Exact saddle points in complexified path integrals

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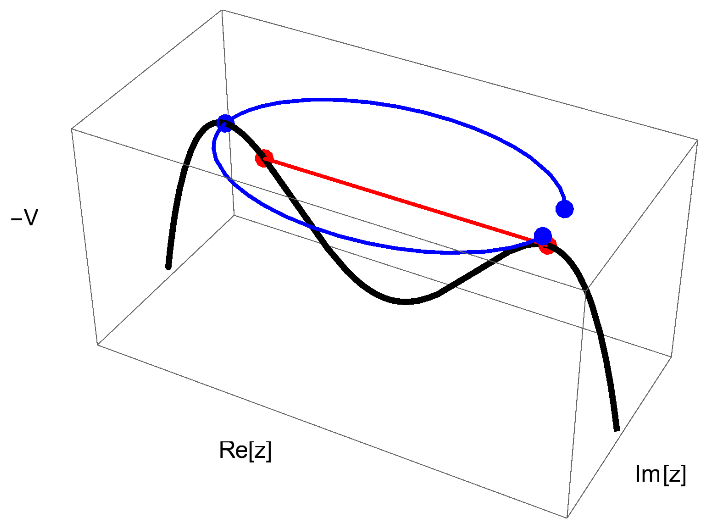


with Behtash, Dunne, Sulejmanpasic, and Ünsal

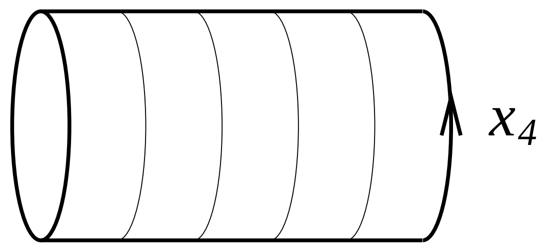
arXiv:1510.00978 & arxiv:1510.00978

Motivation: Exact results and non-trivial phases from approximate saddle points

Consider $SU(2)$ gauge theory on $R^3 \times S_1$

$$\mathcal{L} = -\frac{1}{4g^2} F^a_{\mu\nu} F^{a\,\mu\nu} - \frac{i}{g^2} \lambda^a \sigma \cdot D^{ab} \lambda^b + \frac{m}{g^2} \lambda^a \lambda^a$$

