Stochastic Fluid Dynamics and the QCD Critical Point

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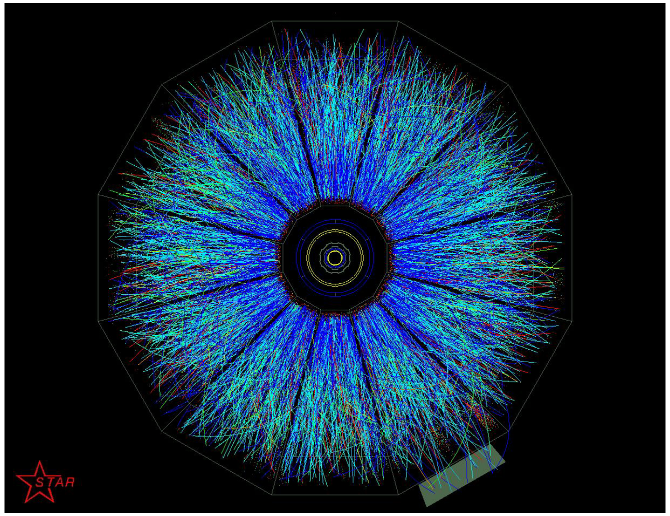
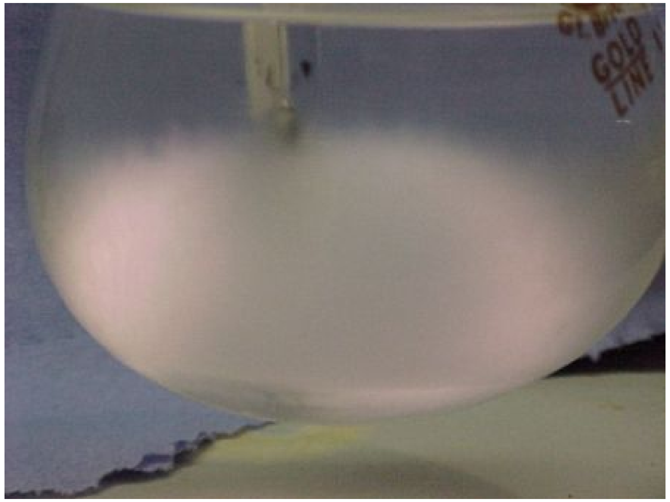
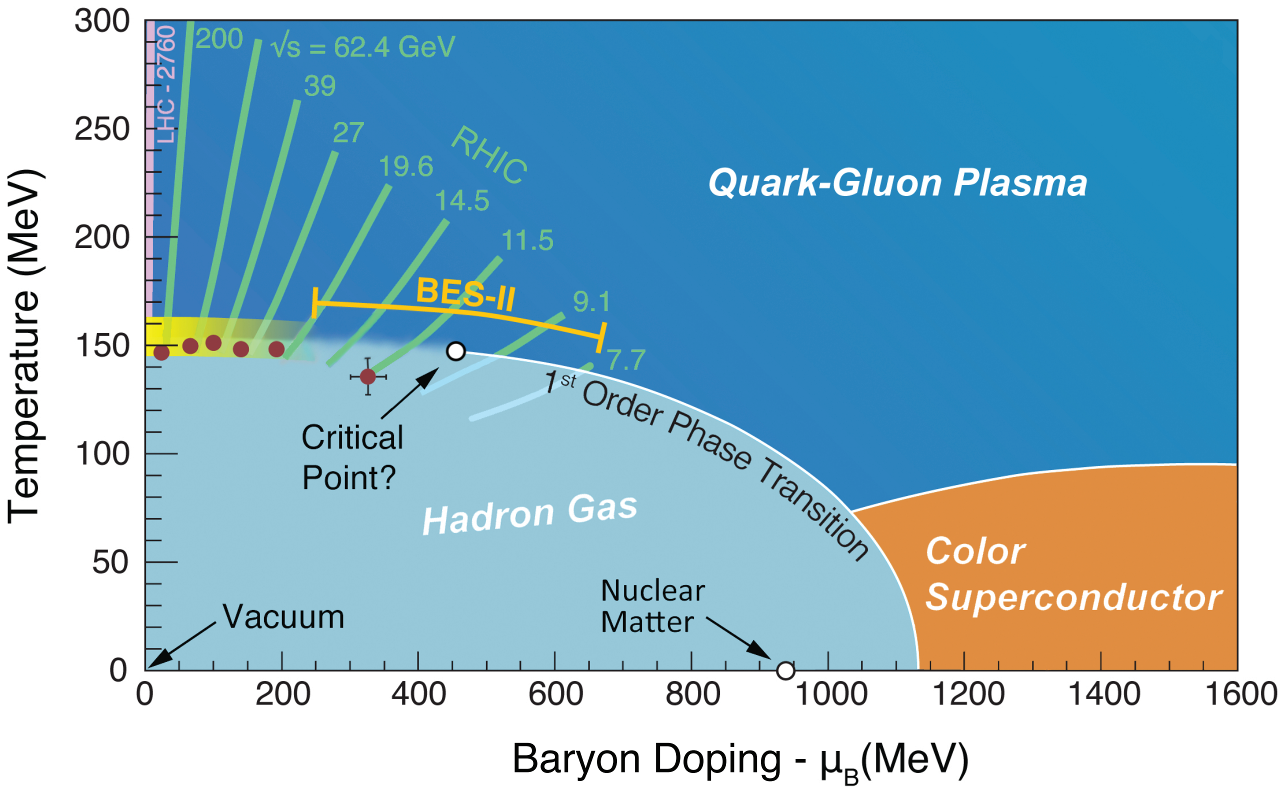


