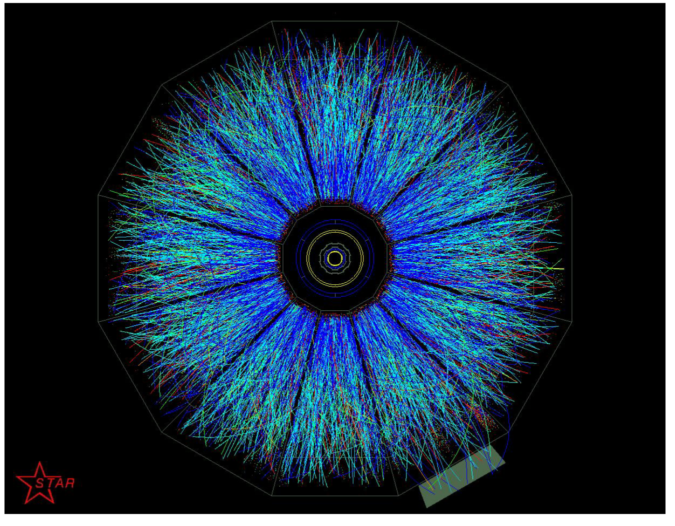
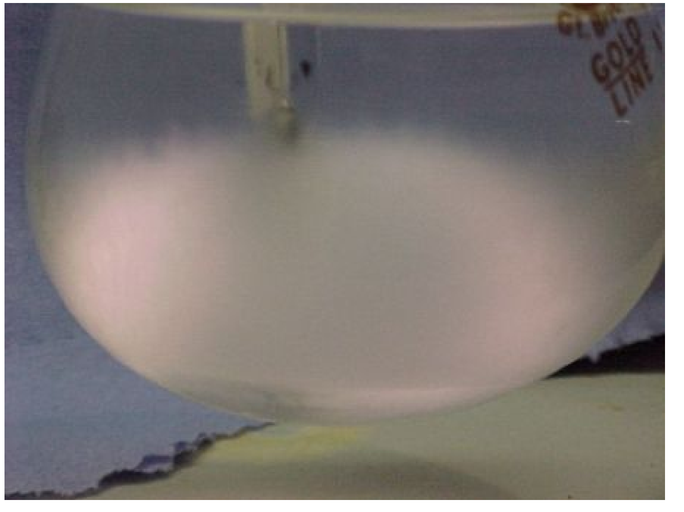
Fluctuations in Fluid Dynamics

Thomas Schäfer

North Carolina State University

BEST COLLABORATION





with S. Bass, M. Bluhm, M. Nahrgang, M. Martinez

Why consider fluctuations?

For consistency: Satisfy fluctuation-dissipation relations.

Fluid dynamics as an EFT: Fluctuations determine non-analyticities in (ω, k), and encode the resolution dependence of low energy parameters (such as transport coefficients).

Role of fluctuations enhanced in nearly perfect fluids $(\eta/s \lesssim 1)$.

Fluctuations are dominant near critical points.

Part I: Non-relativistic Fluids

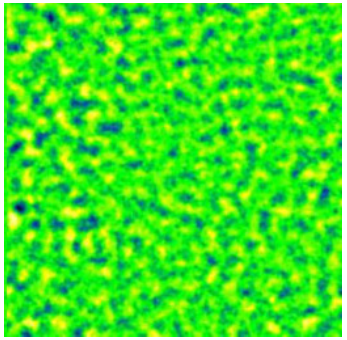
Main application: Ultracold Fermi Gases

The unitary Fermi gas is a scale invariant, strongly interacting, non-relativistic fluid ($(\eta/s)_{min} < \hbar/k_B$). Can detune from unitarity to study scale breaking, and tune temperature to study classical to quantum transition, including transition to superfluid.