$$A^a_\mu(0) = A^a_\mu(L)$$

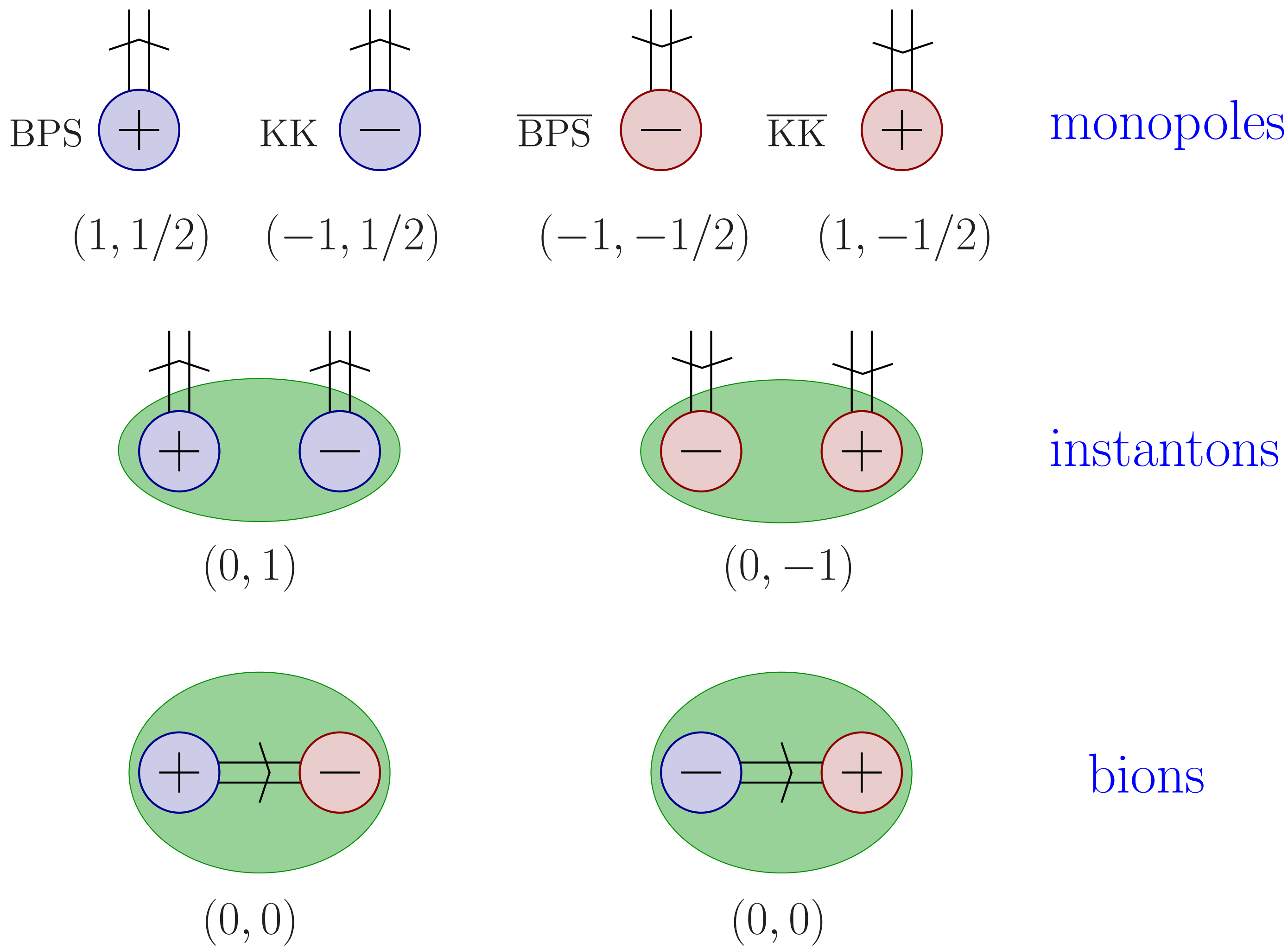


Small S_1: Confinement can be studied using semi-classical methods, based on monopole-instantons, instantons, and bions.

Low energy fields: Holonomy $b \sim \Delta\theta$ and dual photon σ

Topological objects

$$(Q_M, Q_{top}) = \left(\int_{S_2} B \cdot d\Sigma, \int_{R^3 \times S_1} F\tilde{F}\right)$$



Note: BPS/KK topological charges in Z_2 symmetric vacuum. Also have $(2, 0)$ (magnetic) bions.

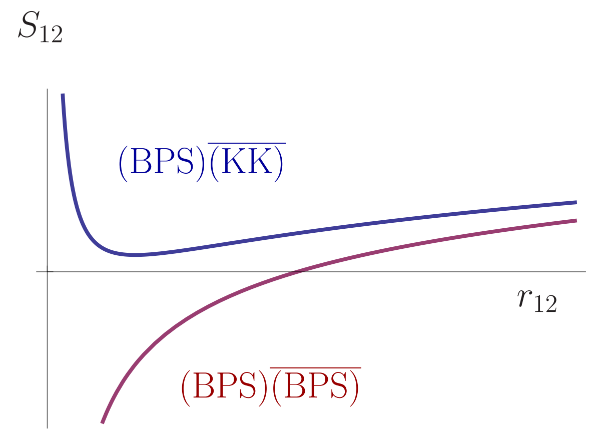
Effective potential

Instantons and monopoles: Exact solutions, but $V(b,\sigma) = 0$.

