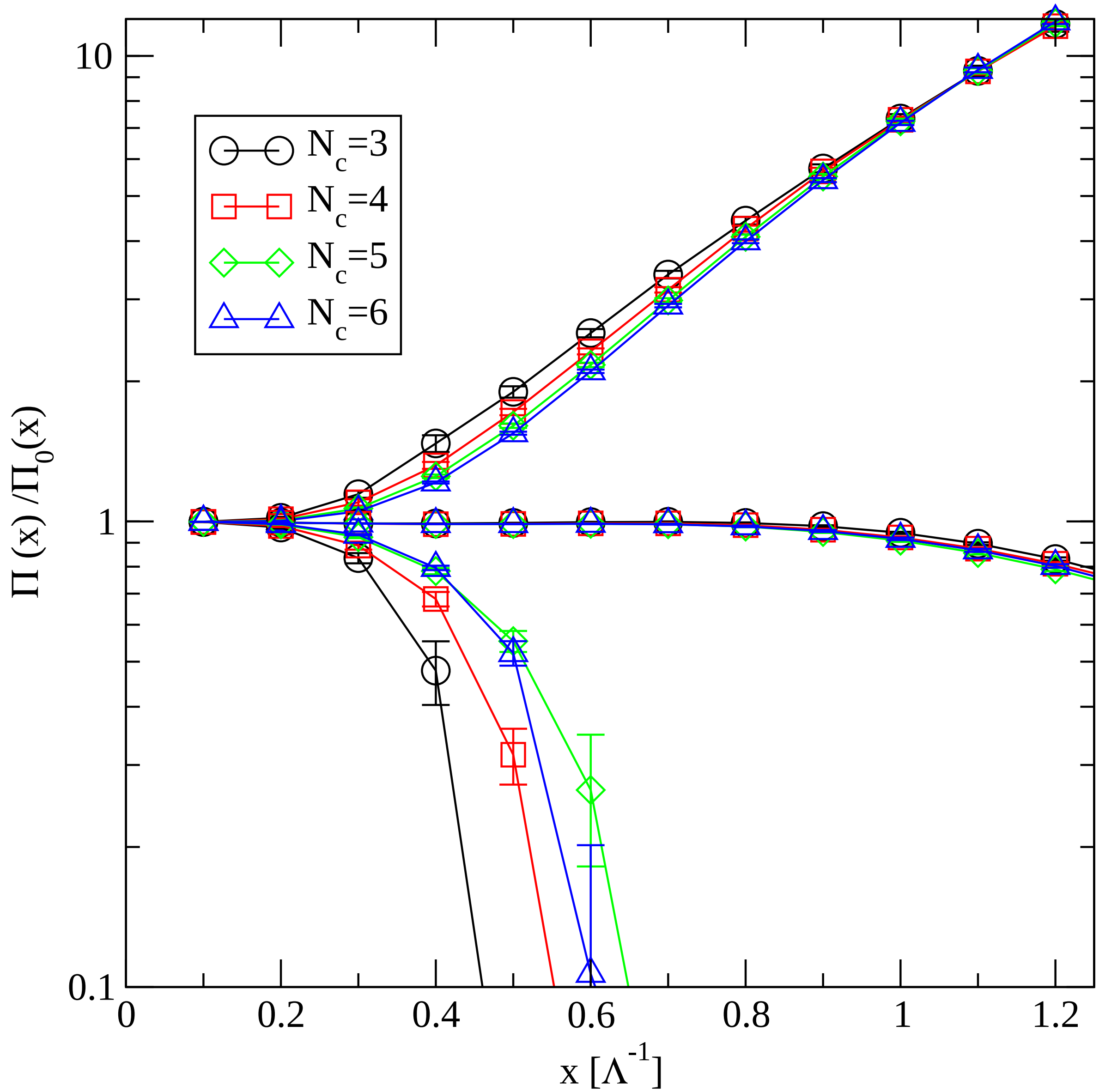
also true for topological charge

$$\langle Q^2\rangle = \frac{4}{b-r(b-4)}\langle N\rangle \sim O(1)$$

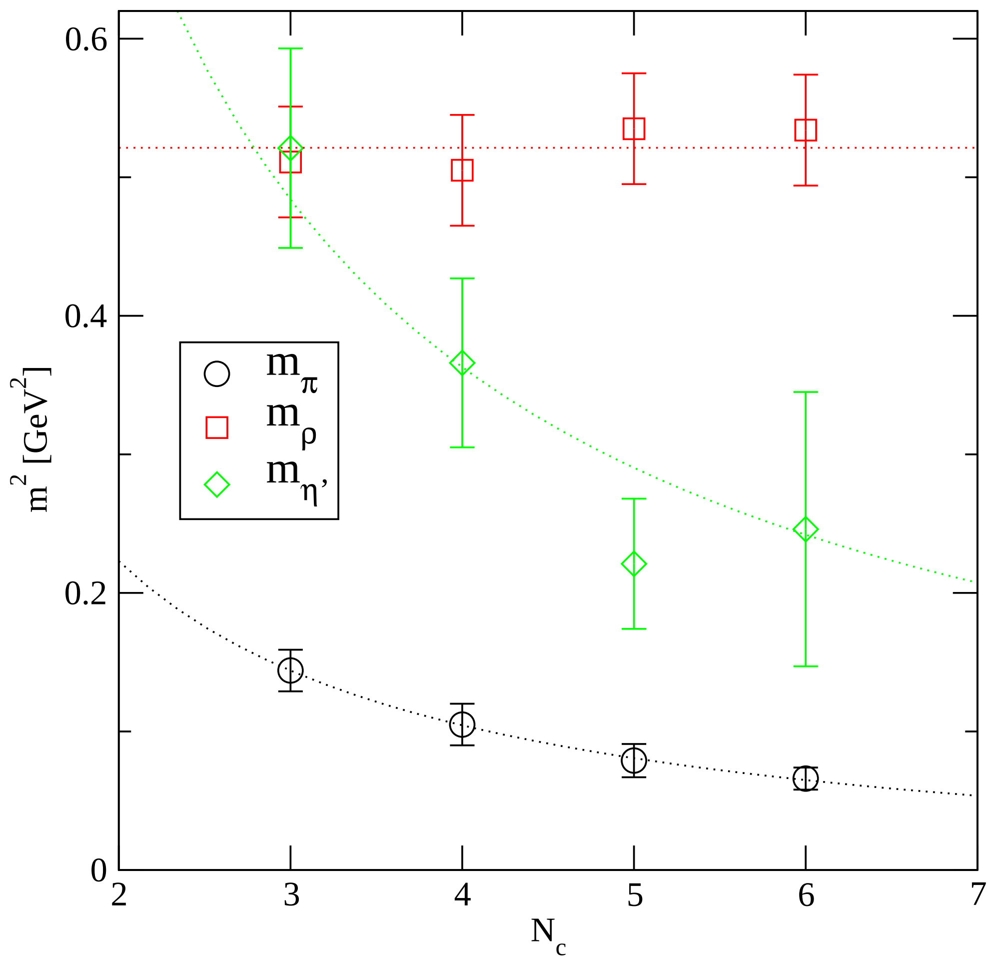