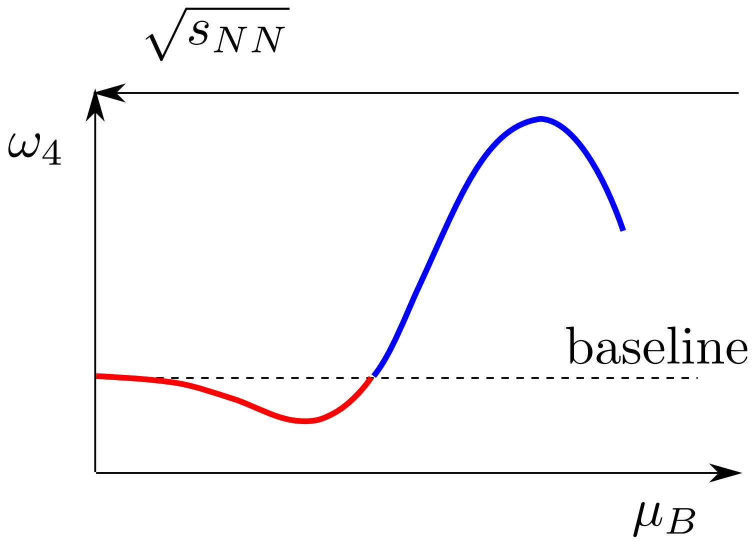


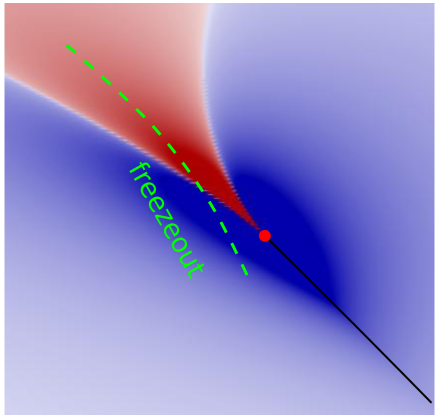
RHIC beam energy scan

Can we experimentally locate the QCD phase transition, either by detecting a critical point, or by identifying a first order transition?



Basic discovery idea: Study fluctuation observables. Expect non-monotonic variation of 4th order gallant near Ising critical point.





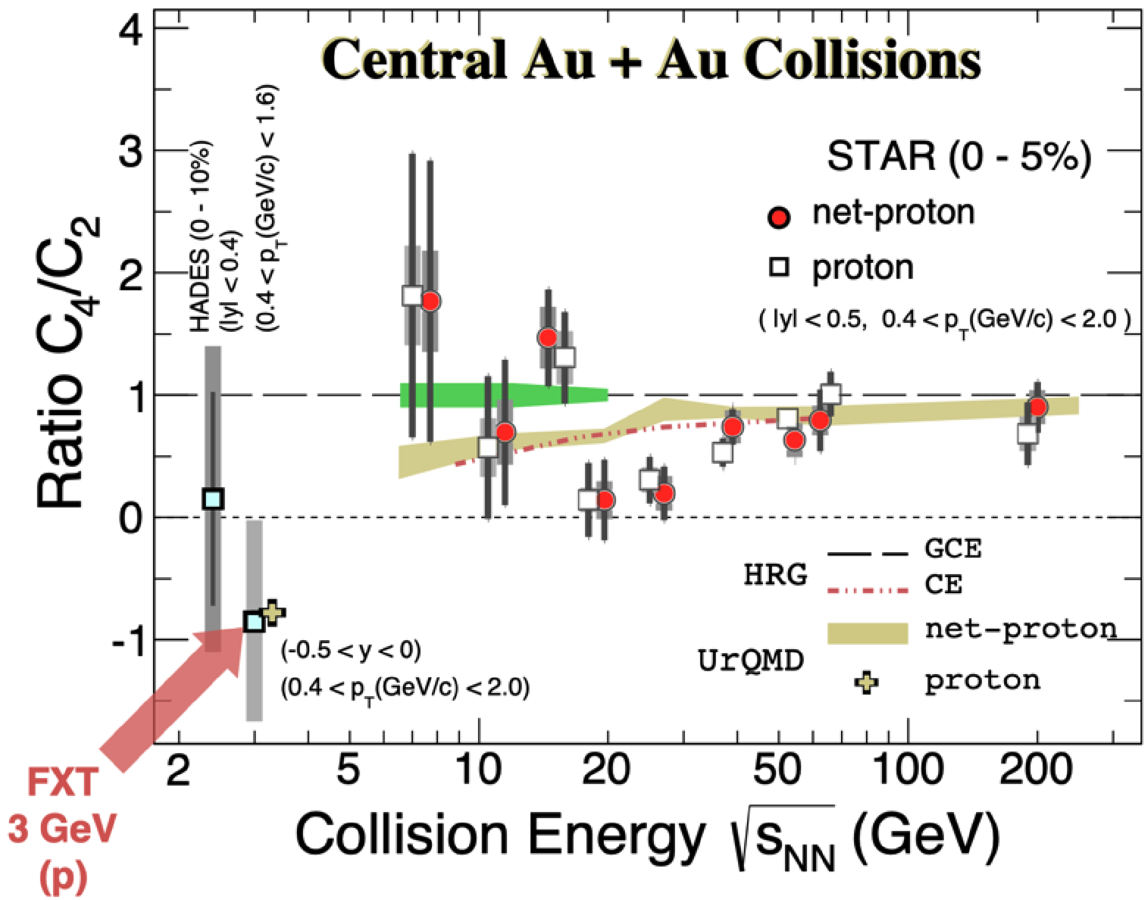
Real world may well be more complicated:

- Finite size and finite expansion rate effects.
- Non-equilibrium effects (memory, critical slowing).
- Freezeout, resonances, global charge conservation, etc.

Motivates dynamical studies.