Bions: Approximate solutions

$$V_{BPS,\overline{BPS}} \sim e^{-2b} e^{-2S_0} \int d^3r\, e^{-S_{12}(r)}$$

$$S_{12}(r) = \frac{4\pi L}{g^2 r}(q_m^1 q_m^2 - q_b^1 q_b^2) + 4\log(r)$$



Saddle point integral after analytic continuation $g^2 \to -g^2$ (BZJ)

$$V(b,\sigma) \sim \frac{M_{PV}^6 L^3 e^{-2S_0}}{g^6}\left[\cosh\left(\frac{8\pi}{g^2}(\Delta\theta - \pi)\right) - \cos(2\sigma)\right]$$

Center symmetric vacuum $\mathrm{tr}(\Omega) = 0$ preferred

Mass gap for dual photon $m_\sigma^2 > 0$ ($\to$ confinement)

Questions

Vacuum energy at confining minimum $\Delta\theta = \pi$, $\sigma = 0$

$$\mathcal{E} \sim -\left[e^{-s_0} - e^{-S_0}\right] = 0$$

How can we get a cancellation? Does tunneling not always lower the ground state energy?

How can we get the exact low energy potential? The bion term does not have any real saddle points.

Complexified Path Integral

Consider complexified version of path integral in QM

$$Z = \int_\Gamma Dz \, e^{-\frac{1}{\hbar} S[z(t)]}, \quad S[z(t)] = \int dt \left(\frac{1}{2} \dot{z}^2 + V(z) \right),$$

Integration cycle Γ with $dim(\Gamma) = dim(R^n)$

Critical points: Holomorphic Newton equation

$$\frac{\delta S}{\delta z} = 0 \Rightarrow \frac{d^2 z}{dt^2} = +\frac{\partial V}{\partial z}.$$

Real and imaginary parts: $V(z) = V_{\rm r}(x,y) + iV_{\rm i}(x,y)$

$$\frac{d^2 x}{dt^2} = +\frac{\partial V_{\rm r}}{\partial x} \quad , \quad \frac{d^2 y}{dt^2} = -\frac{\partial V_{\rm r}}{\partial y},$$

The sign is crucial! This is not 2d classical mechanics.

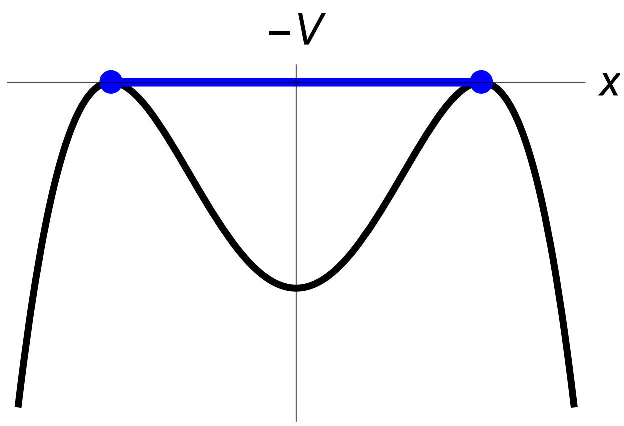
Supersymmetric quantum mechanics

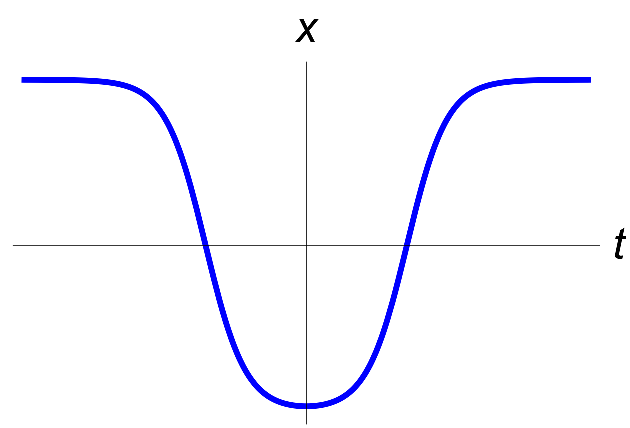
SUSY QM with superpotential $\mathcal{W}(x)$

$$S = \frac{1}{g}\int dt\left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}(\mathcal{W}')^2 + \left[\bar{\psi}\dot{\psi} + gp\mathcal{W}''\bar{\psi}\psi\right]\right),$$

p=1 is the supersymmetric theory.