Beyond gradients: Hydrodynamic fluctuations

Hydrodynamic variables fluctuate

$$\langle \delta v_i(x,t)\delta v_j(x',t)\rangle = \frac{T}{\rho}\delta_{ij}\delta(x-x')$$



Linearized hydrodynamics propagates fluctuations as shear or sound

$$\langle \delta v_i^T \delta v_j^T\rangle_{\omega,k} = \frac{2T}{\rho}(\delta_{ij} - \hat{k}_i\hat{k}_j)\frac{\nu k^2}{\omega^2 + (\nu k^2)^2} \qquad \textit{shear}$$

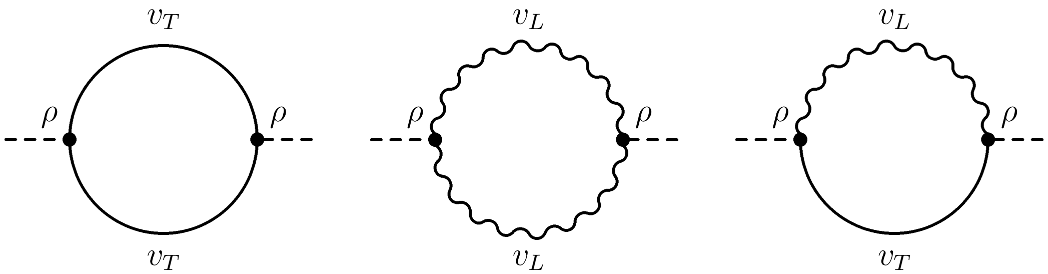
$$\langle \delta v_i^L \delta v_j^L\rangle_{\omega,k} = \frac{2T}{\rho}\hat{k}_i\hat{k}_j\frac{\omega k^2\Gamma}{(\omega^2 - c_s^2k^2)^2 + (\omega k^2\Gamma)^2} \qquad \textit{sound}$$

$v = v_T + v_L$: $\nabla \cdot v_T = 0, \nabla \times v_L = 0$ $\qquad$ $\nu = \eta/\rho, \quad \Gamma = \frac{4}{3}\nu + \ldots$

Hydro Loops: "Breakdown" of second order hydro

Correlation function in hydrodynamics

$$G_S^{xyxy} = \langle\{\Pi^{xy}, \Pi^{xy}\}\rangle_{\omega,k} \simeq \rho_0^2 \langle\{v_x v_y, v_x v_y\}\rangle_{\omega,k}$$



Match to response function in $\omega \to 0$ (Kubo) limit

$$G_R^{xyxy} = P + \delta P - i\omega[\eta + \delta\eta] + \omega^2 \left[\eta\tau_\pi + \delta(\eta\tau_\pi)\right]$$

with

$$\delta P \sim T\Lambda^3 \qquad \delta\eta \sim \frac{T\rho\Lambda}{\eta} \qquad \delta(\eta\tau_\pi) \sim \frac{1}{\sqrt{\omega}} \frac{T\rho^{3/2}}{\eta^{3/2}}$$

Hydro Loops: RG and "breakdown" of 2nd order hydro

Cutoff dependence can be absorbed into bare parameters. Non-analytic terms are cutoff independent.

Fluid dynamics is a "renormalizable" effective theory.

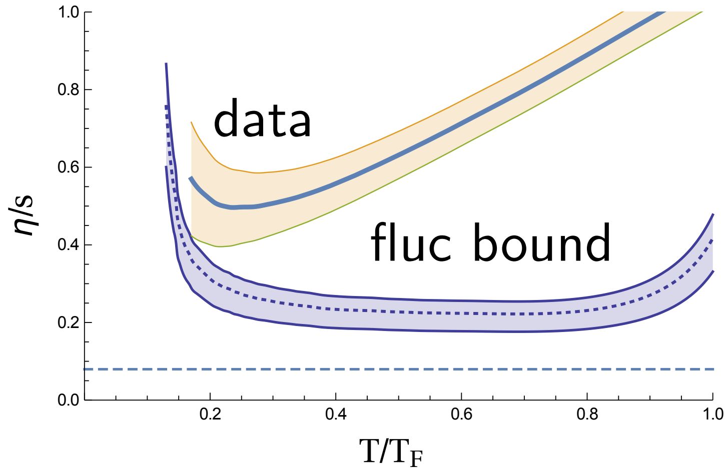
Small η enhances fluctuation corrections: $\delta\eta \sim T \left(\frac{\rho}{\eta}\right)^2 \left(\frac{P}{\rho}\right)^{1/2}$

Small η leads to large $\delta\eta$: There must be a bound on η/n.