global observables: $(N/V), \langle \bar{q}q \rangle, \chi_{top}$



meson correlation functions (π, ρ, η')



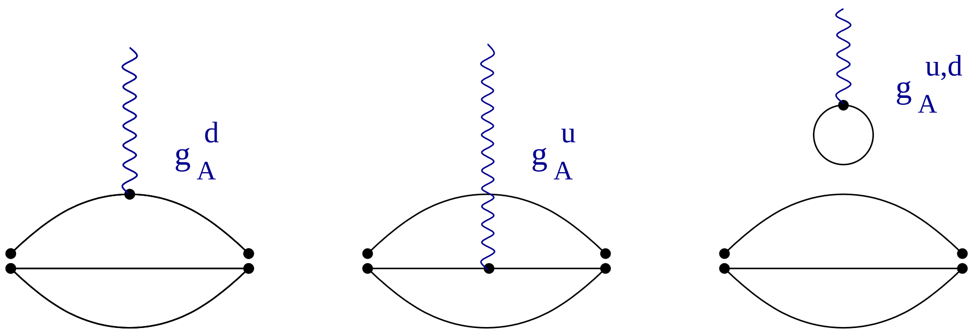
meson masses: $m_\pi^2, m_\rho^2 \sim 1,\ m_{\eta'}^2 \sim 1/N_c$