Figure from Bzdak et al. [1906.00936]

RHIC beam energy scan, BESI



BESII data have been taken, and are being analyzed.

Dynamical Theory

What is the dynamical theory near the critical point?

The basic logic of fluid dynamics still applies. Important modifications:

- Critical equation of state.
- Stochastic fluxes, fluctuation-dissipation relations.
- Possible Goldstone modes (chiral field in QCD?)

Outline:

1. Static universality: Realistic EOS with Ising universality
   (a) Bulk viscosity near the critical point
2. Dynamic universality: Model H in a static background
   (a) Critical relaxation rate
   (b) Multiplicative noise
3. Hydrokinetics in an expanding background
4. Numerical approaches to stochastic diffusion

1. Equilibrium fluctuations

Consider an Ising-like system with order parameter ψ. Fluctuations governed by an entropy functional

$$Prob[\psi,\epsilon] \sim \exp(S[\psi,\epsilon]) \qquad\qquad S = \int d^3x\, s(\psi,\epsilon)$$

energy density ϵ, order parameter ψ

Conjugate variables

$$x^A = (\epsilon,\psi) \qquad\qquad X_A = -\frac{\partial s}{\partial x^A} = (r,h)$$

reduced temperature r, magnetic field h

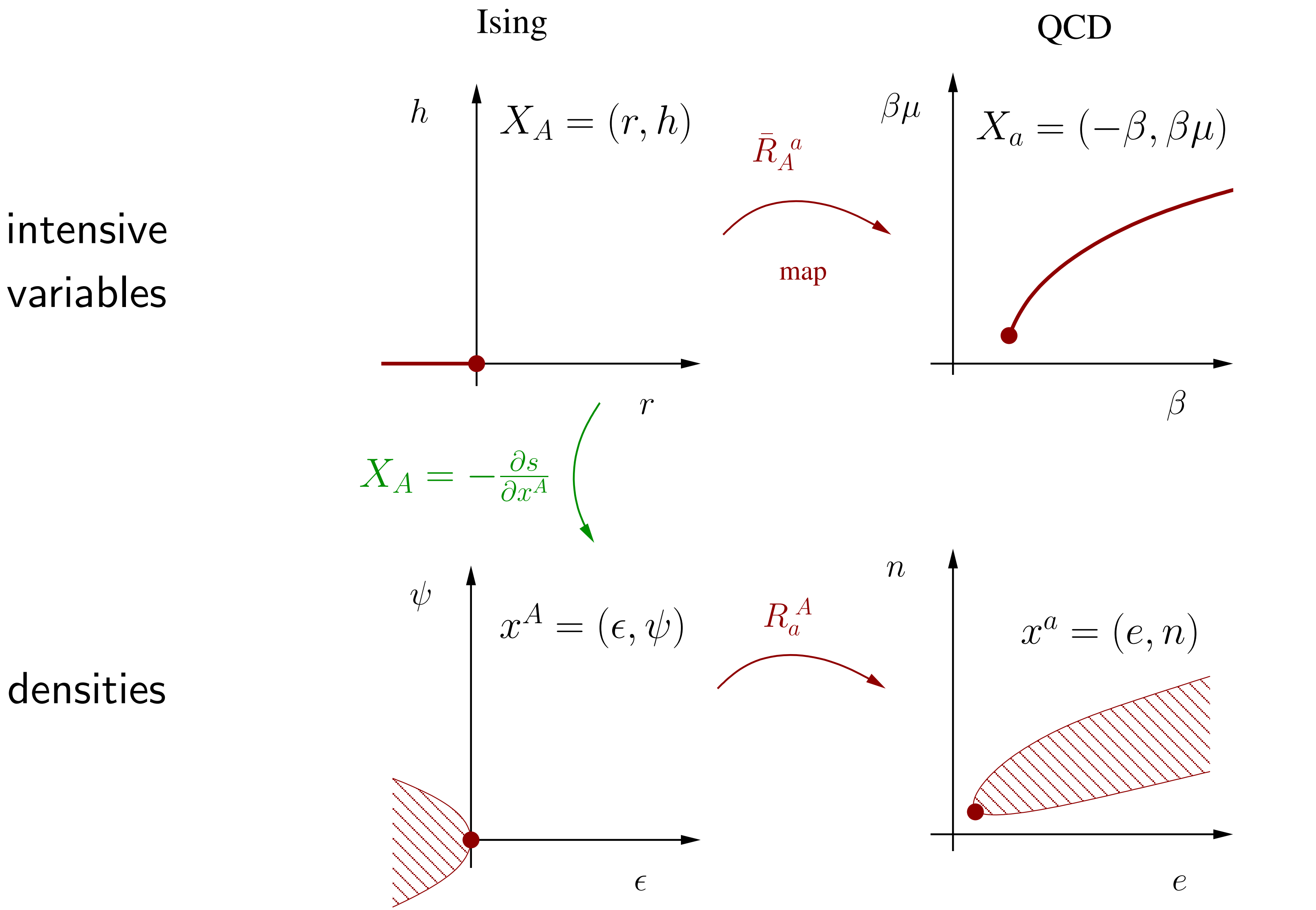
QCD: Canonical pair

$$x^a = (e,n) \qquad\qquad X_a = (-\beta,\beta\mu)$$

energy density e, baryon density n

inverse temperature β, chemical potential μ

Mapping the Ising EOS to QCD



$$R_b^A = (\partial x^A)/(\partial x^b) \qquad \bar{R}_A^b = (\partial X_A)/(\partial X_b) \qquad R_b^A \bar{R}_C^b = \delta_C^A$$