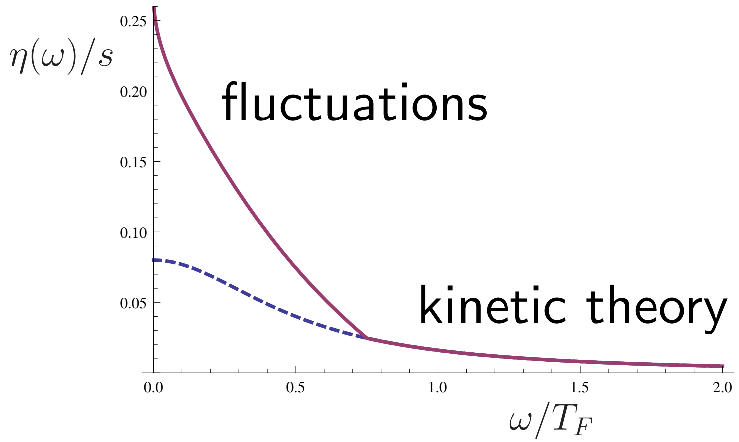
Relaxation time diverges: $\delta(\eta\tau_\pi) \sim \frac{1}{\sqrt{\omega}} \left(\frac{\rho}{\eta}\right)^{3/2}$

2nd order hydro without fluctuations inconsistent.

Fluctuation induced bound on η/s



$(\eta/s)_{min} \simeq 0.2$



spectral function

non-analytic $\sqrt{\omega}$ term

Schaefer, Chafin (2012), see also Kovtun, Moore, Romatschke (2011)

Fluctuation induced bulk stresses

Kubo relation for bulk viscosity

$$\zeta = -\lim_{\omega\to 0} \mathrm{Im}\frac{1}{9\omega}\int dtd^3x\, e^{-i\omega t}\,\langle[\Pi_{ii}(t,x),\Pi_{jj}(0)]\Theta(t)\rangle$$

Scale invariance not manifest

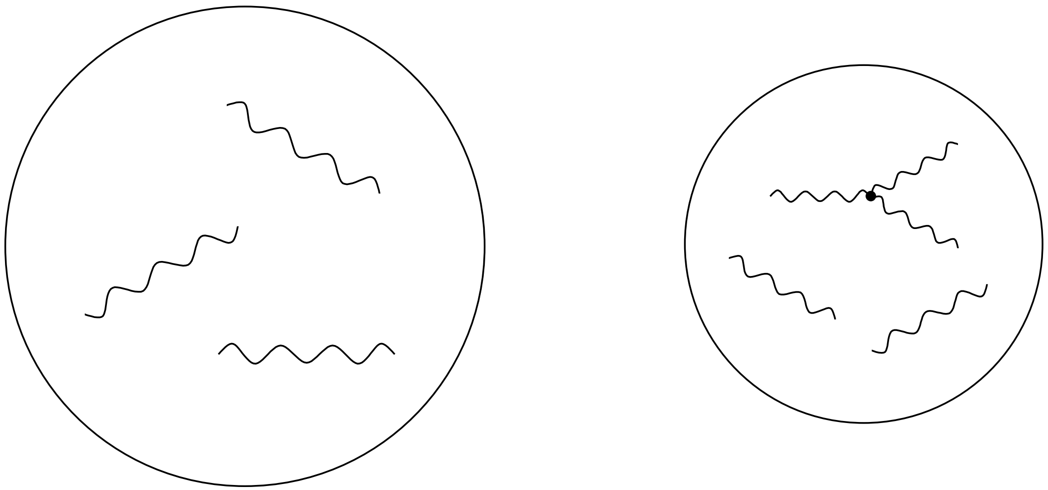
May use conservation of energy $\partial_t\mathcal{E} + \vec{\nabla}\cdot\vec{\jmath}^{\,\epsilon} = 0$ to rewrite Kubo formula

$$\zeta = -\lim_{\omega\to 0} \mathrm{Im}\frac{1}{\omega}\langle[\mathcal{O}(t,x),\mathcal{O}(0)]\rangle_{\omega k} \qquad\qquad \mathcal{O} = \frac{1}{3}\Pi_{ii} - \frac{2}{3}\mathcal{E}$$

and consider coupling to fluctuations of ρ and T

$$\mathcal{O} = \mathcal{O}_0 + a_{\rho\rho}(\Delta\rho)^2 + a_{\rho T}\Delta\rho\Delta T + a_{TT}(\Delta T)^2 + \ldots\,.$$

Fluctuation induced bulk stresses



Fluctuation contribution to bulk spectral function ($A_i \sim (P - \frac{2}{3}\mathcal{E})^2$):

$$\zeta(\omega) = \zeta(0) - \left(\frac{A_T}{(2D_T)^{3/2}} + \frac{A_\Gamma}{\Gamma^{3/2}} \right) \frac{\sqrt{\omega}}{36\sqrt{2}\pi} \, .$$

