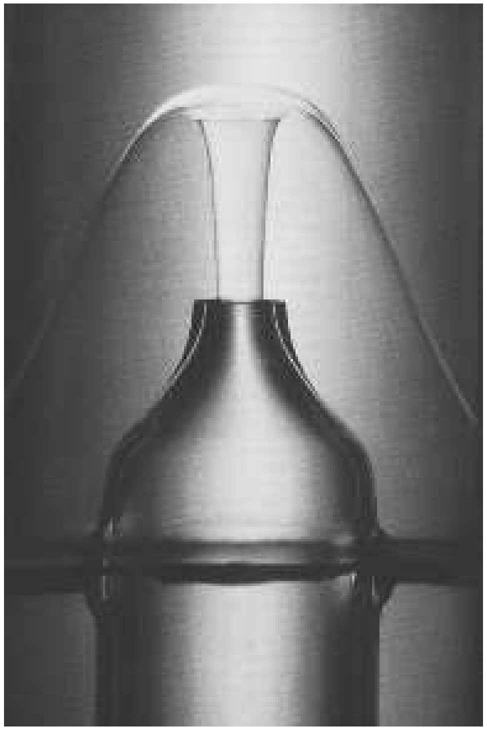
(Super) Fluid Dynamics

Thomas Schaefer, North Carolina State University

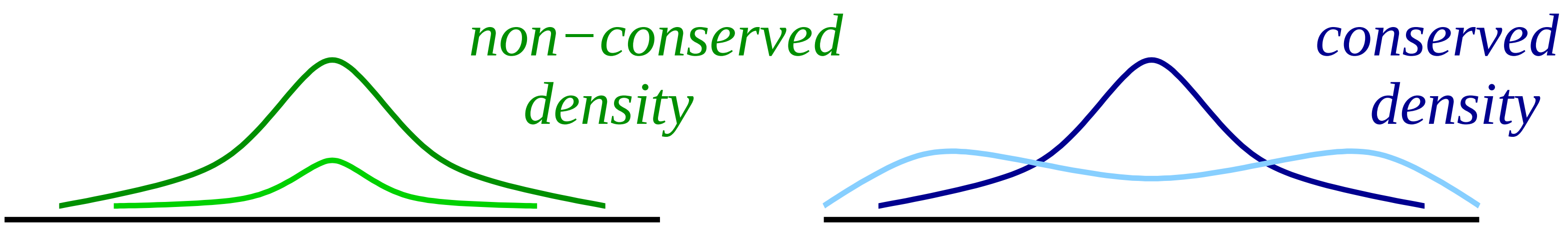


Hydrodynamics

Hydrodynamics (undergraduate version): Newton's law for continuous, deformable media.

Fluids: Gases, liquids, plasmas, . . .

Hydrodynamics (postmodern): Effective theory of non-equilibrium long-wavelength, low-frequency dynamics of any many-body system.



$\tau \sim \tau_{micro}$ $\qquad\qquad$ $\tau \sim \lambda$

$\tau \gg \tau_{micro}$: Dynamics of conserved charges.

Water: $(\rho, \epsilon, \vec{\pi})$

Effective theories for fluids (Unitary Fermi Gas, $T > T_F$)

$$\mathcal{L} = \psi^\dagger \Big(i\partial_0 + \frac{\nabla^2}{2M} \Big) \psi - \frac{C_0}{2} (\psi^\dagger \psi)^2$$

$$\frac{\partial f_p}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f_p = C[f_p] \qquad \omega < T$$

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0 \quad \omega < T \frac{s}{\eta}$$

Gradient expansion (simple non-relativistic fluid)

Simple fluid: Conservation laws for mass, energy, momentum

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \vec{j}^{\,\rho} = 0$$

$$\frac{\partial \epsilon}{\partial t} + \vec{\nabla} \vec{j}^{\,\epsilon} = 0$$

$$\frac{\partial \pi_i}{\partial t} + \frac{\partial}{\partial x_j} \Pi_{ij} = 0$$

Ward identity: mass current $=$ momentum density

$$\vec{j}^{\,\rho} \equiv \rho \vec{v} = \vec{\pi}$$

Constitutive relations: Gradient expansion for currents

Energy momentum tensor

$$\Pi_{ij} = P\delta_{ij} + \rho v_i v_j + \eta \left(\partial_i v_j + \partial_j v_i - \frac{2}{3} \delta_{ij} \partial_k v_k \right) + O(\partial^2)$$

Gradient expansion, Kubo formula

Consider background metric $g_{ij}(t,x) = \delta_{ij} + h_{ij}(t,x)$. Linear response

$$\delta\Pi^{xy} = -\frac{1}{2} G_R^{xyxy} h_{xy}$$

Harmonic perturbation $h_{xy} = h_0 e^{-i\omega t}$

$$G_R^{xyxy} = P - i\eta\omega + \ldots$$

Kubo relation: $$\eta = -\lim_{\omega\to 0}\left[\frac{1}{\omega}\mathrm{Im}G_R^{xyxy}(\omega,0)\right]$$

Gradient expansion: $\omega \leq \dfrac{P}{\eta} \simeq \dfrac{s}{\eta}T$.

Superfluid hydrodynamics

Spontaneous symmetry breaking: $\langle \Psi \rangle = v_0 e^{i\theta}$.

Goldstone boson is a new hydro mode: $\vec{v}_s = \frac{\hbar}{m} \vec{\nabla}\theta$

$$\partial_t \vec{v}_s + \frac{1}{2}\vec{\nabla}(v_s^2) = -\vec{\nabla}\mu$$

Momentum density: $\pi_i = \rho_n v_{n,i} + \rho_s v_{s,i}$

$$\rho = \rho_n + \rho_s \qquad \rho_s = \frac{1}{2}\frac{\partial F}{\partial w^2} \qquad \vec{w} = \vec{v}_n - \vec{v}_s$$

Stress tensor and energy current

$$\begin{aligned} \Pi_{ij} &= P\delta_{ij} + \rho_n v_{n,i} v_{n,j} + \rho_s v_{s,i} v_{s,j} \\ \vec{j}^{\epsilon} &= sT\vec{v}_n + \left(\mu + \frac{1}{2}v_s^2\right)\vec{\pi} + \rho_n \vec{v}_n \vec{v}_n \cdot \vec{w} \end{aligned}$$

Superfluid hydrodynamics

Dissipative stresses

$$\delta\Pi_{ij} = -\eta\left(\nabla_i v_{n,j} + \nabla_j v_{n,i} - \frac{2}{3}\delta_{ij}\vec{\nabla}\cdot\vec{v}_n\right)$$

$$-\,\delta_{ij}\Big(\zeta_1\vec{\nabla}\left(\rho_s\left(\vec{v}_s-\vec{v}_n\right)\right) + \zeta_2\left(\vec{\nabla}\cdot\vec{v}_n\right)\Big)$$