Note: $m_{\eta'}^2 \sim 1/N_c$ even though $(N/V) \sim N_c$

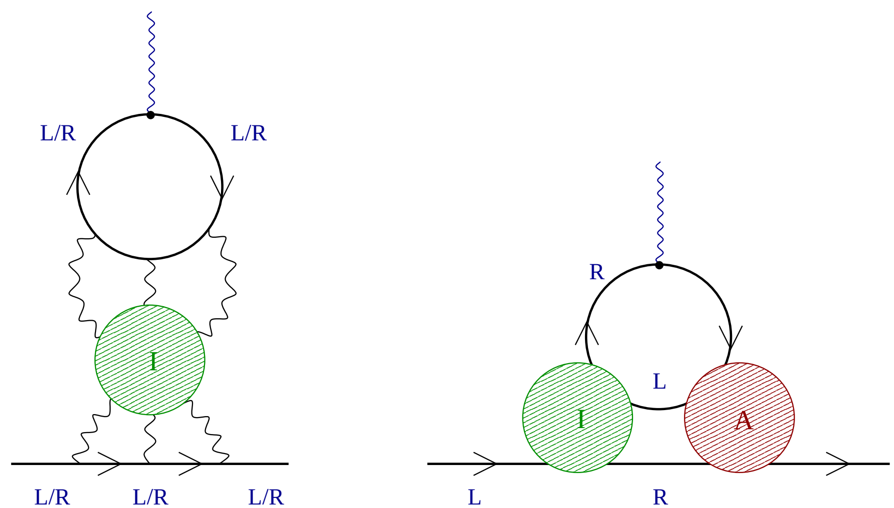
Nucleon Spin

- polarized DIS implies large OZI violation

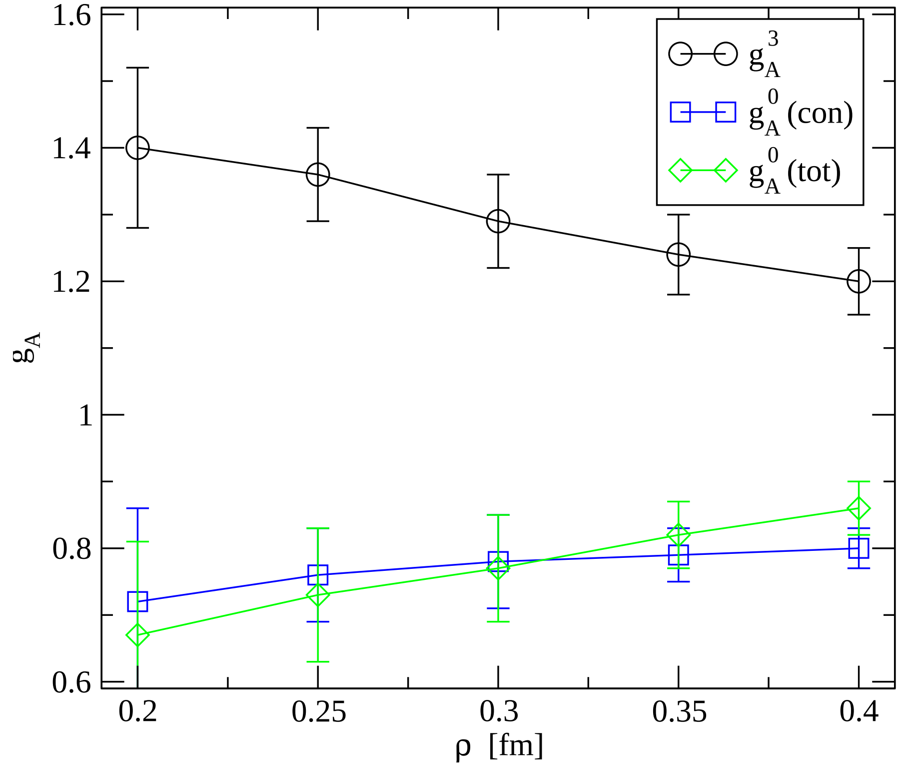


$$g_A^0 \simeq 0.25$$