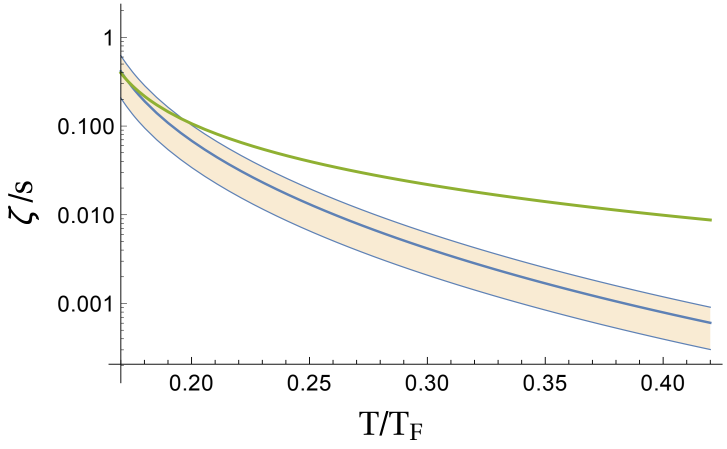
Fluctuation bound

$$\zeta_{min} = \left(\frac{A_T}{2D_T^2} + \frac{\sqrt{5}A_\Gamma}{\sqrt{3}\Gamma^2} \right) \sqrt{\frac{T}{m}} \, .$$

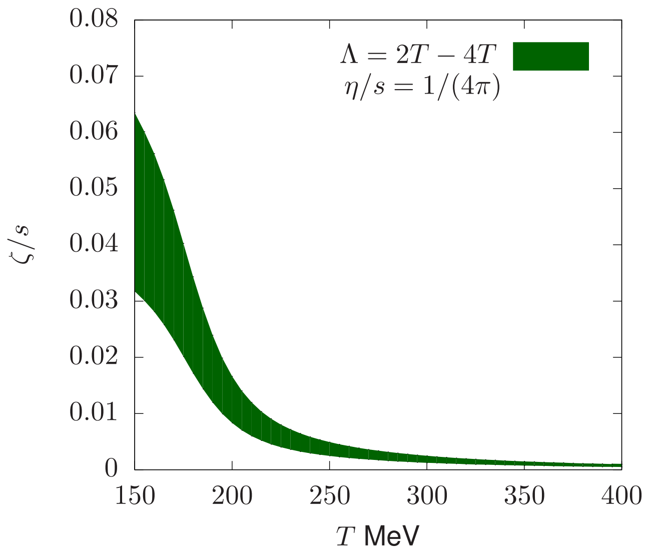
Consider $\lambda/a \sim 1$. Get $\zeta/s \gtrsim 0.1$

Fluctuation induced bound on ζ/s

(Detuned) Unitary Fermi Gas



Quark Gluon Plasma



Non-relativistic fluid, M. Martinez, T. S. (2017); relativistic fluid, Akamatsu et al. (2018)

See also Kovtun, Yaffe (2003)

Digression: Diffusion

Consider a Brownian particle

$$\dot{p}(t) = -\gamma_D p(t) + \zeta(t) \qquad \langle \zeta(t)\zeta(t') \rangle = \kappa\delta(t - t')$$

drag (dissipation) white noise (fluctuations)

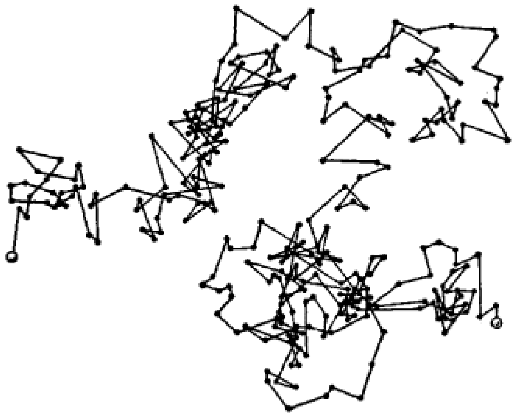
For the particle to eventually thermalize

$$\langle p^2 \rangle = 2mT$$

drag and noise must be related

$$\kappa = \frac{mT}{\gamma_D}$$

Einstein (Fluctuation-Dissipation)