BEST equation of state

Parotto et al. write

$$P(T,\mu_B) = T^4 \sum_n c_{2n}^{reg}(T)(\beta\mu_B)^{2n} + P^{crit}(T,\mu_B)$$

where c_{2n}^{reg} is adjusted to reproduce lattice $\chi_n^B(T)$ and the critical part is determined by a linear map to the Ising EOS (parameterized by Zinn-Justin)

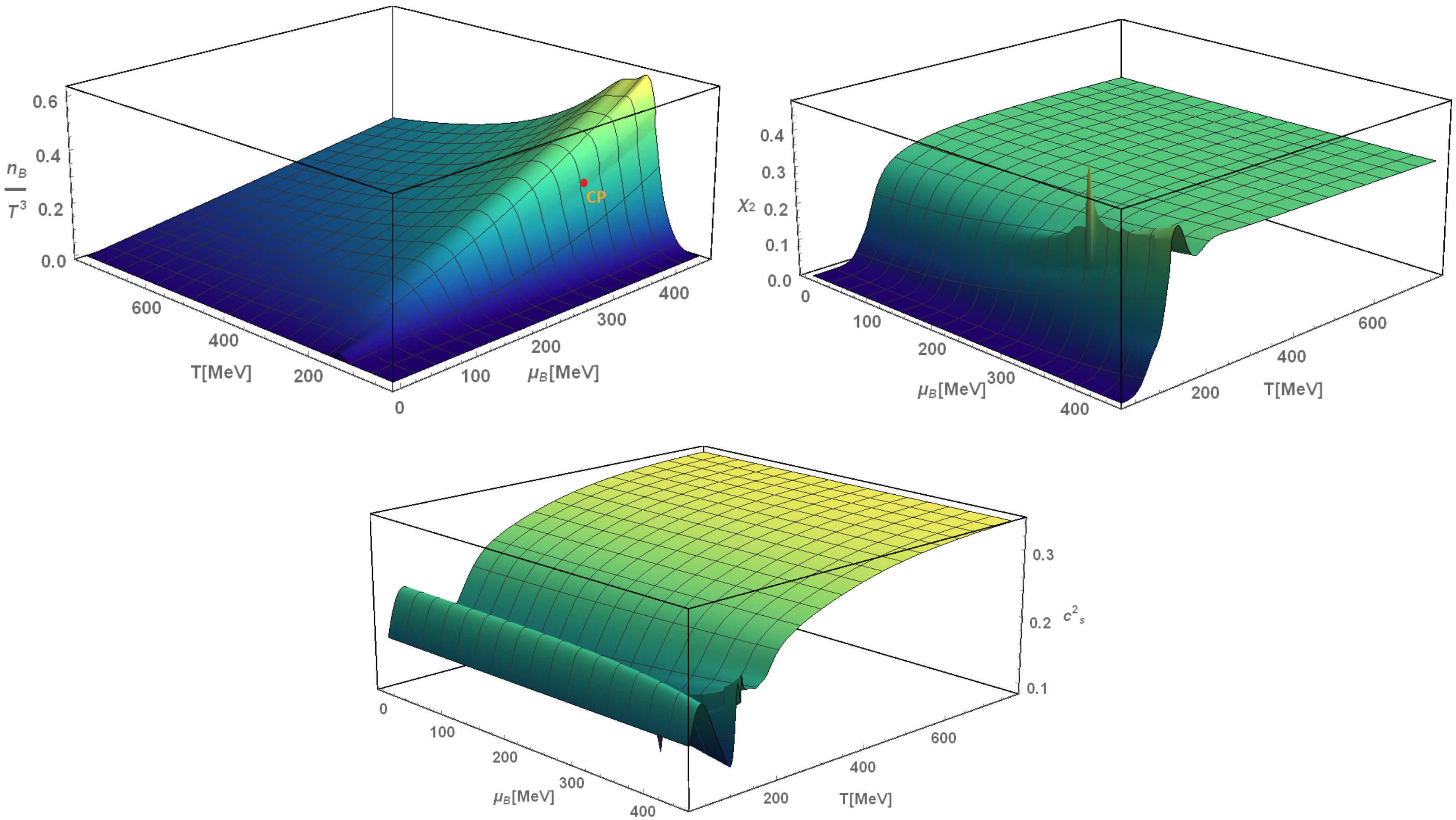
$$\begin{aligned} \frac{T-T_c}{T_c} &= \bar{w}\,(r\bar{\rho}\sin\alpha_1 + h\sin\alpha_2) \\ \frac{\mu-\mu_c}{T_c} &= \bar{w}\,(-r\bar{\rho}\cos\alpha_1 - h\cos\alpha_2) \end{aligned}$$

parameters

$$(\mu_c, T_c, \bar{w}, \bar{\rho}, \alpha_1, \alpha_2)$$

Connect to hadron gas at low T, and impose thermodynamic constraints.

A critical equation of state for QCD



Baryon density, compressibility, speed of sound.

Parotto et al. [1805.05249]

Application: Critical bulk viscosity

Bulk viscosity from order parameter relaxation

$$\zeta \sim (\gamma n T R_n^\epsilon)^2 \int d^3k \; \frac{2T\chi_k^2}{-i\omega + 2\Gamma_k}\bigg|_{\omega\to 0} \sim (\gamma n T R_n^\epsilon)^2 \xi^4$$

Critical bulk viscosity

$$\frac{\zeta}{s} = \sin^2(\alpha_1)\left(\frac{4\pi}{s/\eta}\right)\left(\frac{\xi}{\xi_0}\right)^{2.8} \begin{cases} 3.4\cdot 10^{-2} & r > 0 \\ 2.2\cdot 10^{-1} & r < 0 \end{cases}$$

$z \simeq 3$ dynamical critical exponent.

$\sin(\alpha_1)$: angle between Ising r and QCD temperature.