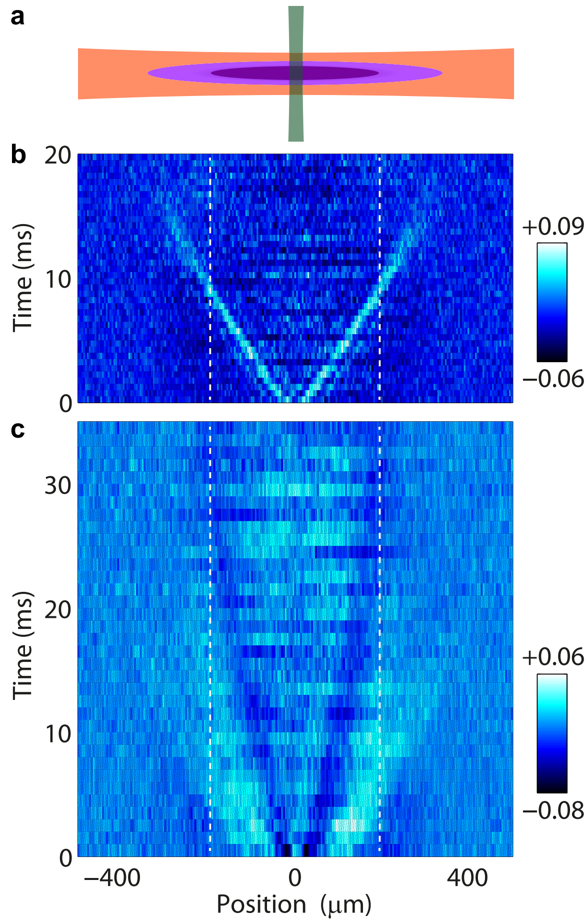
Equation of motions for v_s: $\dot{v}_s + \frac{1}{2}\nabla(v_s^{\,2}) = -\nabla(\mu + H)$ with

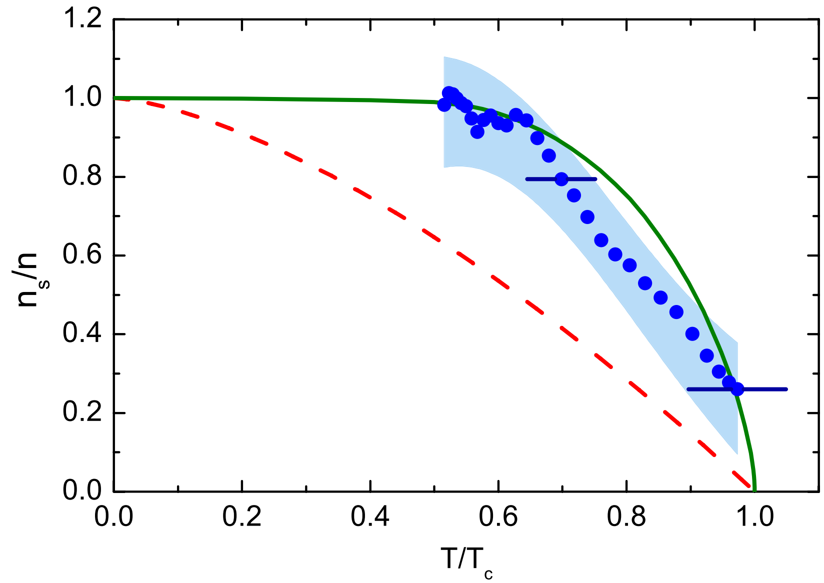
$$H = -\zeta_3\vec{\nabla}\left(\rho_s\left(\vec{v}_s-\vec{v}_n\right)\right) - \zeta_4\vec{\nabla}\cdot\vec{v}_n$$

Conformal symmetry: $\boxed{\zeta_1 = \zeta_2 = \zeta_4 = 0}$

Son (2007)

Superfluid Hydrodynamics: Second Sound





1st (top) 2nd sound (bottom)

in unitary Fermi gas

Superfluid mass fraction

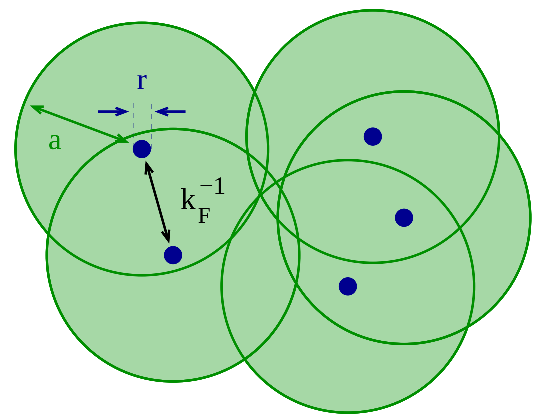
CAG, He, BEC (th)

Grimm et al. (2013)

In the following, I will concentrate on the unitary Fermi gas. This system is, essentially, equivalent to a dilute neutron gas (at densities $\rho \sim (0.1 - 1.0)\rho_0$).

dilute: $r\rho^{1/3} \ll 1$

strongly correlated: $a\rho^{1/3} \gg 1$



The results can be extended, without too much effort, to np pairing, $^3P_2$ pairing, and CFL quark matter (relativistic superfluid hydro).

Fermi gas at unitarity: Field Theory

Non-relativistic fermions at low momentum

$$\mathcal{L}_{\rm eff} = \psi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2M} \right) \psi - \frac{C_0}{2} (\psi^\dagger \psi)^2$$

Unitary limit: $a \to \infty$, $\sigma \to 4\pi/k^2$ ($C_0 \to \infty$)

This limit is smooth (HS-trafo, $\Psi = (\psi_\uparrow, \psi_\downarrow^\dagger)$)

$$\mathcal{L} = \Psi^\dagger \left[i\partial_0 + \sigma_3 \frac{\vec{\nabla}^2}{2m} \right] \Psi + \left(\Psi^\dagger \sigma_+ \Psi \phi + h.c. \right) - \frac{1}{C_0} \phi^* \phi \; ,$$

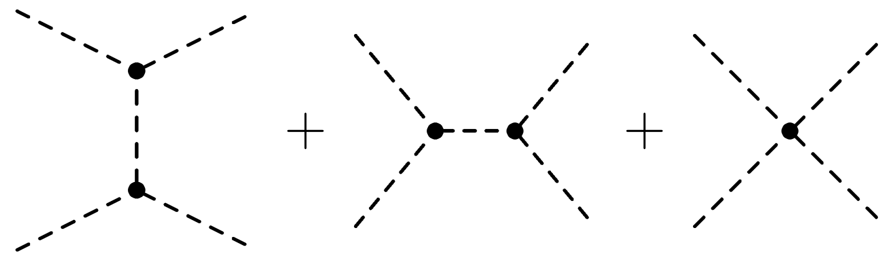
Low T ($T < T_c \sim \mu$): Pairing and superfluidity

Low T: Phonons Goldstone boson $\psi\psi = e^{2i\varphi}\langle\psi\psi\rangle$

$$\mathcal{L} = c_0 m^{3/2}\left(\mu - \dot{\varphi} - \frac{(\vec{\nabla}\varphi)^2}{2m}\right)^{5/2} + \ldots$$