Hydrodynamic equation for critical mode

Equation of motion for critical mode ϕ ("model H")

$$\frac{\partial \phi}{\partial t} = D\nabla^2 \frac{\delta \mathcal{F}}{\delta \phi} - g\vec{\nabla}\phi \cdot \frac{\delta \mathcal{F}_T}{\delta \vec{\pi}} + \zeta_\phi$$

Diffusive Reactive White Noise

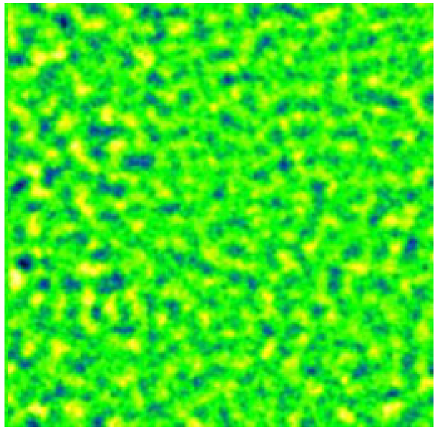
Free energy functional: Order parameter ϕ, momentum density $\vec{\pi} = \rho\vec{v}$

$$\mathcal{F} = \int d^d x \left[\frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{r}{2}\phi^2 + \lambda\phi^4 + \frac{1}{2}\vec{\pi}^2\right]$$

Fluctuation-Dissipation relation

$$\langle \zeta_\phi(x,t)\zeta_\phi(x',t')\rangle = 2DT\delta(x-x')\delta(t-t')$$

ensures $P[\phi] \sim \exp(-\mathcal{F}[\phi]/T)$

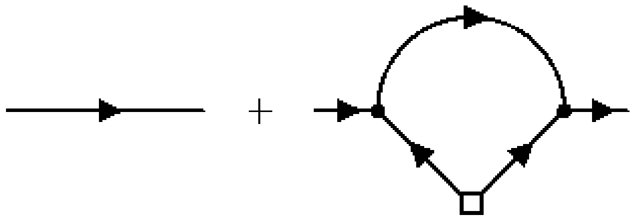


Linearized analysis (non-critical fluid)

Navier-Stokes equation: $\partial_0 \vec{v} + \nu \nabla^2 \vec{v} = mode\ couplings + noise$

Linearized propagator: $\langle \delta v_i^T \delta v_j^T \rangle_{\omega,k} = \frac{1}{\rho} \frac{-\nu k^2 P_{ij}^T}{-i\omega + \nu k^2}$ $\qquad \nu = \frac{\eta}{\rho}$

Fluctuation correction:



Renormalized viscosity:

$$\eta = \eta_0 + c_\eta \frac{T\rho\Lambda}{\eta_0} - c_\tau \sqrt{\omega} \frac{T\rho^{3/2}}{\eta_0^{3/2}}$$

Hydro is a renormalizable stochastic field theory

Linearized analysis (critical fluid)

Consider order parameter mode

$$\partial_0 \phi = -D\nabla^2 \frac{\delta \mathcal{F}}{\delta \phi} + mode\ couplings + \zeta_\phi$$

$$\mathcal{F} = \int d^3x \left\{ \frac{1}{2}(\nabla\phi)^2 + \frac{r}{2}\phi^2 + \lambda\phi^4 + \frac{1}{2}\vec{\pi}^2 \right\}$$

Dispersion relation $i\omega = Dq^2(r+q^2) + \ldots$

Use $r \sim \xi^{-2}$. Relaxation time for modes $q \sim \xi^{-1}$:

$$\tau \sim \xi^z \quad (z=4) \qquad\qquad \text{"Critical slowing down"}$$

A more sophisticated analysis gives $z \simeq 3$ and

$$\eta \sim \xi^{0.05} \qquad\qquad \kappa \sim \xi^{0.9} \qquad\qquad \zeta \sim \xi^{2.8}$$

Part II: Relativistic Fluids

Main application: QGP at RHIC

Expectation: If there is a critical endpoint in the QCD phase diagram, then the dynamical universality class is that of model H (liquid-gas).
Consider simplifications: Purely diffusive dynamics of order parameter mode (model B), or coupled dynamics away from critical points, and truncated at second order moments.