[Note: For $\sin(\alpha_1) \sim 0$ get $\zeta/s \sim (n/s)^2$.]

Amplitude ratio $(\gamma_-/\gamma_+)^2 \simeq 6$.

Martinez, T.S., Skokov [1906.11306]; see also Stephanov & Yin [1712.10305], An et al. [1912.13456], and Onuki, PRE 55 403 (1997).

2. Hydrodynamic equation for critical mode

Equation of motion for critical mode ψ coupled to momentum density $\vec{\pi}$ ("model H")

$$\frac{\partial\psi}{\partial t} = \kappa\nabla^2\frac{\delta\mathcal{F}}{\delta\psi} - g\vec{\nabla}\psi\cdot\frac{\delta\mathcal{F}}{\delta\vec{\pi}_T} + \zeta_\psi$$

Diffusion Advection Noise

Free energy functional: Order parameter ψ, momentum density $\vec{\pi} = w\vec{v}$

$$\mathcal{F} = \int d^dx\,\left[\frac{1}{2w}\vec{\pi}^2 + \frac{\gamma}{2}(\vec{\nabla}\psi)^2 + \frac{m^2}{2}\psi^2 + \frac{u}{4}\psi^4\right] \qquad D = m^2\kappa$$

Noise average (noise kernel $L = DT\nabla^2$)

$$\langle O\rangle = \frac{1}{Z}\int D\zeta_\psi\; O(\psi(x,t))\;\exp\left(-\frac{1}{4}\int d^3x\,\zeta_\psi L^{-1}\zeta_\psi\right)$$

Model H: Effective Action

MSRJD: Write noise average as an effective action

$$Z_{MSR} = \int D\psi D\tilde{\psi} D\pi D\tilde{\pi} \, \exp\left(- \int d^4x \mathcal{L} \right)$$

$$\mathcal{L} = \tilde{\psi}\left(\partial_t - D\nabla^2\right)\psi + \tilde{\pi}_T\left(\partial_t - \nu\nabla^2\right)\pi_T \qquad \text{Diffusion}$$

$$-\tilde{\psi} D T \nabla^2 \tilde{\psi} - \tilde{\pi}_T \nu T \nabla^2 \tilde{\pi}_T \qquad \text{Noise}$$

$$+\frac{1}{w}\tilde{\psi}\pi\cdot\nabla\psi + u\tilde{\psi}D\nabla^2\psi^3 + \ldots \qquad \text{Advection \& Interaction}$$

Consider background fluid at rest, $\psi_0 = const$, $\vec{\pi}_0 = 0$:

- Gaussian action $\Psi_a G^{-1}_{ab}\Psi_b$ with $\Psi_a = (\tilde{\psi}, \psi)$. G_{ab} has the analytic structure of Keldysh Green fct.
- $\mathcal{T}$-invariance: Detailed balance and Fluctuation-Dissipation relations.

Martin, Siggia, Rose, Phys. Rev. A 8, 423 (1973).

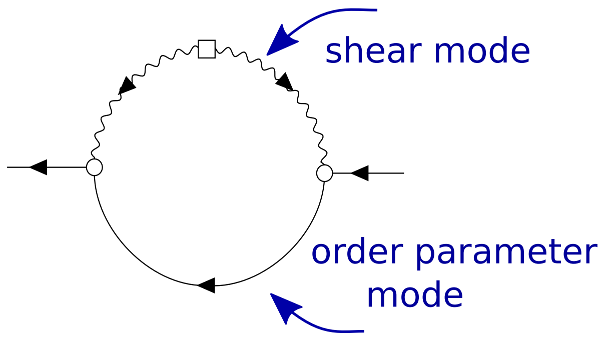
Model H: Critical Dynamics

Non-critical fluids: Gradient expansion $k\xi \ll 1$.

Critical fluids: RG analysis, study possible fixed points.

"Mode Coupling" approximation: Use bare shear viscosity, and static susceptibility χ_k

$$G^{-1}(\omega, k) = i\omega - Dk^2 - \delta\Gamma_k$$



Order parameter relaxation rate ("Kawasaki function").

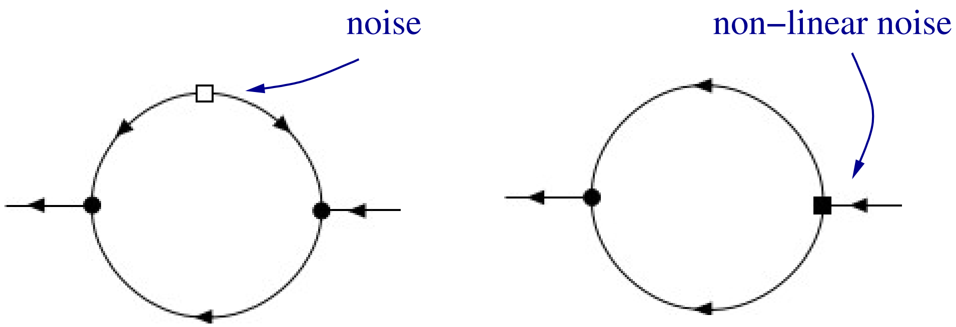
$$\Gamma_k = \frac{T\xi^{-3}}{6\pi\eta_0} K(k\xi) \qquad K(x) = \frac{3}{4}\left[1 + x^2 + \left(x^3 + x^{-1}\right)\arctan(x)\right] .$$

Dynamic critical exponent: $\Gamma_{\xi^{-1}} \sim \xi^{-z}$ with $z = 3$

New and non-classical interactions

Other interactions: Field dependent diffusion/viscosity, $\kappa = \kappa_0(1 + \lambda_D \psi)$.

$$\mathcal{L}_{int} = -\frac{D_0 \lambda'}{2} \left(\nabla^2 \tilde{\psi}\right) \psi^2 - \frac{D_0 \lambda_D}{m^2} \left(\vec{\nabla} \tilde{\psi}\right)^2 \psi$$



Coupling constant related by fluctuation-dissipation relation ($\mathcal{T}$-invariance)

Contribute to (non-critical) order parameter relaxation

$$\Sigma(\omega, k) = \frac{\lambda'}{32\pi} \left(i\lambda' \omega k^2 + \lambda_D \left[i\omega - Dk^2\right] k^2\right) \sqrt{k^2 - \frac{2i\omega}{D}}$$

(non-critical) Kawasaki function not modified.