$$g_A^8 \simeq 0.65$$

- instanton effects: anomaly, $u_L \to u_R d_R \bar{u}_L$ vertex



Numerical Study

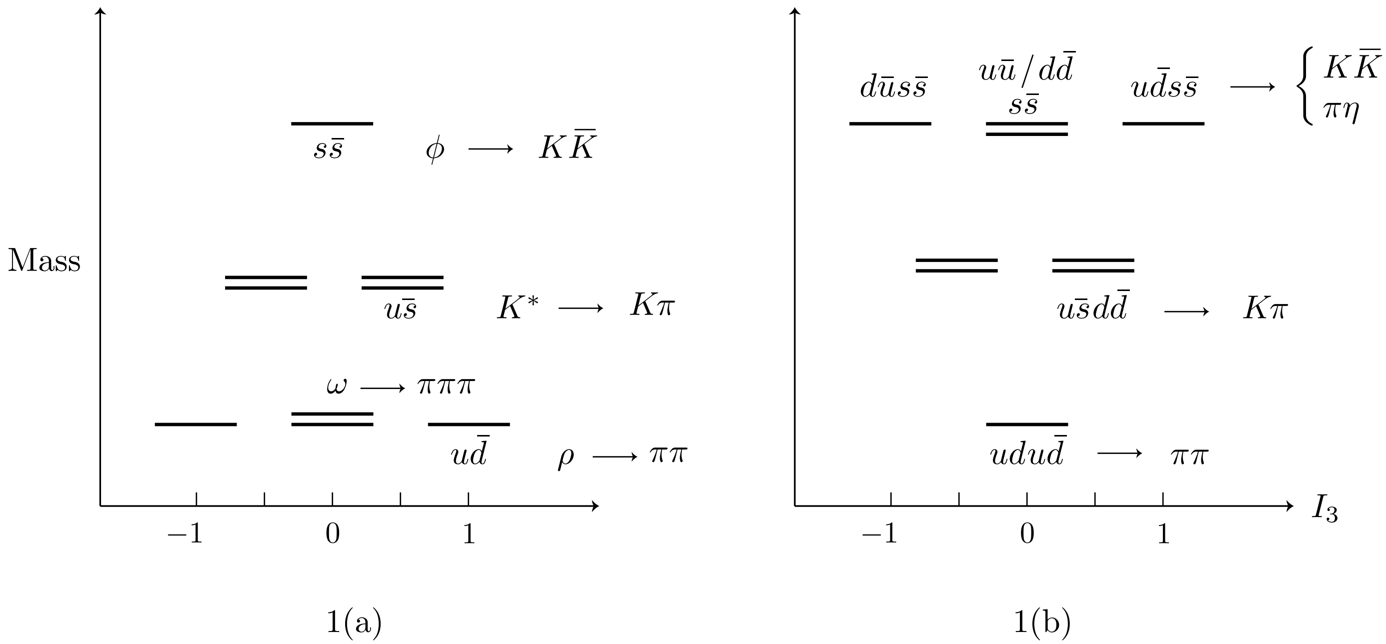


$g_A^3 \simeq 1.25$ agrees with experiment