Numerical Simulation: Stochastic Diffusion

Stochastic diffusion equation

$$\partial_t n_B(x,t) = \Gamma \nabla^2 \left(\frac{\delta \mathcal{F}}{\delta n_B} \right) + \nabla \cdot J(x,t)$$

$$\vec{J}(x,t) = \sqrt{2T\Gamma}\,\vec{\zeta}(x,t) \qquad \langle \zeta_i(x,t)\zeta_j(x',t')\rangle = \delta(x-x')\delta(t-t')\delta_{ij}$$

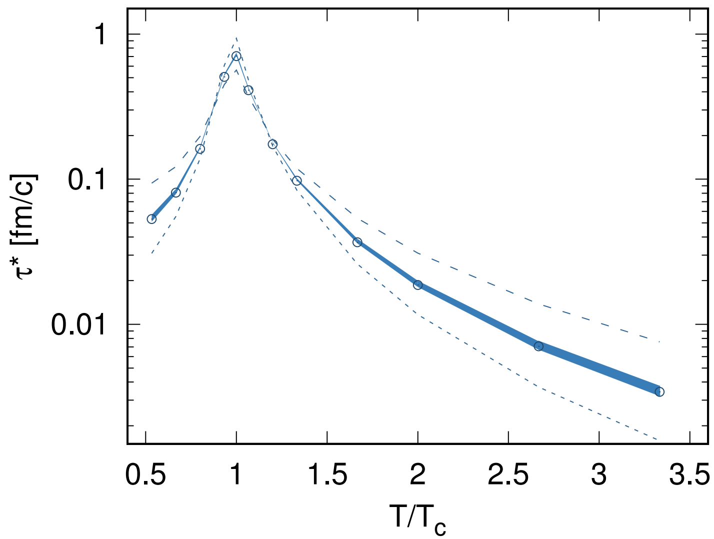
Free energy functional

$$\mathcal{F}[n_B] = T \int d^3x \left(\frac{m^2}{2n_c^2} (\Delta n_B)^2 + \frac{K}{2n_c^2} (\nabla n_B)^2 \right.$$

$$\left. + \frac{\lambda_3}{3n_c^3} (\Delta n_B)^3 + \frac{\lambda_4}{4n_c^4} (\Delta n_B)^4 + \frac{\lambda_6}{6n_c^6} (\Delta n_B)^6 \right)$$

Scale $m^2 \sim \xi^{-2}$, $\lambda_3 \sim \xi^{-3/2}$ etc., parameterize $\xi(t)$ with $t = \frac{T-T_c}{T_c}$.

Numerical results (diffusion in expanding critical fluid)



Dynamical scaling: Consider correlation function

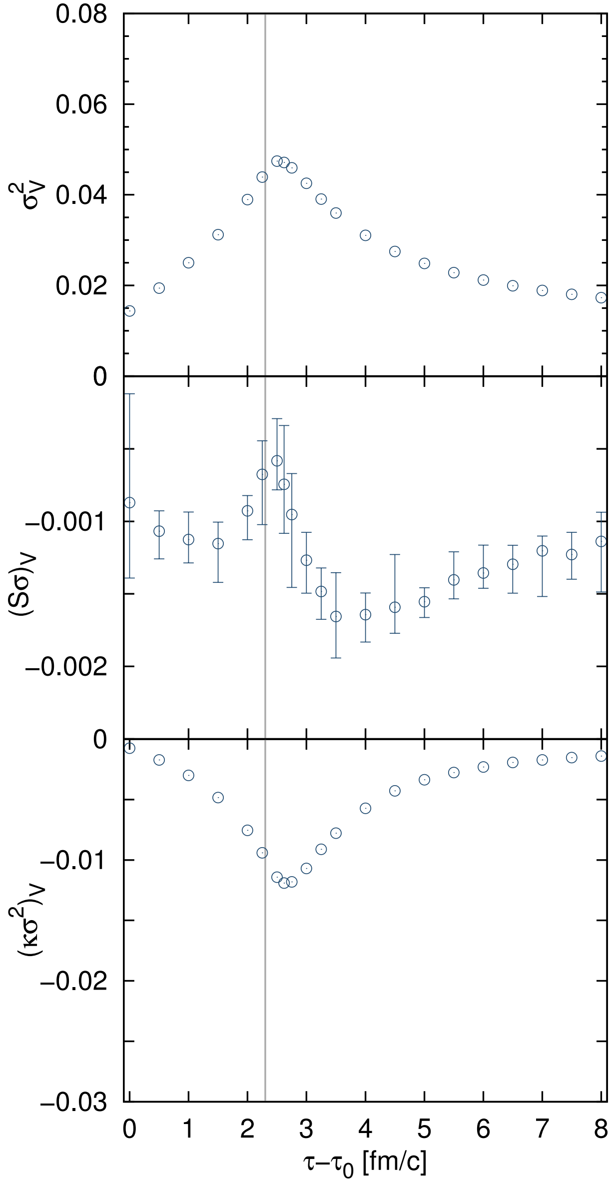
$$C_2(t) = \langle \Delta n_B(k,0) \Delta n_B(-k,t) \rangle \text{ for } k = k^* \sim \xi^{-1}$$

