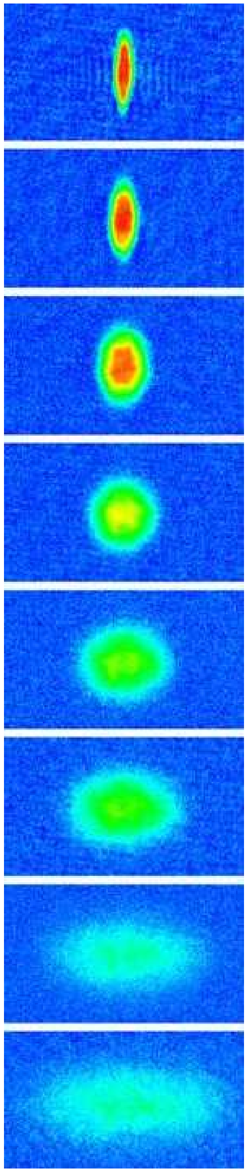
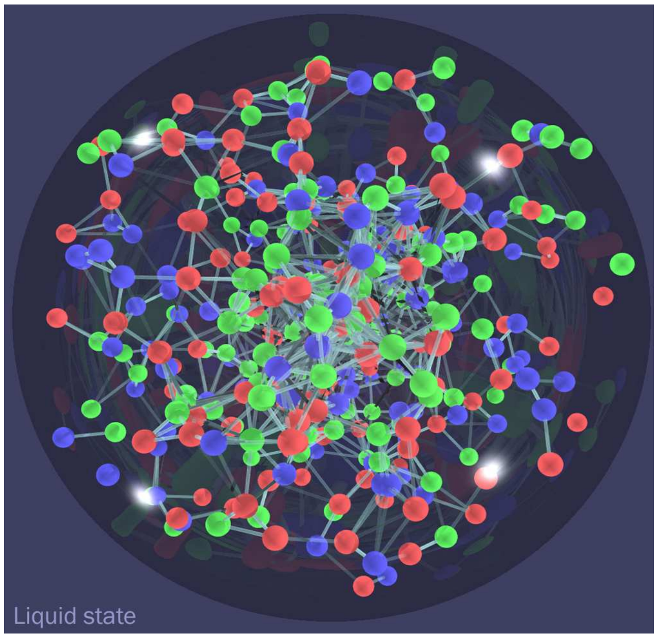
Effective Field Theory for Non-Relativistic Fermions in the Unitary Limit

Thomas Schaefer

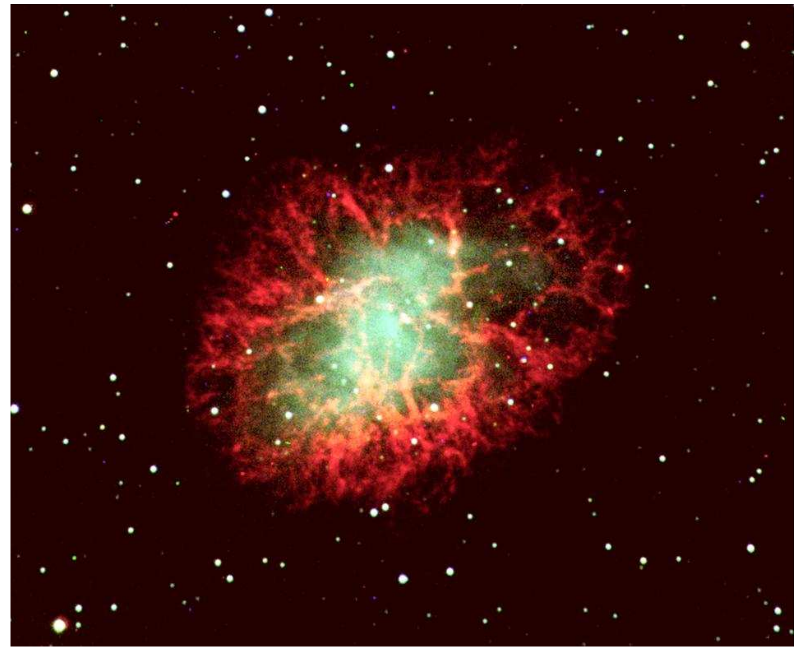
North Carolina State University

Perfect Liquids





sQGP (T=180 MeV)



Neutron Matter (T=1 MeV)

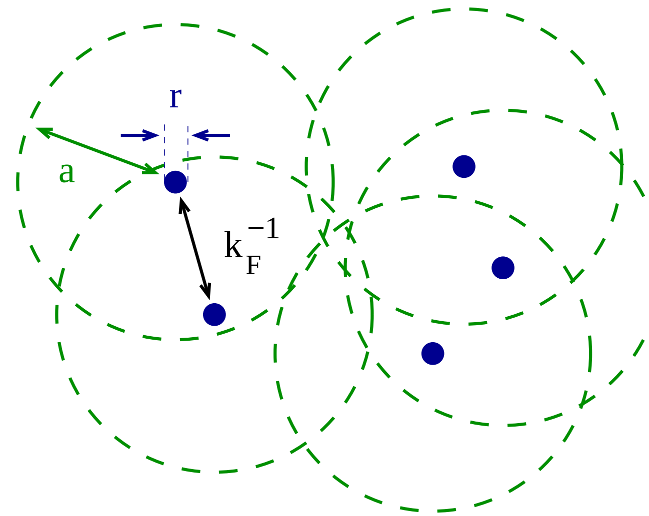
Trapped Atoms (T=5 peV)