$g_A^0 \simeq 0.75$ too large

Very little OZI violation

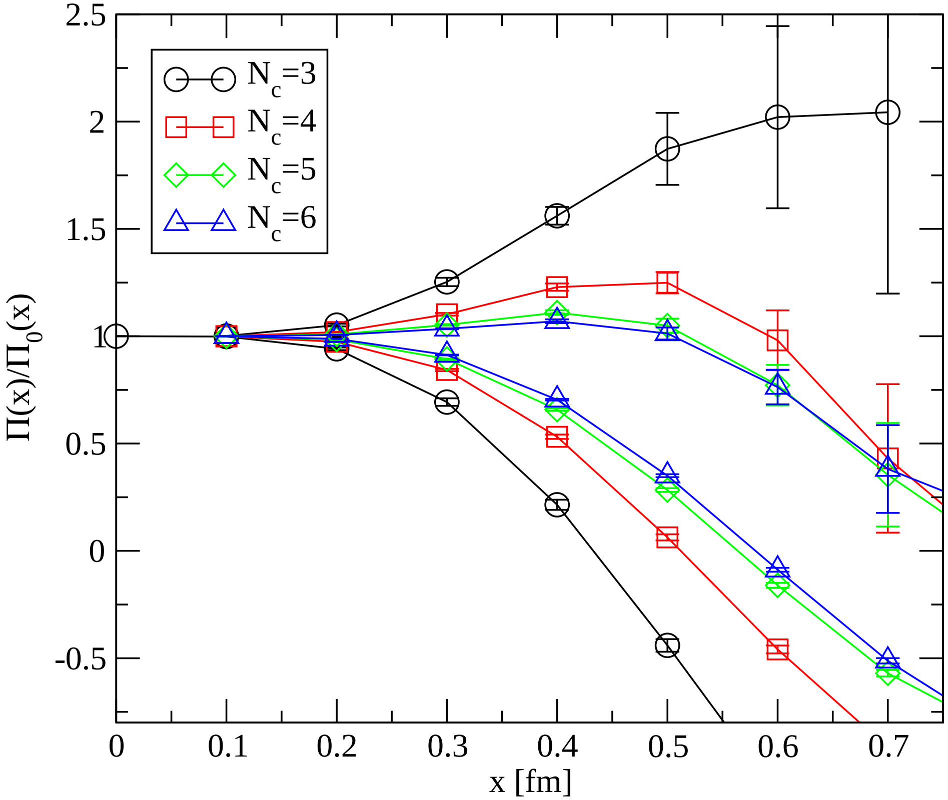
Light Scalars

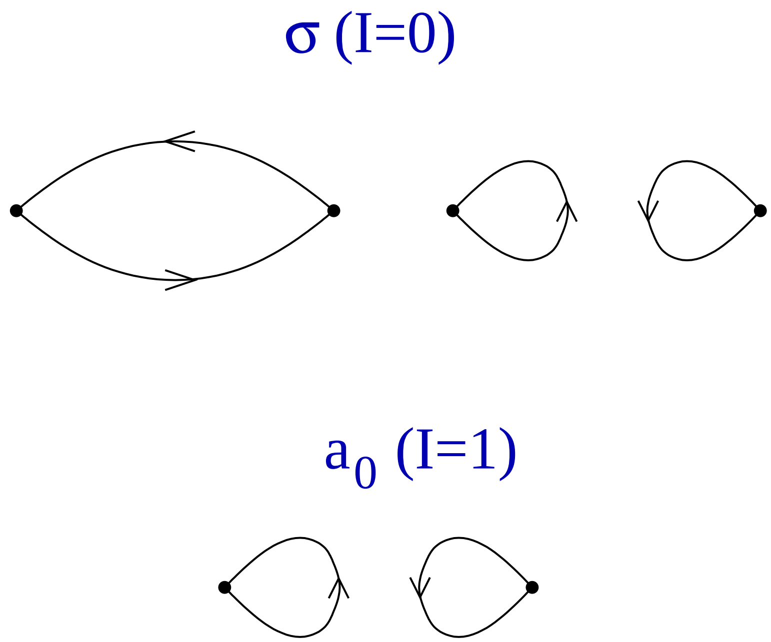
light scalars can be interpreted as $(qq)(\bar{q}\bar{q})$ states (Jaffe)