Determine decay rate $C_2(t) \sim \exp(-t/\tau^*)$.

Blue line: Expectation for $z = 4$.

M. Nahrgang et al. (2018)

Numerical results
(diffusion in expanding critical fluid)



Variance

Skewness

Kurtosis

M. Nahrgang et al. (2018)

Analytic study: Hydro tails in Bjorken geometry

Consider linearized stochastic dynamics about some fluid backround (Bj)

Determine eigenmodes: two sound $\phi_\pm$, three diffusive modes ϕ_d, ϕ_{T_i}.

Deterministic equation for 2-point functions C_{ab} after noise average.

$$\partial_0 C + [\mathcal{A}, C] + \{\mathcal{D}, C\} = \mathcal{P}C + C\mathcal{P}^\dagger + \mathcal{N}$$

evolution+reactive + diffusive = sources + noise-correlator

Mixed representation: $C_{ab}(\tau, \vec{k})$. Local quantities after momentum integration.

Contain divergences, can be renormalized by subtraction in homogeneous system.

Akamatsu et al. (2016)

Homogeneous System

Coupled equation for two-pont function of hydro modes

$$\partial_0 C_{\pm\pm} + \frac{4}{3}\gamma\, k^2\, C_{\pm\pm} = \mathcal{P}_{\pm\pm} C_{\pm\pm} + \mathcal{N}_{\pm\pm}$$

$$\partial_0 C_{T_l T_l} + 2\,\gamma\, k^2\, C_{T_l T_l} = \mathcal{P}_{T_l T_l} C_{T_l T_l} + \mathcal{N}_{T_l T_l}$$

$$\partial_0 C_{dd} + 2D\, k^2\, C_{dd} = \mathcal{N}_{dd}$$

$$\partial_0 C_{dT_l} + (\gamma + D)\, k^2\, C_{dT_l} = \mathcal{P}_{dT_l} C_{T_l T_l} + \mathcal{P}_{T_l d} C_{dd} + \mathcal{P}_{T_l T_l} C_{dT_l}$$

Off-diagonal couplings important for diffusive tails

$$G_R(\omega) = -i\omega[\sigma + \delta\sigma] - (1+i)\omega^{3/2}\,\frac{\chi T}{(D+\gamma)^{3/2}}$$

M. Martinez, T. S. (2018)

Expanding System

Coupled equations in Bj geometry

$$\partial_\tau C_{\pm\pm} \; + \; \frac{4}{3}\gamma\, k^2\, C_{\pm\pm} = -\frac{2 + c_s^2 + \cos^2\theta_K}{\tau} C_{\pm\pm} \mp \frac{\hat{K}\cdot E}{\bar{w}}\frac{1+c_s^2}{c_s^2} C_{\pm\pm} + \mathcal{N}_{\pm\pm}$$

$$\partial_\tau C_{T_1T_1} \; + \; 2\,\gamma\, k^2\, C_{T_1T_1} = -\frac{2}{\tau} C_{T_1T_1} + \mathcal{N}_{T_1T_1}$$

$$\partial_\tau C_{dd} \; + \; 2D\, k^2\, C_{dd} = -\frac{2}{\tau} C_{dd} + \mathcal{N}_{dd}$$

$$\partial_\tau C_{dT_1} \; + \; (\gamma + D)\, k^2\, C_{dT_1} = -frac2\tau C_{T_1T_1} + \frac{1}{\bar{w}}\hat{e}_{T_1}\cdot E(C_{dd} - c_s C_{T_1T_1}$$

Asymptotic solution (as $\gamma k^2 \tau \gg 1$)

$$C^{as}_{dT_1} = \frac{\chi}{\bar{w}}\frac{1}{(D+\gamma)K^2\tau}\hat{e}_{T_1}\cdot E$$

renormalizes $\sigma \to \sigma + \delta\sigma$. $C - C^{as}$ generates hydro tails in expanding fluid.