Universality

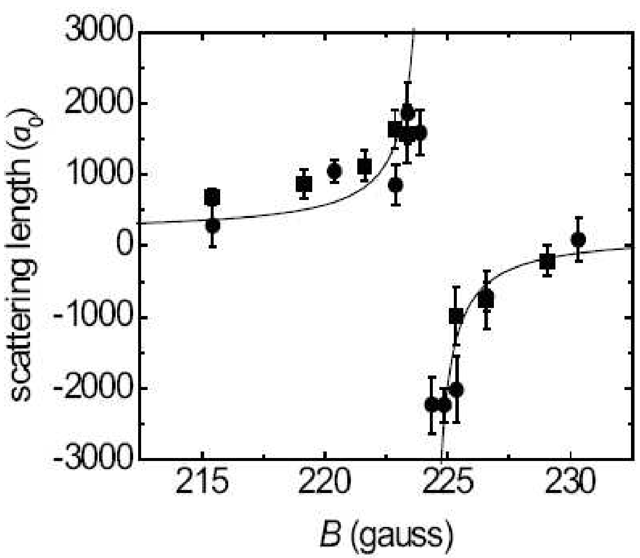
What do these systems have in common?

dilute: $r\rho^{1/3} \ll 1$

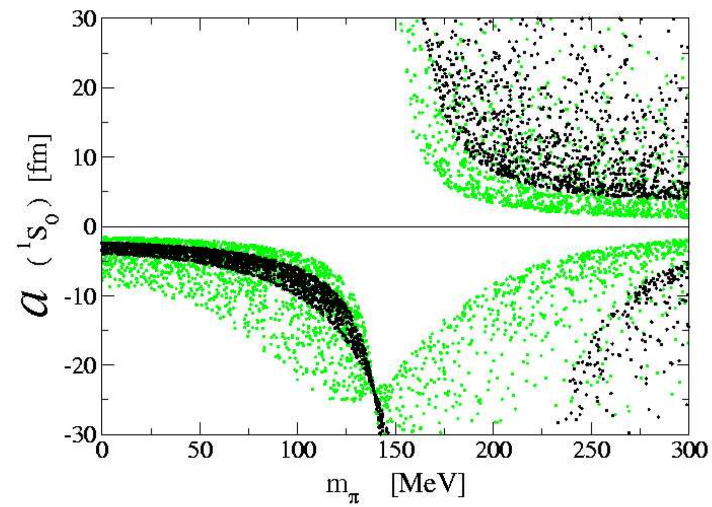
strongly correlated: $a\rho^{1/3} \gg 1$



Feshbach Resonance in ^{6}Li



Neutron Matter



Universal Equation of State

Consider limiting case ("Bertsch" problem)

$$(k_F a) \to \infty \qquad\qquad (k_F r) \to 0$$

Universal equation of state

$$\frac{E}{A} = \xi \left(\frac{E}{A}\right)_0 = \xi \frac{3}{5} \left(\frac{k_F^2}{2M}\right)$$

How to find ξ?

Numerical Simulations

Experiments with trapped fermions

Analytic Approaches

Effective Field Theory

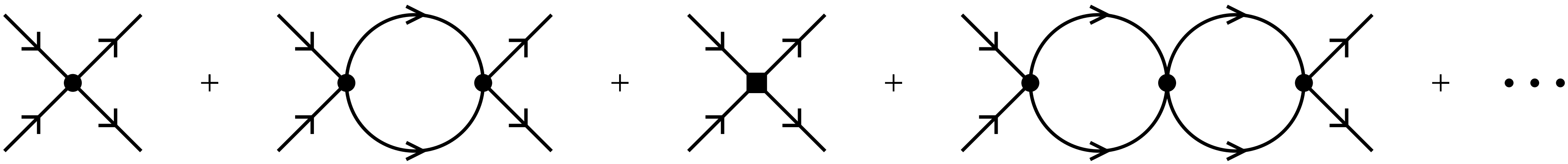
Effective field theory for pointlike, non-relativistic neutrons

$$\mathcal{L}_{\text{eff}} = \psi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2M} \right) \psi - \frac{C_0}{2} (\psi^\dagger \psi)^2 + \frac{C_2}{16} \left[(\psi\psi)^\dagger (\psi \overleftrightarrow{\nabla}^2 \psi) + h.c. \right] + \ldots$$

Scattering amplitude

$$\mathcal{A}_l = \frac{1}{p \cot \delta_l - ip} \qquad\qquad p \cot \delta_0 = -\frac{1}{a} + \frac{1}{2}\Lambda^2 \sum_n r_n \left(\frac{p^2}{\Lambda^2} \right)^{n+1}$$