explains "inverted" mass spectrum and strong coupling to $(\pi\pi), (K\bar{K})$

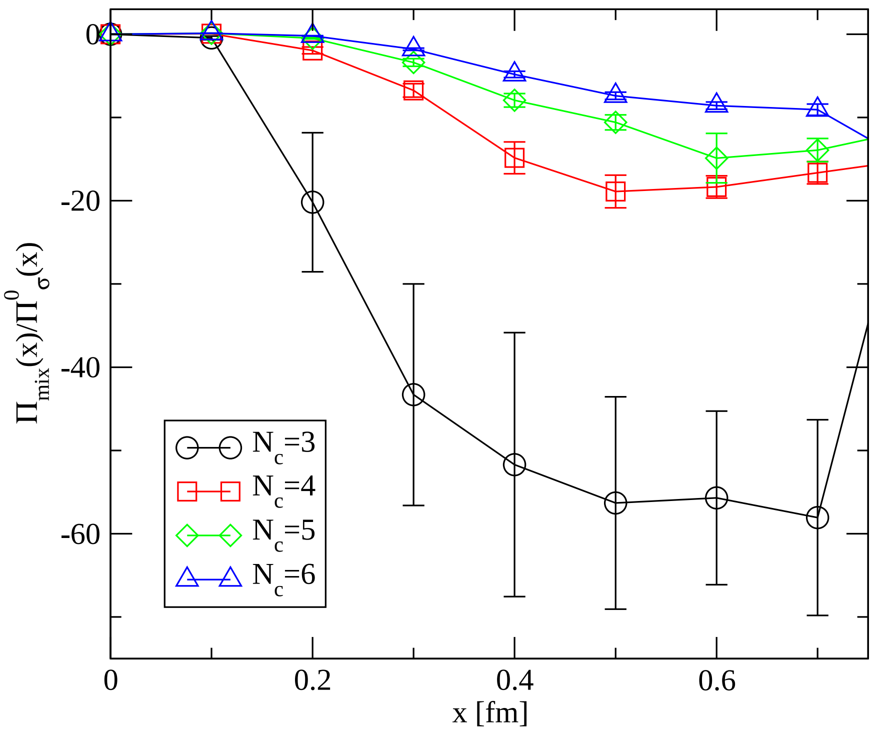
scalar correlation functions

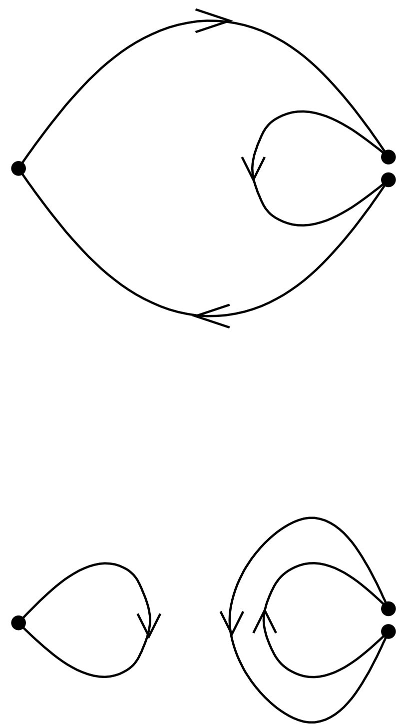




$m_\sigma \rightarrow 1$ GeV as $N_c \rightarrow \infty$

Tetraquark-meson $(qq)(\bar{q}\bar{q}) - (\bar{q}q)$ mixing





σ becomes "normal" $(\bar{q}q)$ scalar as $N_c \to \infty$

Summary

- nice example for semi-classical instanton liquid: QCD at large baryon density.

can be studied on the lattice

- instanton liquid can have a smooth large N_c limit

$$\left(\frac{N}{V}\right) = O(N_c), \qquad \chi_{top} = O(1), \qquad m^2_{\eta'} = O(1/N_c)$$

- smooth large N_c limit does not imply that $N_c = 3$ is similar to $N_c = \infty$ in all regards

η', scalar mesons, nucleon spin and strangeness