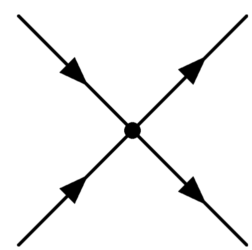
Viscosity dominated by $\varphi + \varphi \to \varphi + \varphi$

$$\eta = A\frac{\xi^5}{c_s^3}\frac{T_F^8}{T^5}$$



High T: Atoms Cross section regularized by thermal momentum

$$\eta = \frac{15}{32\sqrt{2}}(mT)^{3/2}$$

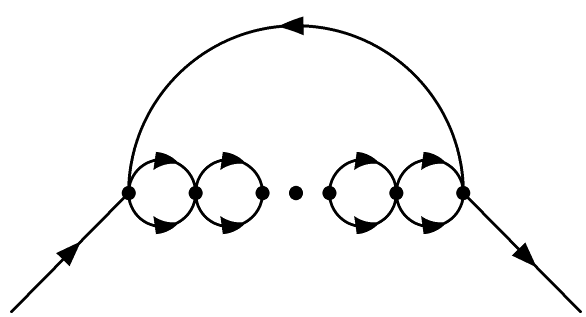


Bulk viscosity and conformal symmetry breaking

Conformal symmetry breaking (thermodynamics, normal phase)

$$1 - \frac{2\mathcal{E}}{3P} = \frac{\langle \mathcal{O}_{\mathcal{C}} \rangle}{12\pi m a P} \sim \frac{1}{6\pi}\, n\lambda^3\, \frac{\lambda}{a}$$

How does this translate into $\zeta \neq 0$? Momentum dependent $m^*(p)$.



$$Im\,\Sigma(k) \sim zT\sqrt{\frac{T}{\epsilon_k}}Erf\left(\sqrt{\frac{\epsilon_k}{T}}\right) \ll T$$

$$Re\,\Sigma(k) \sim zT\frac{\lambda}{a}\sqrt{\frac{T}{\epsilon_k}}F_D\left(\sqrt{\frac{\epsilon_k}{T}}\right)$$

Bulk viscosity

$$\boxed{\zeta = \frac{1}{24\sqrt{2}\pi}\lambda^{-3}\left(\frac{z\lambda}{a}\right)^2}$$

$$\zeta \sim \left(1 - \frac{2\mathcal{E}}{3P}\right)^2 \eta$$

$\zeta_1 - \zeta_4$ in superfluid phase, Escobedo et al (2009).

Thermal conductivity

Superfluids are very efficient conductors of heat, by a process usually called superfluid convection.

There is a non-zero (but difficult to observe) diffusive contribution

$$\vec{\jmath}^{\,\epsilon} = -\kappa \vec{\nabla} T$$

The calculation of κ is subtle, because quasi-particles with linear dispersion $E_p \sim c_s p$ do not contribute. [Roughly, linear qp's always transport momentum together with energy.]

The dominant process is phonon splitting, made possible by non-linear terms in the dispersion relation.

$$\kappa = \frac{128}{3\pi}\frac{\gamma^2}{g_3^2}\frac{T^2}{c_s^2} D_H = \frac{256\sqrt{2}}{25\pi^3\xi^2 m}(mT)^{3/2}\left(\frac{T}{T_F}\right)^2 D_H$$

Normal phase $\kappa \sim m^{1/2} T^{3/2}$

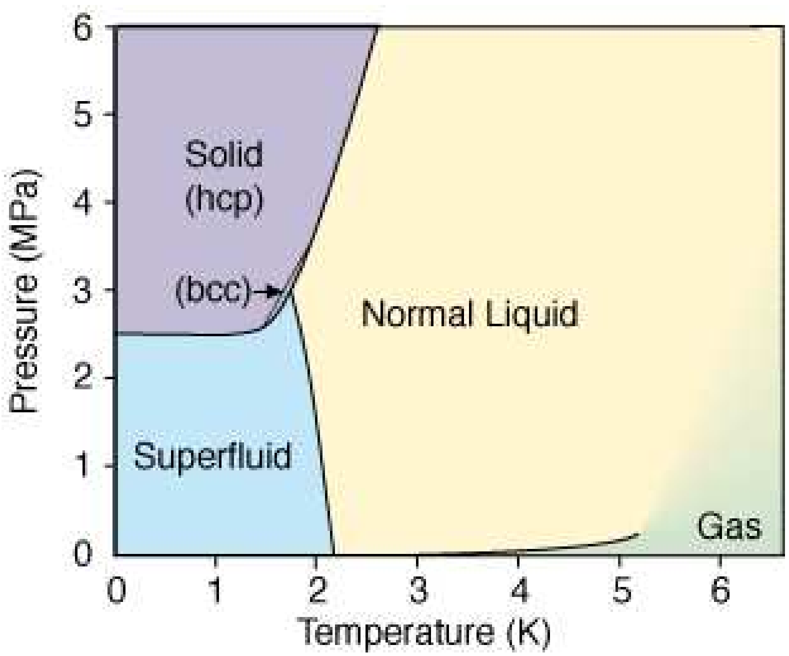
Liquid Helium

Bosons, van der Waals + short range repulsion

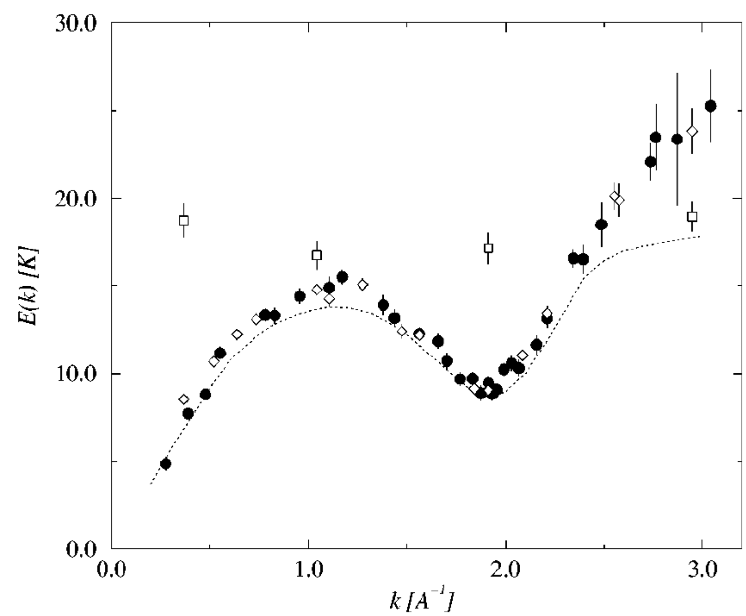
$$S = \int \Phi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2M} \right) \Phi + \int\int \left(\Phi^\dagger \Phi \right) V(x-y) \left(\Phi^\dagger \Phi \right)$$

with $V(x) = V_{sr}(x) - c_6/x^6$. Note: $a = 189 a_0 \gg a_0$

Phase Diagram



Excitations



Low T: Phonons and Rotons Effective lagrangian

$$\mathcal{L} = \varphi^*(\partial_0^2 - v^2)\varphi + i\lambda\dot{\varphi}(\vec{\nabla}\varphi)^2 + \ldots$$

$$+ \varphi^*_{R,v}(i\partial_0 - \Delta)\varphi_{R,v} + c_0(\varphi^*_{R,v}\varphi_{R,v})^2 + \ldots$$