Consider $W(x) = \frac{1}{3}x^3 - x$. Bosonic potential is the double-well:





Instantons have fermion zero modes. E_0 determined by IA-pairs.

Ground state energy can be computed using SUSY

$$E_0 = \langle 0|H|0\rangle \sim \langle 0|Q,Q|0\rangle \sim |\langle 0|Q|1\rangle|^2 \sim e^{-2S} > 0$$

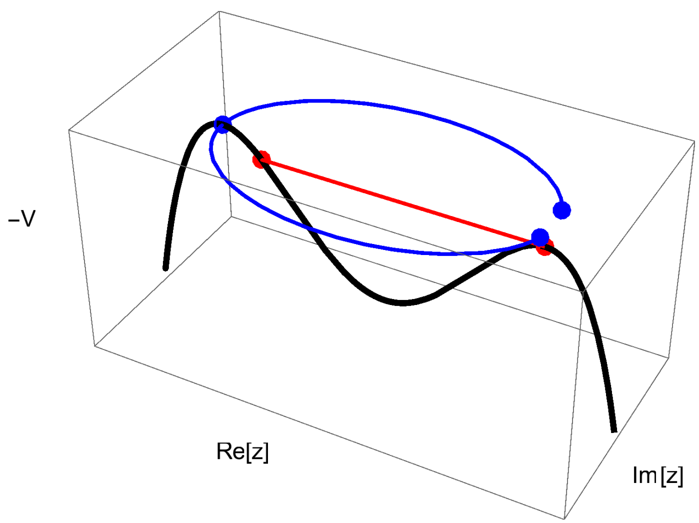
Graded formulation

Construct fermionic Hilbert space. Bosonic potentials $V_\pm$

$$V_\pm(x) = \frac{1}{2}(\mathcal{W}'(x))^2 \pm \frac{pg}{2}\mathcal{W}''(x)$$

Holomorphic Newton equation with quantum modified potential

$$\frac{d^2 z}{dt^2} = W'(z)W''(z) + \frac{pg}{2}W'''(z)\,.$$



tilted double well potential

bounce solution

complex bions

Complex bion solution

$$z_{\rm cb}(t) \;=\; z_1^{\rm cr} - \frac{z_1^{\rm cr} - z_T}{2}\coth\left(\frac{\omega_{\rm cb} t_0}{2}\right)\left[\tanh\left(\frac{\omega_{\rm cb}(t+t_0)}{2}\right) - \tanh\left(\frac{\omega_{\rm cb}(t-t_0)}{2}\right)\right]$$