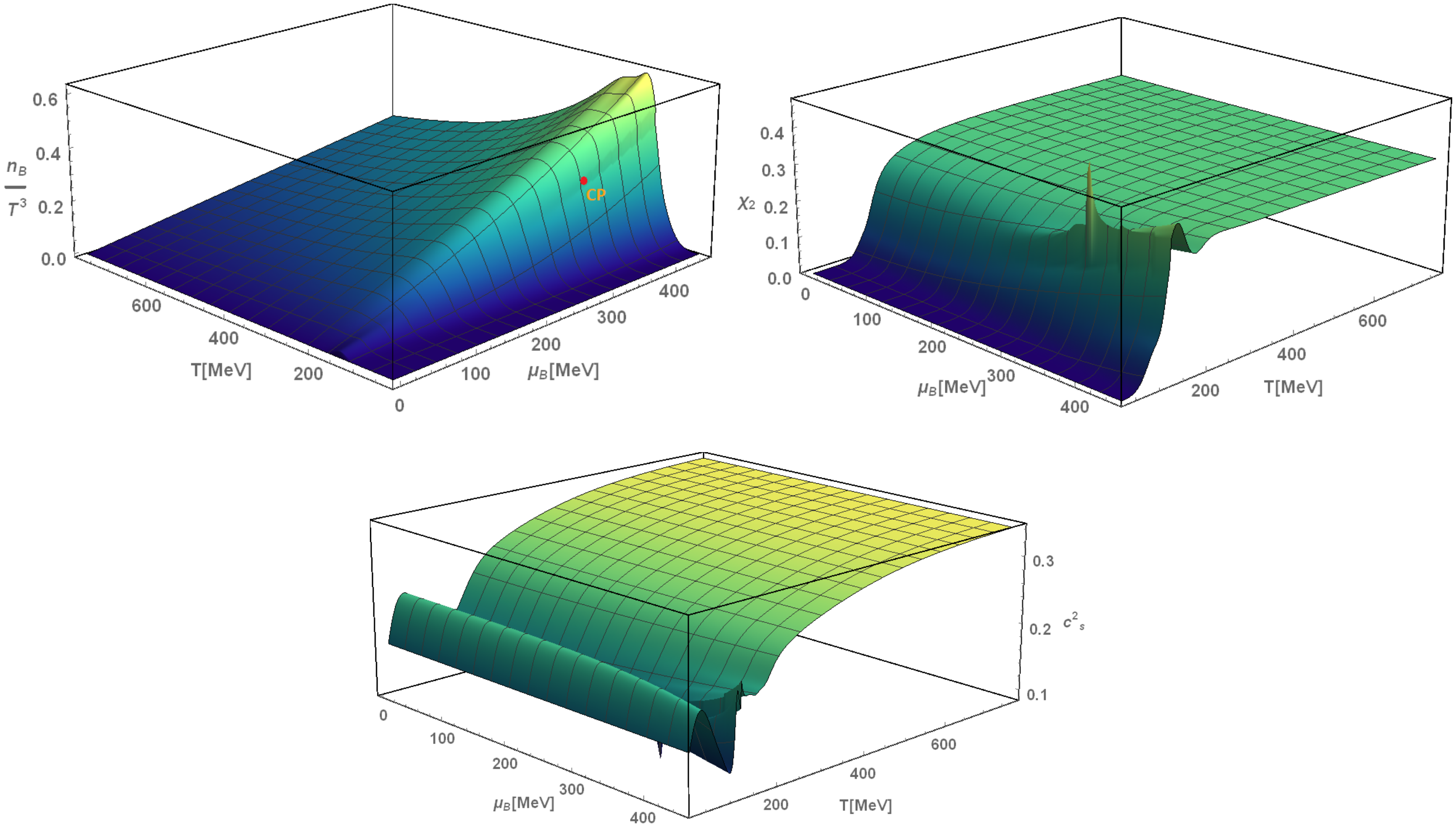
Summary

Obtain higher order cumulants from Gaussian noise and mode couplings.

Find significant finite size effects in correlation length and higher order cumulants.

Full 3d simulations in progress. Finalizing analytic work on two-point correlators.

Outlook



Parotto et al. (2018)