Shear viscosity

$$\eta = \eta_R + \frac{c_{Rph}}{\sqrt{T}}\, e^{\Delta/T} + \frac{c_{ph}}{T^5} + \ldots$$

High T: Atoms Viscosity governed by hard core ($V \sim 1/r^{12}$)

$$\eta = \eta_0(T/T_0)^{2/3}$$

Experiment: Liquid Helium

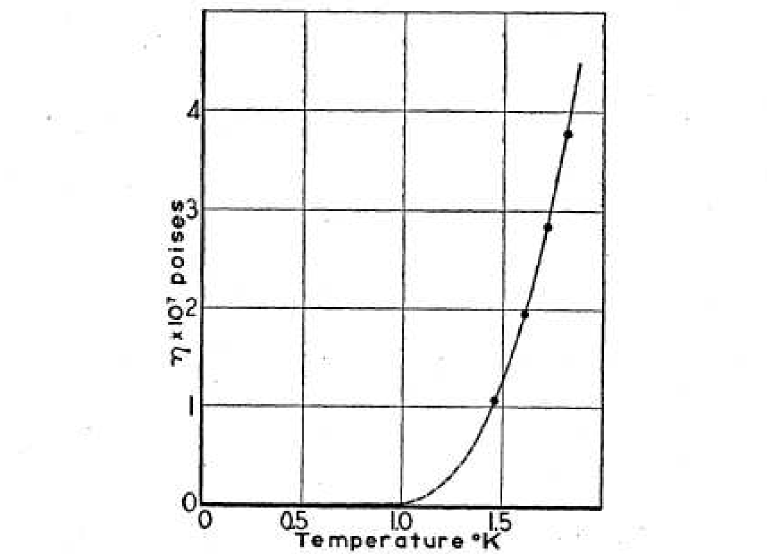
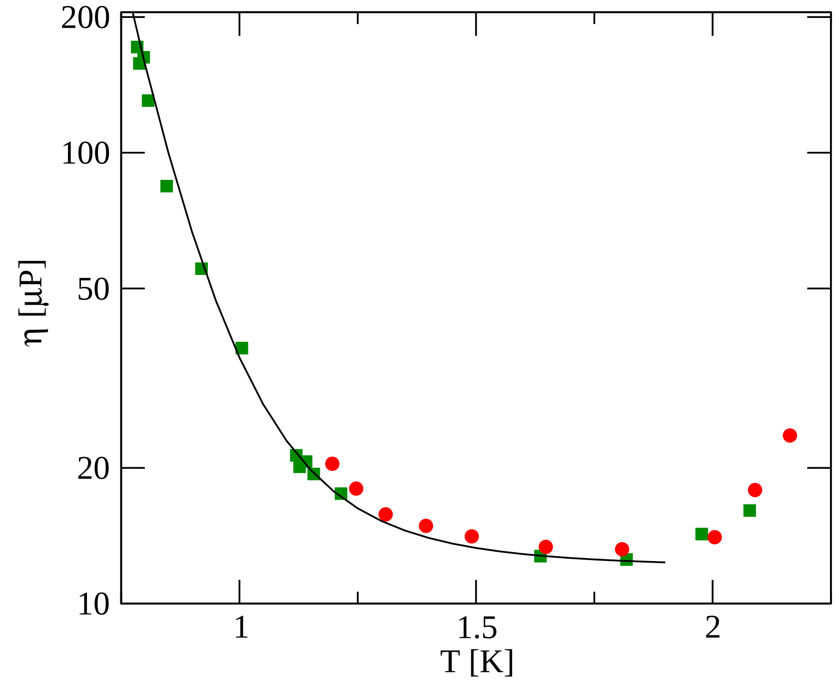


FIG. 1. The viscosity of liquid helium II measured by flow through a 10^{-4} cm channel.



Kapitza (1938)

viscosity vanishes below T_c

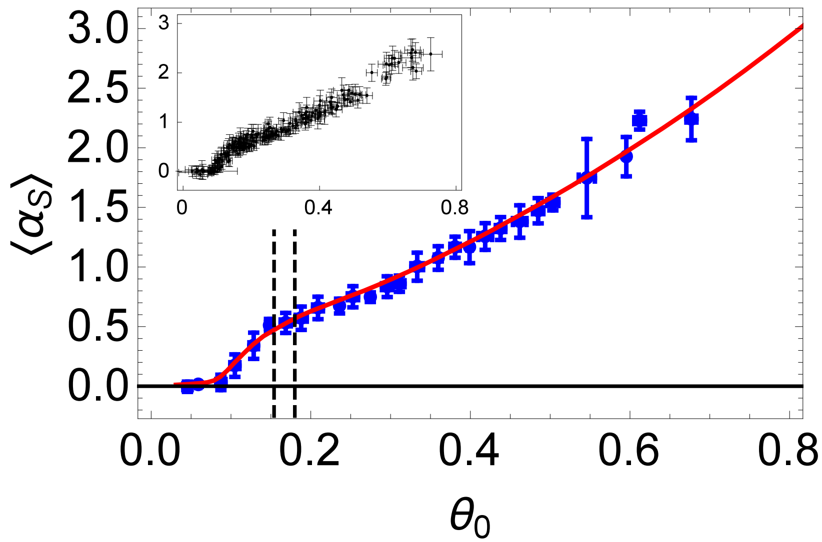
capillary flow viscometer

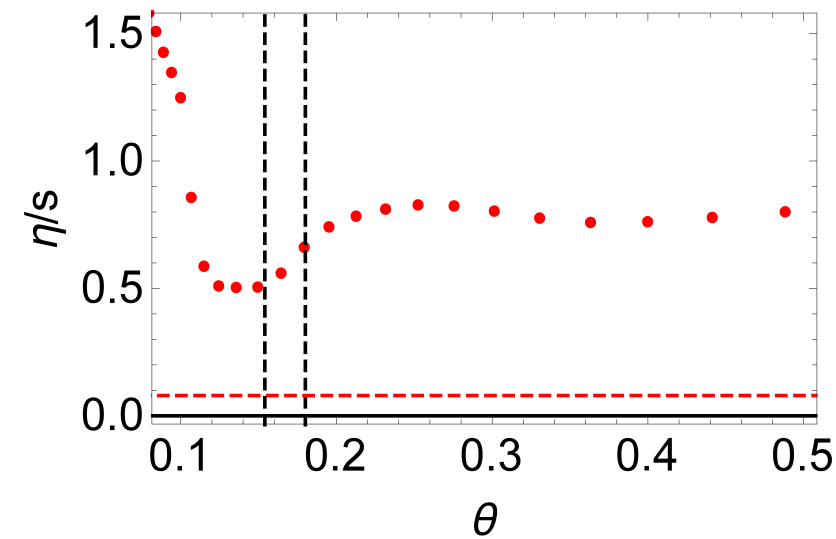
Hollis-Hallett (1955)

roton minimum, phonon rise

rotation viscometer

Experiment: Unitary Fermi Gas (recent update)