$z_1^{\rm cr} = z_{\rm cb}(\pm\infty)$ global maximum of $-V$

$z_T = -z_1^{\rm cr} \pm i\sqrt{pg/(-z_1^{\rm cr})}$ complex turning points

$\omega_{\rm cb} = \sqrt{V''(z_1^{\rm cr})}$ natural frequency at $z_1^{\rm cr}$

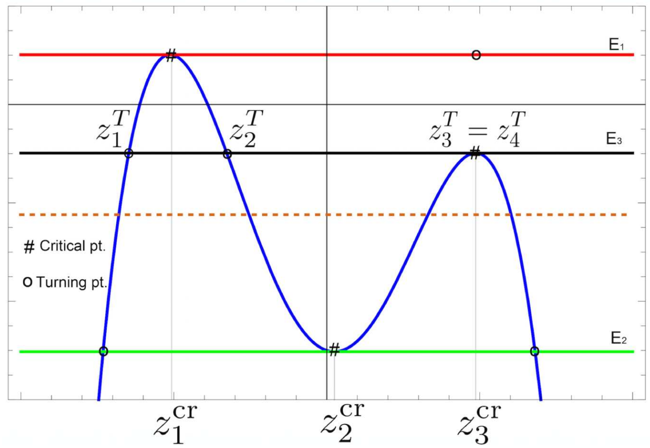
$t_0 = \dfrac{2}{\omega_{\rm cb}}\,\mathrm{arccosh}\left[\dfrac{3\omega_{\rm cb}^2}{\omega_{\rm cb}^2 - V''(z_1^{\rm cr})}\right]^{1/2}$ complex time parameter

$\mathrm{Re}[2t_0] \sim \omega^{-1}\ln(1/pg)$ is the complex bion size

Finding the complex bion

Use energy conservation

$$\int dt = \int \frac{dz}{\sqrt{Q^2}} \qquad Q^2 = 2[V(z) + E]$$



This is a quartic polynomial

$$Q^2 = \prod_k (z - z_k)$$

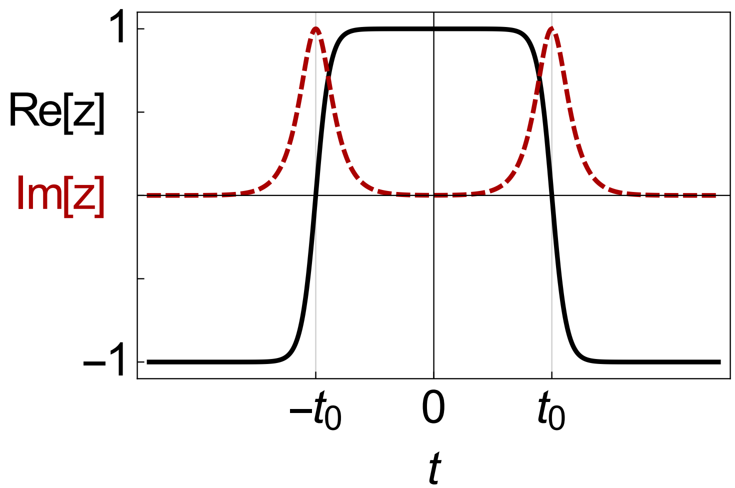
Solution given in terms of elliptic integrals

$$z(t) = z_T + \frac{\frac{1}{2}V'(z_T)}{\mathcal{P}(t; g_2, g_3) - \frac{1}{12}V''(z_T)}$$

Can be extend to complex turning points.

Or: Use analytic continuation in pg.

Complex bion solution



Re[z]: IA pair, size $\ln \frac{16}{pg}$

Im[z]: Complex action

$$S_{\rm cb} \simeq \left(\frac{8}{3g} + p \ln \frac{16}{pg} \right) \pm i\, p\, \pi \, ,$$

Conjugate saddles do not lead to an ambiguity for $p = 1$.

However, $e^{i\pi}$ is a hidden topological angle (HTA) which is crucial for the ground state energy:

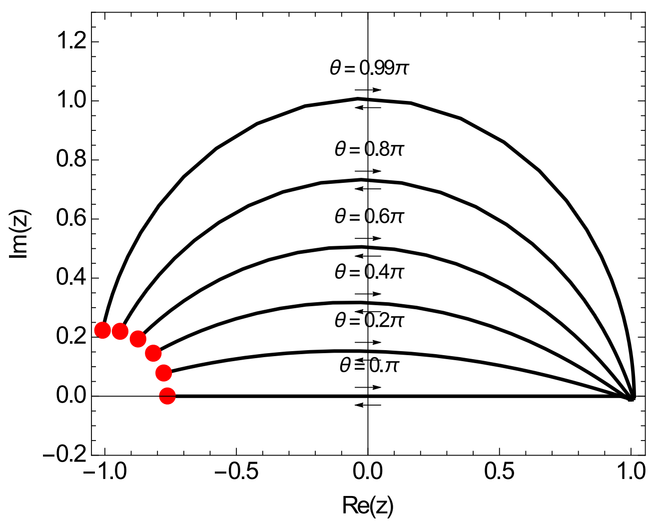
$$E_{gs} \sim -e^{\pm i\pi} e^{-2S_I} \sim +e^{-2S_I} > 0 \, ,$$