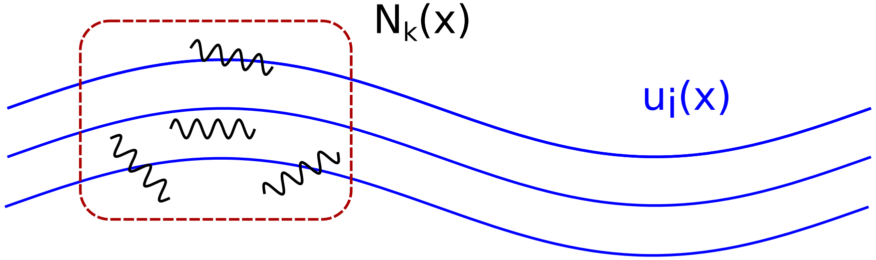
Chao, T.S. [2008.01269], see also Chen-Lin et al. [1811.12540] and Jain & Kovtun [2009.01356]

3. Fluctuations in an expanding fluid

Consider linearized stochastic dynamics about a fluid background.

Determine eigenmodes: two sound $\phi_\pm$, three diffusive modes $\phi_\psi, \phi_{\vec{\pi}_T}$.

Noise average: Consider equal time 2-point fct $W_{ab} = \langle \phi_a(\tau, x)\phi_b(\tau, x')\rangle$.



Wigner function representation: $W_{ab}(\tau, x, k)$. Diagonal component $N_{a,k}(\tau, x)$ is a phase space density of hydro fluctuations.

Akamatsu et al. (2016), Martinez, T.S. (2017).

Critical mode in expanding system

Study transit of critical point: Consider $\hat{s} = s/n$ and follow "mode coupling" philosophy. Use static susceptibility and critical relaxation rate $\Gamma_{\hat{s}}$.

$$\partial_t N_{\hat{s}}(t,k) = -2\Gamma_{\hat{s}}(t,k)\left[N_{\hat{s}}(t,k) - N^0_{\hat{s}}(t,k)\right] + \ldots,$$

$$\Gamma_{\hat{s}}(t,k) = \frac{\lambda_T}{C_p\xi^2}(k\xi)^2(1+(k\xi)^{2-\eta}), \qquad N^0_{\hat{s}}(t,k) = \frac{C_p(t)}{(1+(k\xi)^{2-\eta})},$$

Correlation length $\xi(t) = \xi(n(t), e(t)) = \xi_0 f_\xi(r(t), h(t))$

$$\text{hydro}: \ \frac{\partial_t n}{n} \sim \frac{\partial_t e}{e} \sim \frac{1}{\tau_{exp}} \qquad \text{Ising map}: \ (e,n) \to (r,h)$$

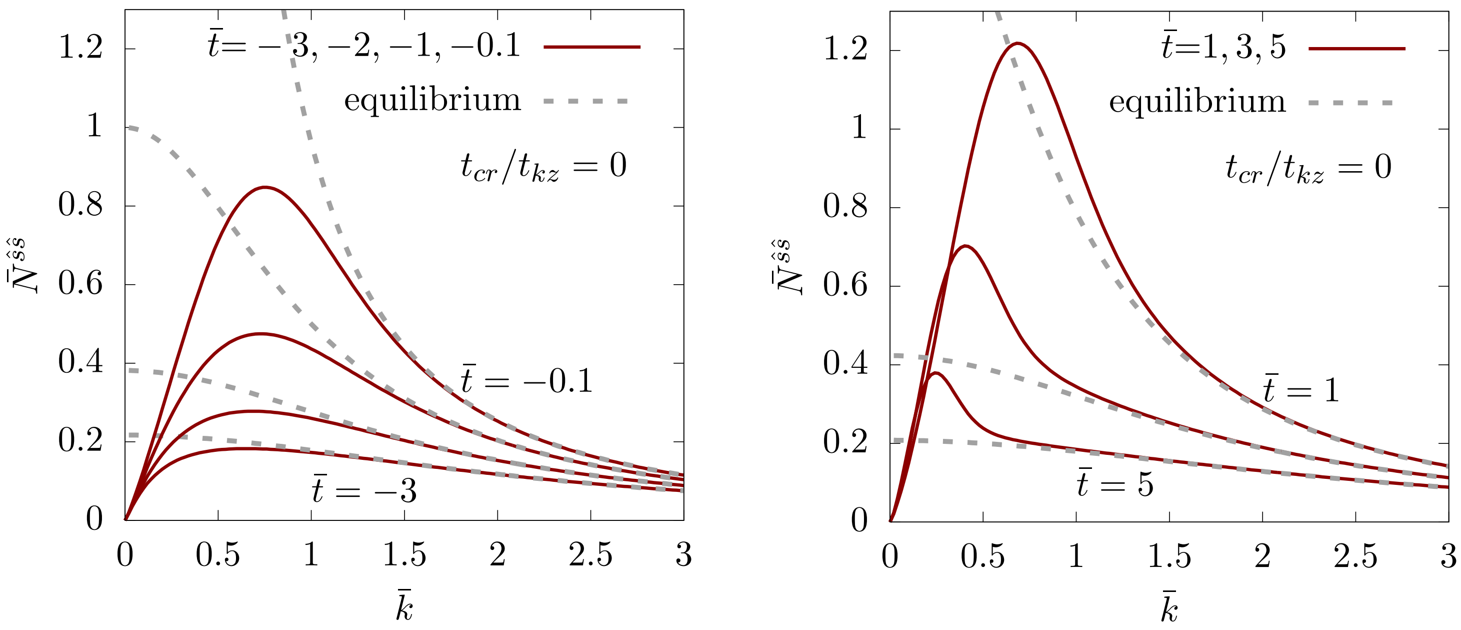
Emergent time scale t_{KZ}: Expansion rate matches relaxation time for modes with $k^* \sim \xi^{-1}$ (modes fall out of equilibrium).