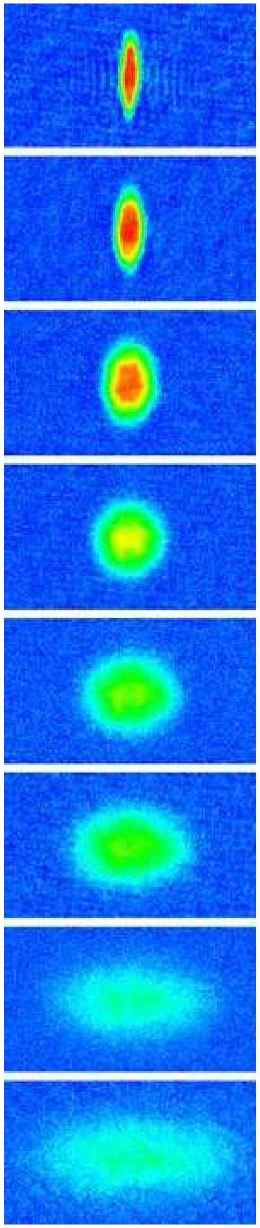


(η/n) drops to zero in superfluid phase

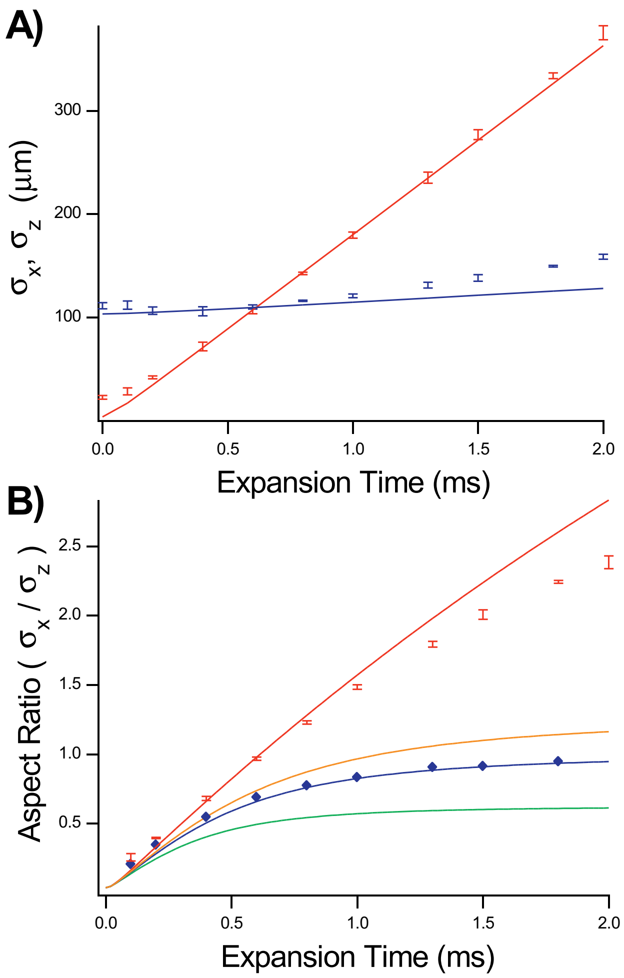
(η/s) has a minimum near T_c

$\theta = (T/T_F)^{3/2}$ (trap center) $\quad \alpha_S = \eta/n \quad$ Joseph et al. (2014)

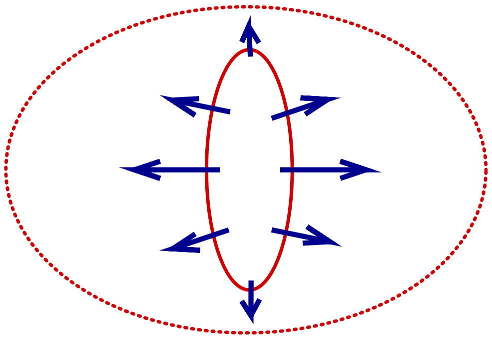
Experiments: Elliptic flow



O'Hara et al. (2002)

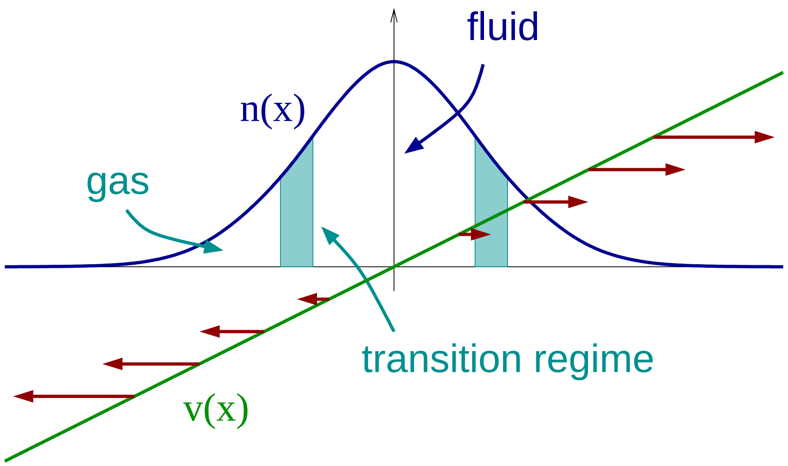


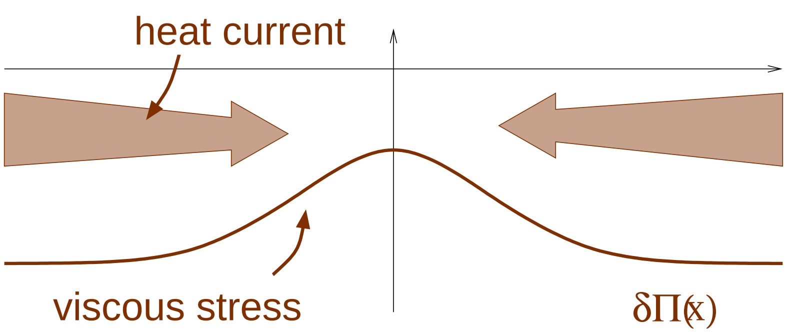
Hydrodynamic expansion converts coordinate space anisotropy to momentum space anisotropy



Determination of $\eta(n,T)$

Measurement of $A_R(t,E_0)$ determines $\eta(n,T)$. But:





The whole cloud is not a fluid.

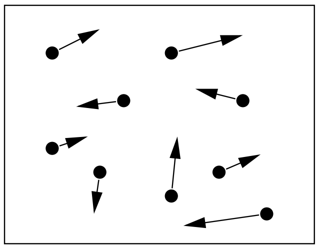
Can we ignore this issue?