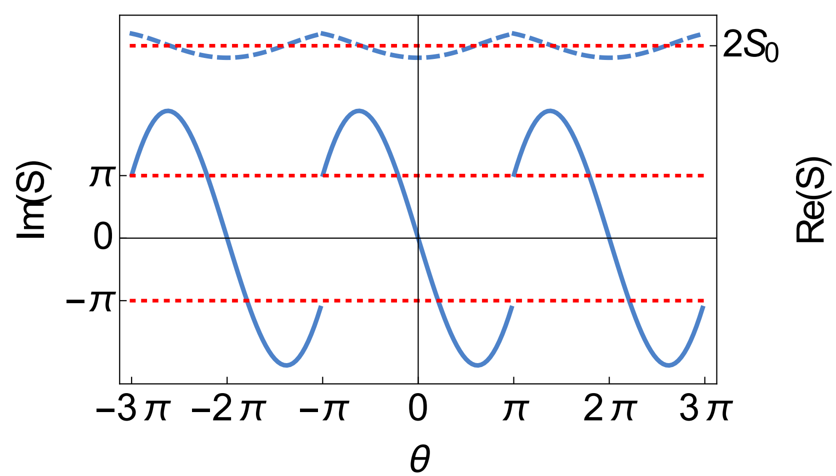
in agreement with the SUSY result.

Complex bion by analytic continuation

The complex bion solution can also be constructed by analytic continuation of the real bounce solution. Consider $p \to pe^{i\theta}$

$$V_\theta(x) = \frac{1}{2}(W'(x))^2 + \frac{pe^{i\theta}g}{2}W''(x)\,.$$





Relation to Lefshetz thimbles

Back to SUSY path integral: QZM integration over IA pairs with separation τ

$$I(N_f,g) = \int_{\Gamma_-^{\rm qzm}} d(m_b\tau) e^{-\left(-\frac{A}{g}e^{-m_b\tau}+N_f m_b\tau\right)}\,,$$

Critical point of the integration is located at a complex separation

$$\tau^* = \frac{1}{2}\left[\ln\left(\frac{16}{gN_f}\right) \pm i\pi\right]$$

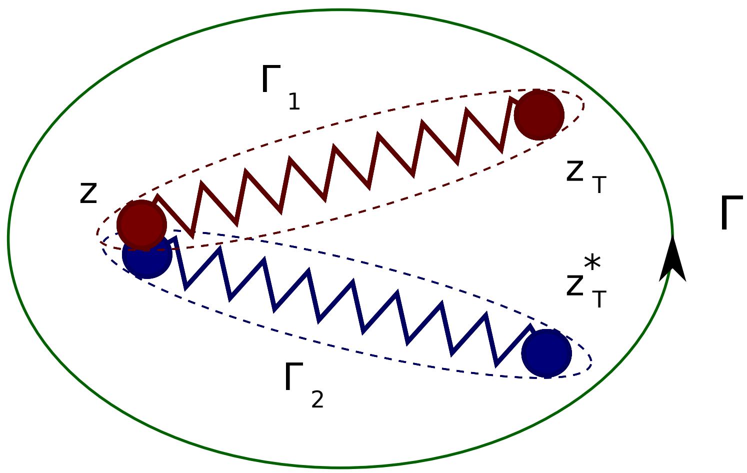
Descent manifold $\Gamma_-^{\rm qzm} = \mathbb{R} \pm i\pi$