Emergent length scale l_{KZ}: $l_{KZ} = \xi(t_{KZ})$. $l_{KZ} \sim 1.6$ fm

Akamatsu, Teaney, Yan, Yi [1811.05081], Martinez, Schaefer, Skokov [1906.11306], see also Berdnikov, Rajagopal [hep-ph/9912274]

Expanding System: Numerical Results

$$\bar{k} = k l_{KZ},\ \bar{t} = t/t_{KZ}$$



before CP　　　　　after CP

Akamatsu, Teaney, Yan, Yi [1811.05081]

4. Stochastic diffusion

Stochastic relaxation equation ("model A")

$$\partial_t \psi = -\Gamma \frac{\delta \mathcal{F}}{\delta \psi} + \zeta \qquad \langle \zeta(x,t)\zeta(x',t')\rangle = DT\delta(x-x')\delta(t-t')$$

Naive discretization

$$\psi(t+\Delta t) = \psi(t) + (\Delta t)\left[-D\frac{\delta \mathcal{F}}{\delta \psi} + \sqrt{\frac{DT}{(\Delta t)a^3}}\theta\right] \qquad \langle \theta^2 \rangle = 1$$

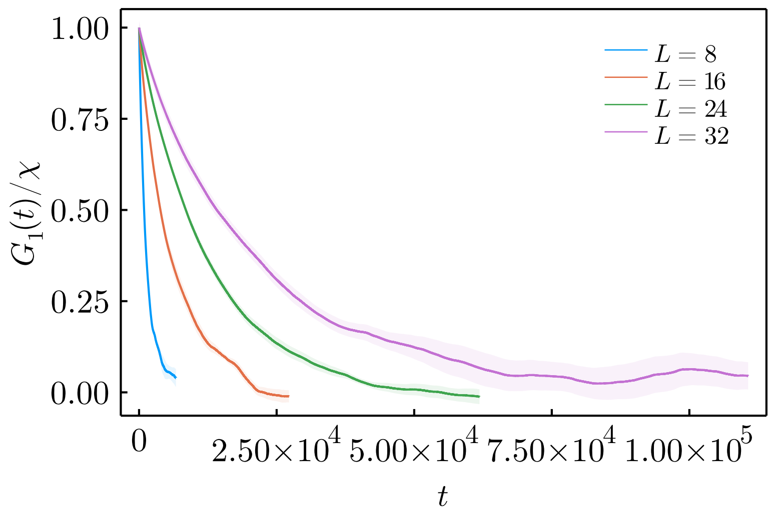
Noise dominates as $\Delta t \to 0$, leads to discretization ambiguities in the equilibrium distribution.

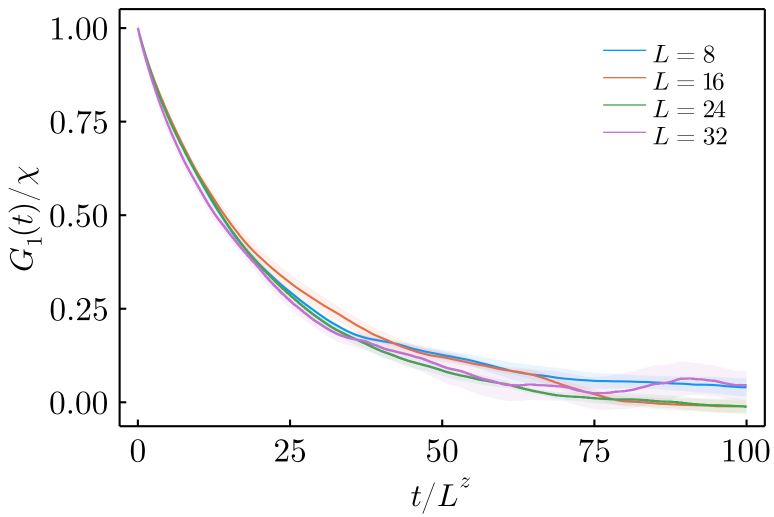
Idea: Add Metropolis step

$$\psi(t+\Delta t) = \psi(t) + \sqrt{2\Gamma(\Delta t)}\theta \qquad p = min(1, e^{-\beta \Delta \mathcal{F}})$$

Dynamic scaling (model A)