No. Hubble flow & low density viscosity $\eta \sim T^{3/2}$ lead to paradoxical fluid dynamics.

$$\dot{Q} = \int \sigma \cdot \delta\Pi = \infty$$

Revisit: Fluid dynamics from kinetic theory

Microscopic picture:
Quasi-particle distribution
function $f_p(x,t)$



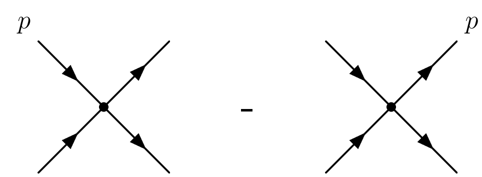
$$\begin{aligned} \rho(x,t) &= \int d\Gamma_p \, m f_p(x,t) \\ \pi_i(x,t) &= \int d\Gamma_p \, p_i f_p(x,t) \\ \Pi_{ij}(x,t) &= \int d\Gamma_p \, p_i v_j f_p(x,t) \end{aligned}$$

Boltzmann equation

$$\left(\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}_x + \vec{F} \cdot \vec{\nabla}_p \right) f_p(t,x,) = C[f_p]$$

Collision term

$$C[f_1] = \int d\Gamma_{234} (f_1 f_2 - f_3 f_4) w(12;34)$$



Fluid dynamics from kinetic theory

Conservation laws (collision term)

$$\int d\Gamma_p \, M_p \, C[f_p] = 0 \qquad\qquad M_p = \{1, p, E_p\}$$

Moments of Boltzmann equation imply fluid dynamic conservation laws

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \vec{\jmath}^{\,\rho} = 0$$

$$\frac{\partial \epsilon}{\partial t} + \vec{\nabla} \vec{\jmath}^{\,\epsilon} = 0$$

$$\frac{\partial \pi_i}{\partial t} + \frac{\partial}{\partial x_j} \Pi_{ij} = 0$$

Need constitutive equations (and equation of state)

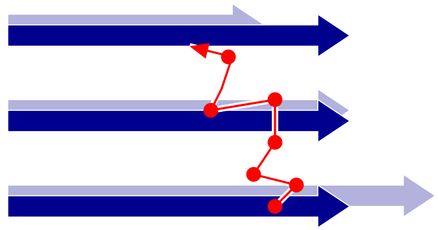
$$\vec{\jmath}^{\,\rho} = ? \qquad \vec{\jmath}^{\,\epsilon} = ? \qquad \Pi_{ij} = ?$$

Kinetic theory: Knudsen expansion

Chapman-Enskog expansion $f = f_0 + \delta f_1 + \delta f_2 + \ldots$

Gradient exp. $\delta f_n = O(\nabla^n)$

$\equiv$ Knudsen exp. $\delta f_n = O(Kn^n)$



Zeroth order result: $f_0 = \exp(-\beta(E_p - \vec{p}\cdot\vec{u} - \mu)) \quad \beta = 1/T$

$$\begin{aligned} \vec{\jmath}^{\,\rho} &= \vec{\pi} = \rho\vec{u} \\ \vec{\jmath}^{\,\epsilon} &= (\mathcal{E} + P)\vec{u} \qquad\qquad P = \frac{2}{3}\mathcal{E} \\ \Pi_{ij} &= \rho u_i u_j + P\delta_{ij} \end{aligned}$$

First order result: $\delta f_1 = -f_0\,\frac{\eta}{PT} v^i v^j \sigma_{ij} + \ldots$

$$\begin{aligned} \delta^{(1)}\Pi_{ij} &= -\eta\sigma_{ij} \\ \delta^{(1)}\jmath_i^{\,\epsilon} &= -\eta u^j\sigma_{ij} - \kappa\nabla_i T \end{aligned}$$

Approaches to dilute regime

Kinetic theory valid, but no expansion $f \simeq f_0 + \delta f + \ldots$.

Combine hydrodynamics & Boltzmann equation. Not straightforward.

Hydrodynamics + non-hydro degrees of freedom ($\mathcal{E}_a$; $a = x, y, z$)

$$\frac{\partial \mathcal{E}_a}{\partial t} + \vec{\nabla} \cdot \vec{\jmath}_a^{\,\epsilon} = -\frac{\Delta P_a}{2\tau} \qquad \Delta P_a = P_a - P$$

$$\frac{\partial \mathcal{E}}{\partial t} + \vec{\nabla} \cdot \vec{\jmath}^{\,\epsilon} = 0 \qquad \mathcal{E} = \sum_a \mathcal{E}_a$$

τ small: Fast relaxation to Navier-Stokes with $\tau = \eta/P$

τ large: Additional conservation laws. Ballistic expansion.

Anisotropic hydro from kinetic theory

Consider modified expansion

$$f = f_A + \delta f_1' + \delta f_2' + \ldots$$

Anisotropic distribution function

$$f_A = \exp\left(-\frac{(p_a - m u_a)^2}{2mT_a} - \frac{\mu}{\bar{T}}\right) \qquad \bar{T} = (\prod T_a)^{1/3}$$

- f_A is an exact solution of the Boltzmann equation in the ballistic limit.
- The viscous stresses and dissipative corrections to the energy current have the same form as in the Chapman-Enskog theory.

Anisotropic Hydrodynamics from kinetic theory

Moments of the Boltzmann equation with $M_p = \{1, \vec{p}, E_P\}$.

Navier-Stokes with $\delta\Pi_{aa} = \Delta P_a$

Moments of the Boltzmann equation with p_a^2

$$\frac{\partial \mathcal{E}_a}{\partial t} + \vec{\nabla} \cdot \vec{\jmath}_a^{\,\epsilon} = -\frac{\Delta P_a}{2\tau} \qquad \Delta P_a = P_a - P$$