Integral related to Gamma-function:

$$I_-(N_f,g) = e^{\pm i\pi N_f}\left(\frac{g}{16}\right)^{N_f}\Gamma(N_f)\,.$$

Quantization of hidden topological angle

Action determined by $S = \frac{1}{g}\int_\Gamma dz\,\sqrt{2(E+V)}$



$$\mathrm{Im}\,S_{\rm cb} = \frac{1}{2i}(S^1_{\rm cb} - S^2_{\rm cb})$$
$$= \frac{1}{2gi}\oint dz\,\sqrt{2(E+V)}$$

Use $2V = (W')^2 + pgW''$ and expand

$$\mathrm{Im}\,S_{\rm cb} = \frac{p}{4}\oint dz\,\frac{W''}{W'} = \frac{p}{2}\oint \frac{dW'}{W'} = ip\pi\,,$$

SUSY theory (p=1): No ambiguity, hidden topological angle

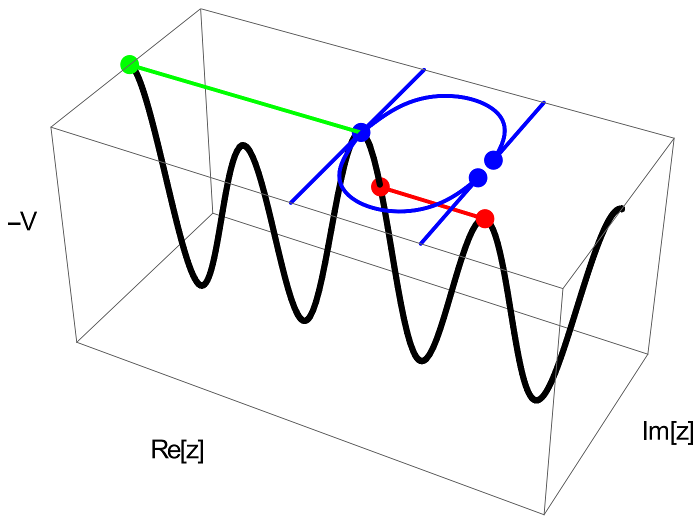
Non-SUSY theory: Ambiguity, related to resurgence.

Periodic potential

Consider the superpotential $W(x) = 4\cos(x/2)$. Then

$$V_{\pm}(x) = 2\sin^2(x/2) \pm \frac{pg}{2}\cos(x/2)\,.$$

It is known that supersymmetry is unbroken, $E_0 = 0$.



tilted sine gordon potential

bounce solution

real bion

complex bions

Complex bion solution

$$z_{\rm cb}(t) = 2\pi \pm 4\left(\arctan e^{-\omega_{\rm cb}(t-t_0)} + \arctan e^{\omega_{\rm cb}(t+t_0)}\right)\,,$$

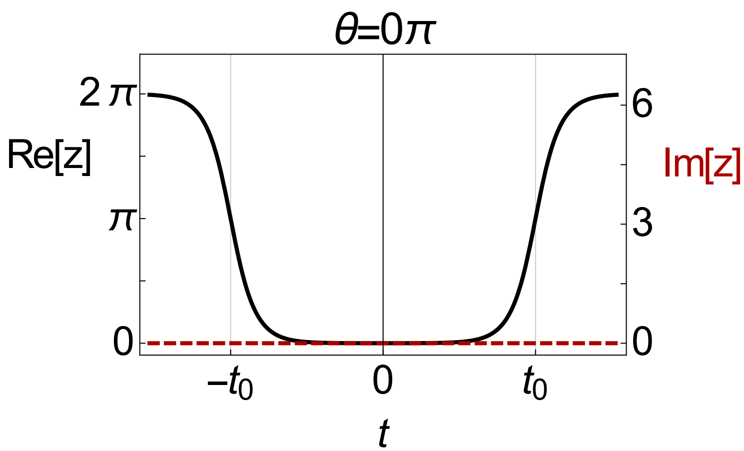
direct integration, or analytic continuation from real bounce