Correlation functions at T_c, $V = L^3$, $L = 8, 16, 24, 32$



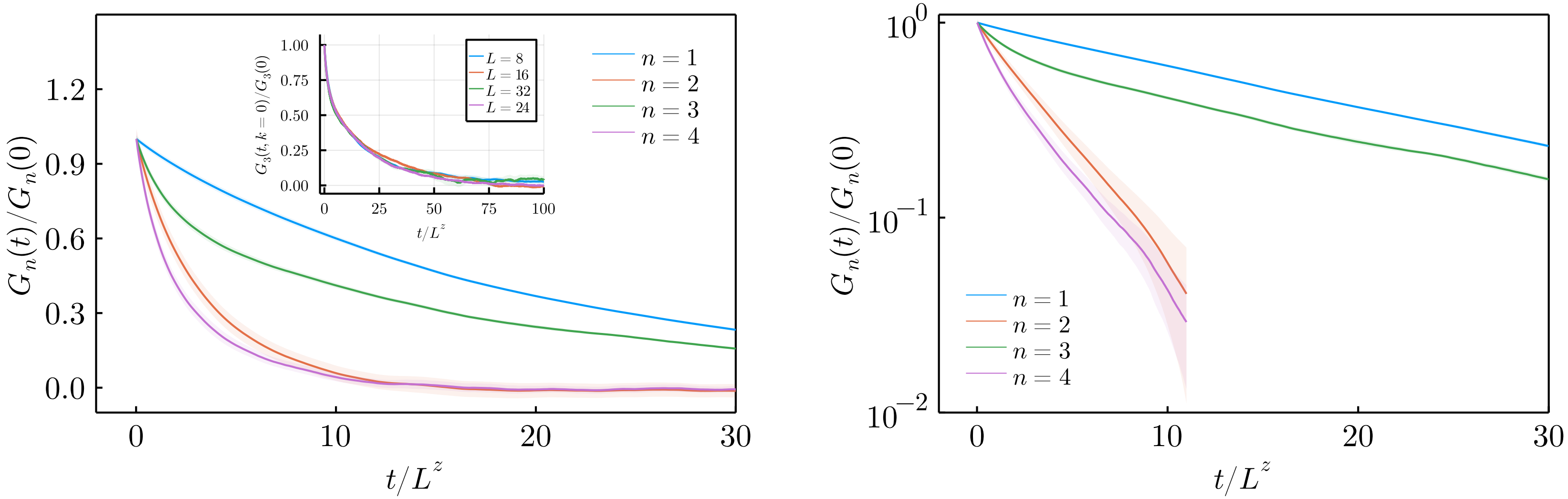


$$G_1(t) = \langle M(0)M(t)\rangle \qquad M(t) = \int d^3x\, \psi(x)$$

Dynamic critical exponent $z = 2.026(56)$.

Correlation functions of higher moments

Correlation functions at T_c

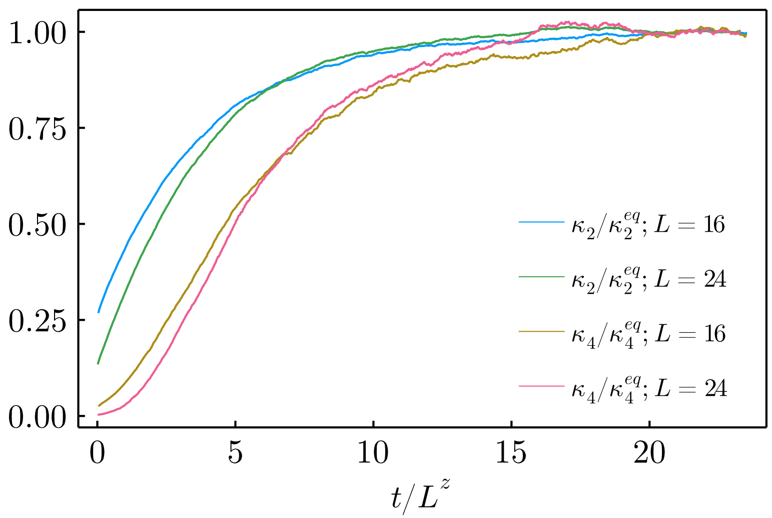


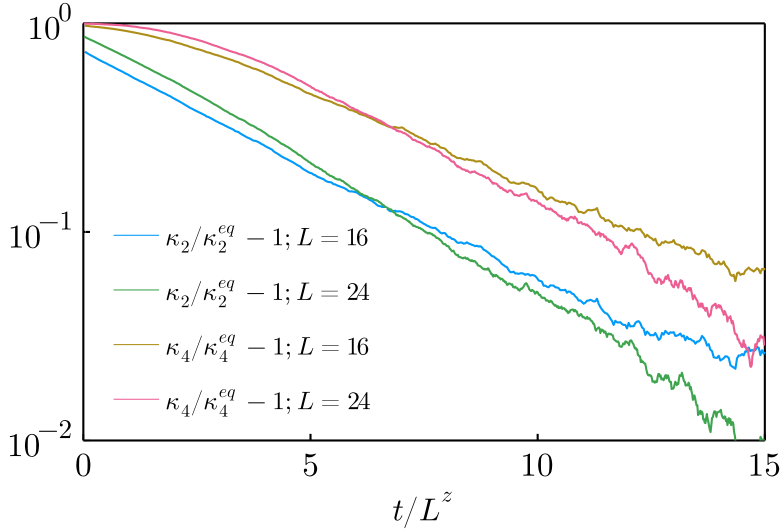
$$G_n(t) = \langle M^n(0) M^n(t) \rangle \qquad M(t) = \int d^3x\, \psi(x)$$

Inset: Dynamic scaling of $G_3(t)$ with $z = 2.026(56)$.

Relaxation after a quench

Thermalize at $T > T_c$. Study evolution at T_c





$$C_n(t) = \langle\langle M^n(t)\rangle\rangle_{M(0)} \qquad M(t) = \int d^3x\, \psi(x)$$

Summary

Dynamical evolution of fluctuations is important.

Model H dynamics in local rest frame: New parameters related to embedding of Ising model, and background correlation length. New results on bulk viscosity and multiplicative noise. New ideas about effective actions on the Keldysh contour.

Dynamics in evolving background: Two basic approaches, "stochastic" or "deterministic", each with their own advantages and disadvantages. Backreaction of fluctuations likely not important. Studies of $C_2(p_\perp, \eta)$ important.

Not discussed: From conserved charges to particles.