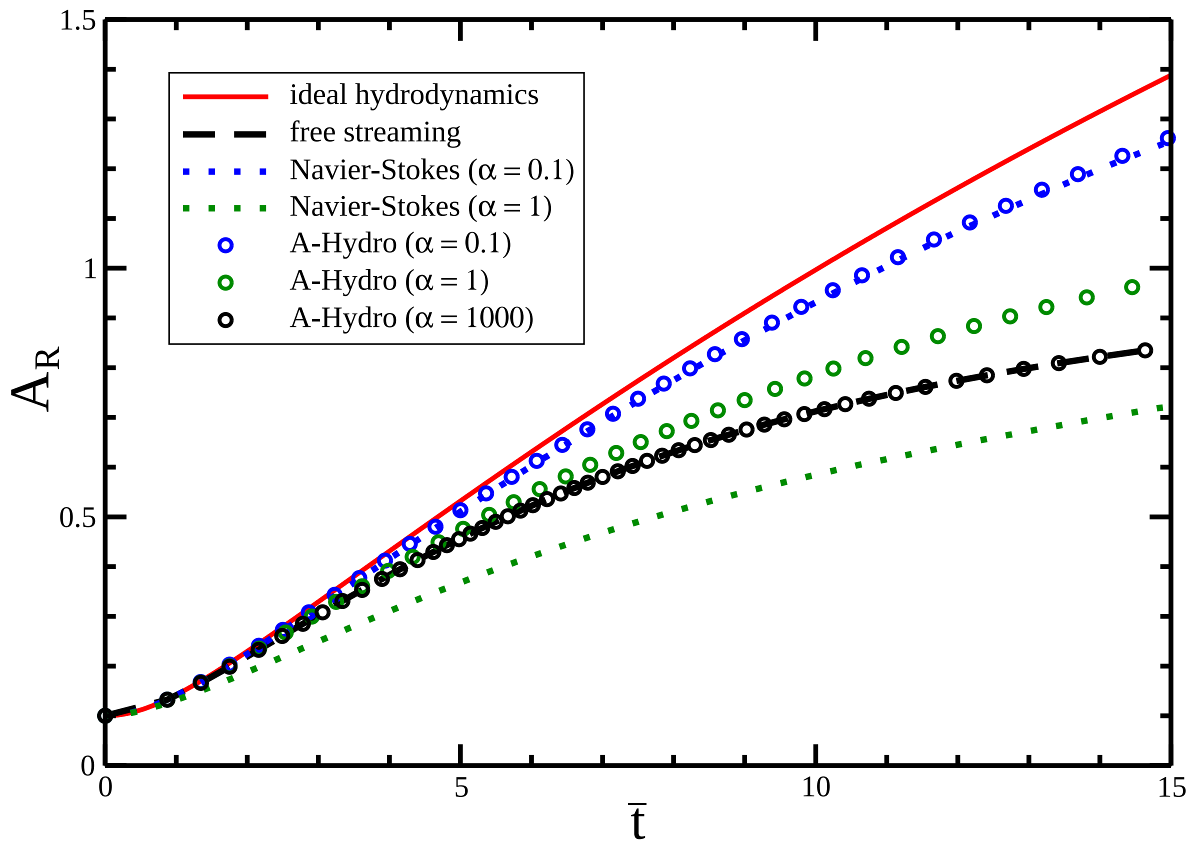
with $P_a = 2\mathcal{E}_a$ $(P = \frac{2}{3}\mathcal{E})$

Solve fluid dynamic equations for small τ

$$\delta\Pi_{aa} = \Delta P_a = -\eta\sigma_{aa}$$

Ballistic limit $\tau \to \infty$: Conservation law for $\mathcal{E}_a$.

Anisotropic Hydrodynamics: Aspect ratio



Consider $\eta = \alpha n$ and $\alpha \in [0, \infty)$

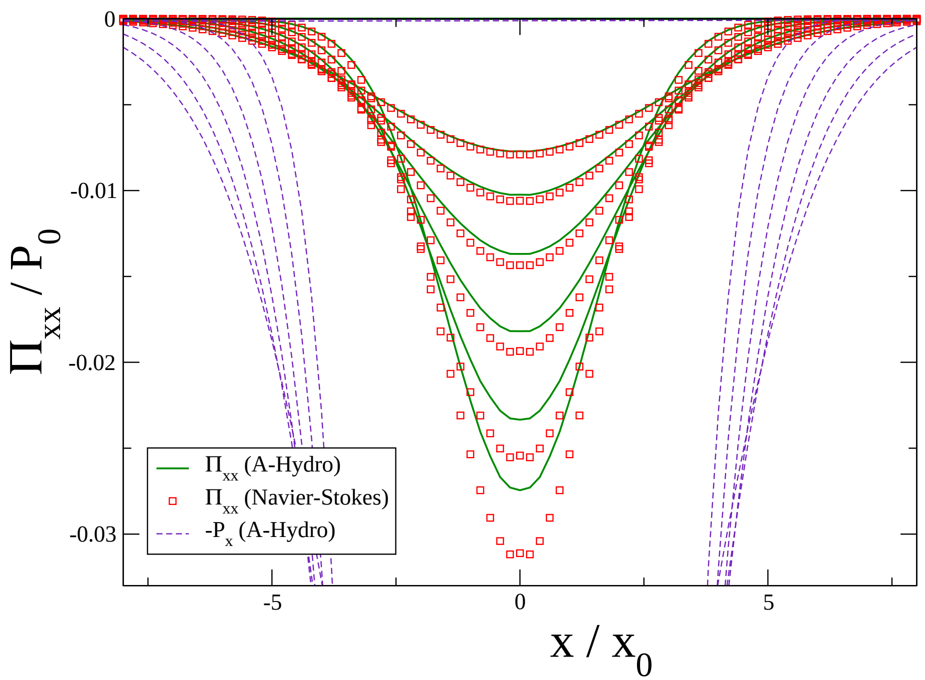
Navier-Stokes: Ideal hydro $\to$ very viscous hydro.

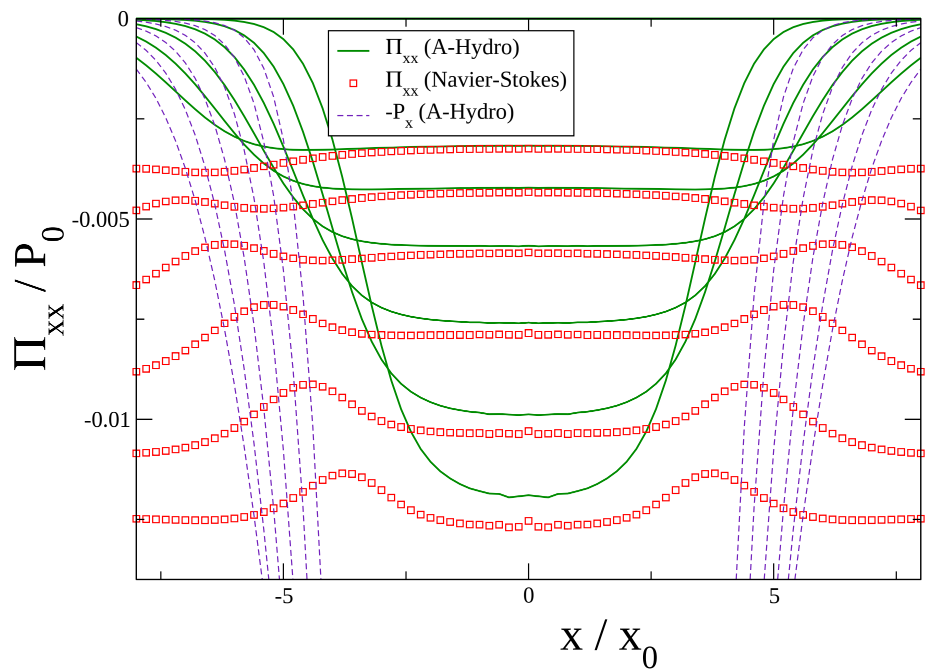
A-hydro: Ideal hydro $\to$ ballistic expansion.