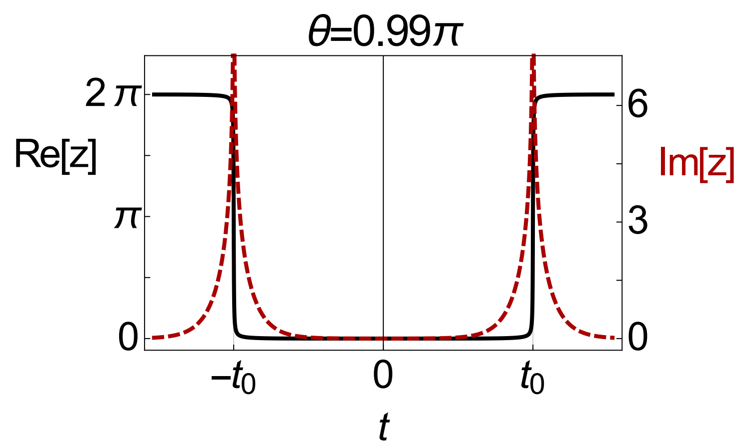
Complex bion, sine gordon potential

The action is finite

$$S_{\rm cb} \simeq \left(\frac{16}{g} + p \ln \frac{32}{pg} + \ldots \right) \pm i\, p\, \pi\,.$$

despite singular behavior at $t = \pm t_0$.





Singularity smoothed out by analytic continuation in θ.

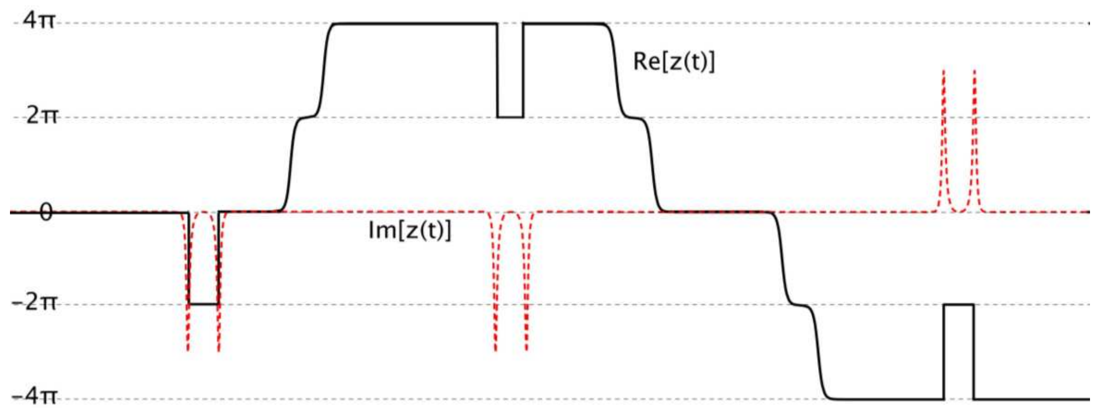
The solution is multi-valued as $\theta \to \pi \pm \epsilon$.

SUSY-QM vacuum: Dilute bion gas

Ground state: Dilute gas of complex and real bions

$$E_{gs} \sim -e^{-S_{\rm cb}} - e^{-S_{\rm rb}} = -e^{\pm i\pi} e^{-2S_{\rm rb}} - e^{-2S_{\rm rb}} = 0\,,$$

consistent with the requirement of supersymmetry.



Summary

Many field theories and quantum mechanical models require complex phases (hidden topological angles) from non-BPS saddle points.

These appear from solutions of complexified path integral. We found exact solutions of holomorphic Newton equations, corresponding to saddle points of complexified path integral. These solutions are potentially singular.

Imaginary part of S is either un-ambiguous, and related to HTA, or ambiguous and related to resurgence.