AVH1 hydro code, M. Bluhm & T.S. (2015)

Anisotropic Hydrodynamics: Evolution of $\delta\Pi_{aa}$

$\eta = \alpha_n n$ $\eta = \alpha_T (mT)^{3/2}$

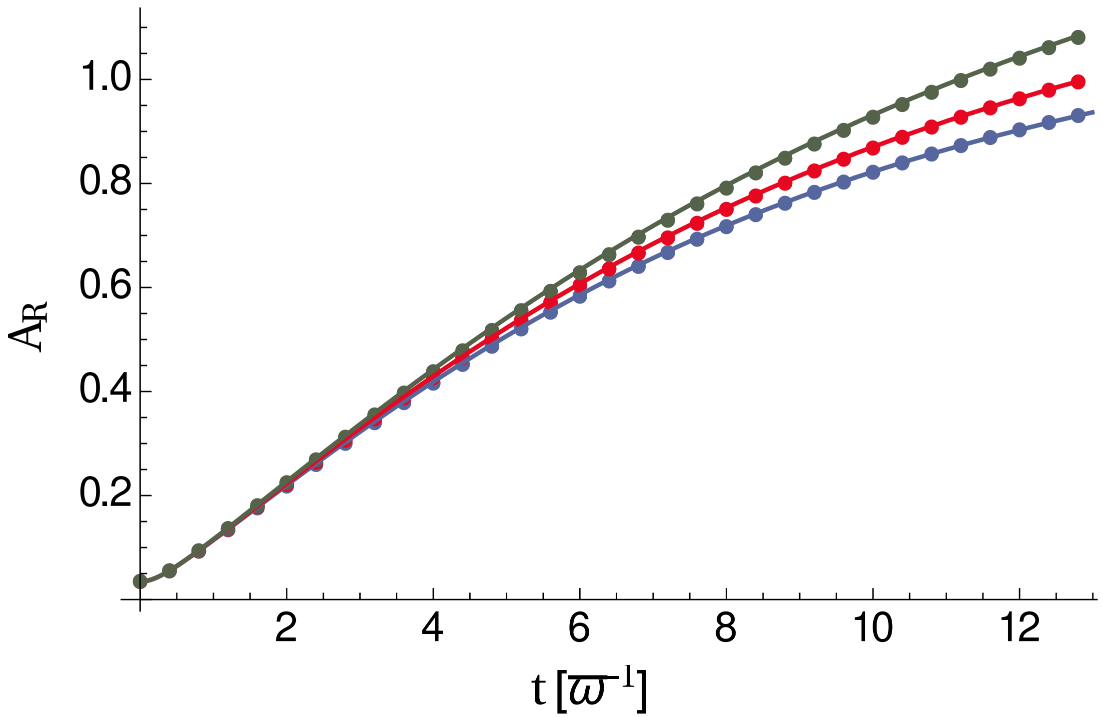




$\delta\Pi_{xx}$(Navier-Stokes) $\delta\Pi_{xx}$(A-Hydro)

AVH1 hydro code, M. Bluhm & T.S. (2015)

Anisotropic Hydrodynamics: Comparison with Boltzmann



$T/T_F =$

$0.79, 1.11, 1.54$

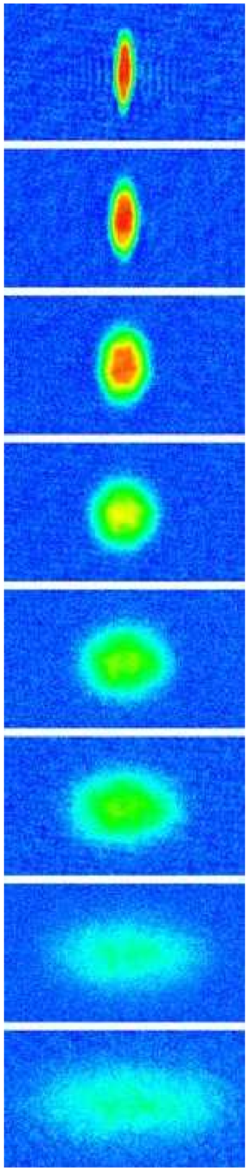
Dots: Two-body Boltzmann equation with full collision kernel

Lines: Anisotropic hydro with η fixed by Chapman-Enskog

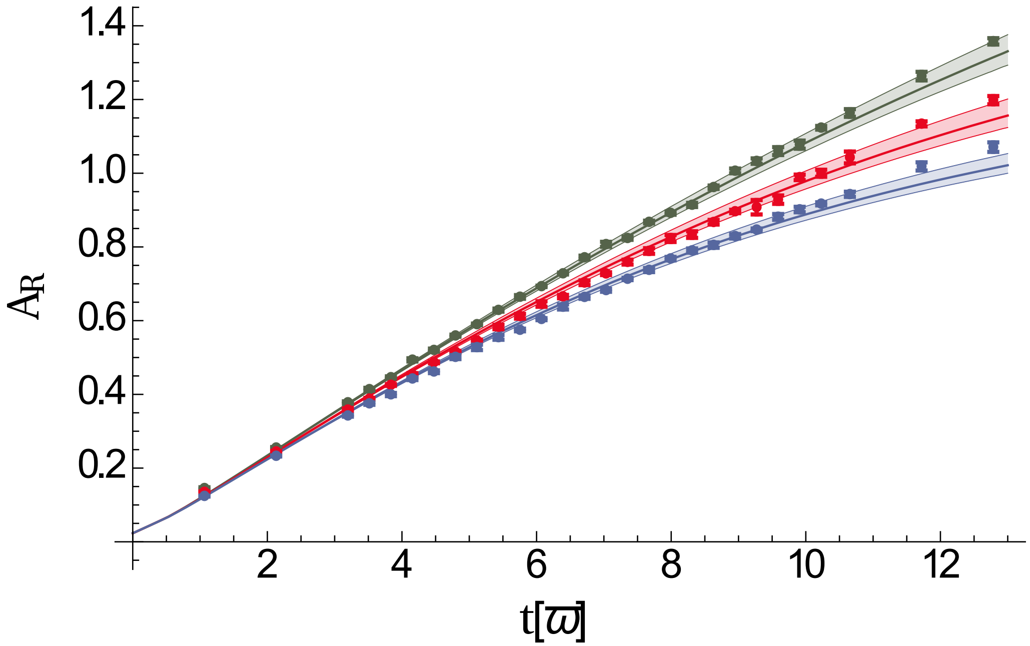
High temperature (dilute) limit: Perfect agreement!

AVH1 hydro code, M. Bluhm & T.S. (2015)

Elliptic flow: High T limit



Quantum viscosity $\eta = \eta_0 \dfrac{(mT)^{3/2}}{\hbar^2}$



Cao et al., Science (2010)

Bluhm et al., PRL (2016)

$T/T_F =$

$0.79, 1.11, 1.54$

fit: $\eta_0 = 0.28 \pm 0.02$

theory: $\eta_0 = \frac{15}{32\sqrt{\pi}} = 0.269$

Outlook

Reanalyze data for $T \gtrsim T_c$. Unfold temperature, density dependence of η/s.

Applications to other transport problems: Diffusion, superfluid hydrodynamics.

Study more complicated flow patterns in shaped traps.