Low energy expansion (natural case)



$$\mathcal{A} = -\frac{4\pi a}{M} \left[1 - iap + (ar_0/2 - a^2)p^2 + O(p^3/\Lambda^3) \right]$$

Modified Expansion

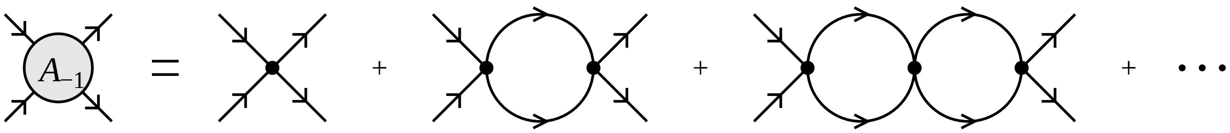
Coupling constants determined by nn interaction

$$C_0 = \frac{4\pi a}{M} \qquad a = -18 \text{ fm} \qquad\qquad C_2 = \frac{4\pi a^2}{M}\frac{r}{2} \qquad r = 2.8 \text{ fm}$$

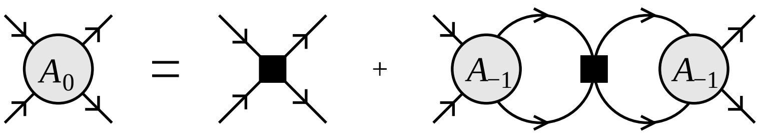
Problem: Large scattering length

$$(ap) \ll 1 \qquad\qquad p \ll 10 \text{ MeV}$$

Need to sum (ap) to all orders. Small parameter $Q \sim (a^{-1}, p, \ldots)$



$$\mathcal{A}_{-1} = -\frac{4\pi}{M}\frac{1}{\frac{1}{a} + ip}$$



$$\mathcal{A}_0 = -\frac{4\pi}{M}\frac{r_0 p^2/2}{(\frac{1}{a} + ip)^2}$$

Power Counting Made Manifest: PDS Scheme

Dimensional regularization in PDS scheme

$$\int \frac{d^3q}{(2\pi)^3} \frac{M}{k^2 - q^2 + i\epsilon} = -\frac{M}{4\pi}(\mu + ik)$$

Low energy constants

$$C_0 = -\frac{4\pi/M}{\mu - 1/a} \sim \frac{1}{Q} \qquad C_2\, k^2 = \frac{4\pi/M}{(\mu - 1/a)^2} \frac{r}{2}\, k^2 \sim Q^0.$$

Scattering matrix

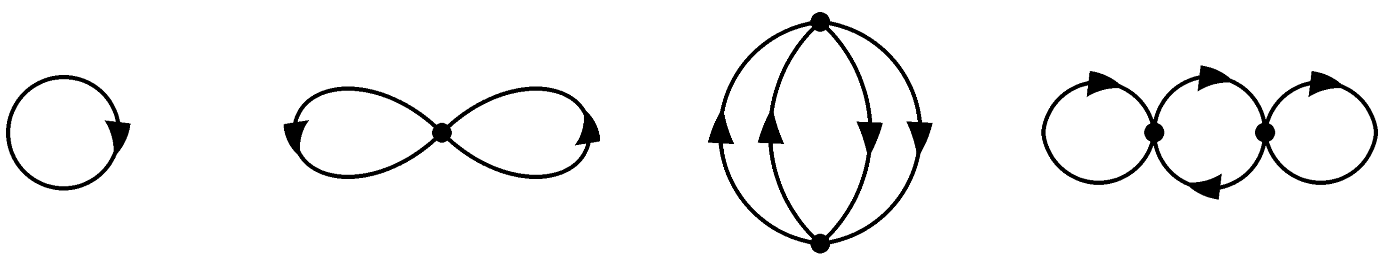
$$T(k) = \frac{C_0 + C_2 k^2}{1 - \frac{M}{4\pi}(\mu + ik)(C_0 + C_2 k^2)}.$$

Low Density Expansion

Finite density: $\mathcal{L} \to \mathcal{L} - \mu\psi^\dagger\psi \Rightarrow$ Modified propagator

$$G_0(k)_{\alpha\beta} = \delta_{\alpha\beta}\Big(\frac{\theta(k-k_F)}{k_0 - k^2/2M + i\epsilon} + \frac{\theta(k_F-k)}{k_0 - k^2/2M - i\epsilon}\Big) \qquad \frac{k_F^2}{2M} = \mu$$

Perturbative expansion

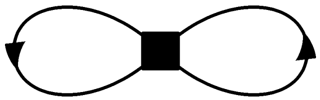


$\epsilon_F\rho$ $\qquad$ $\epsilon_F\rho\,(k_Fa)$ $\qquad$ $\epsilon_F\rho\,(k_Fa)^2$

$$\frac{E}{A} = \frac{k_F^2}{2M}\left[\frac{3}{5} + \left(\frac{2}{3\pi}(k_Fa) + \frac{4}{35\pi^2}(11 - 2\log(2))(k_Fa)^2\right) + \ldots\right]$$

Low Density Expansion: Higher orders

Effective range corrections

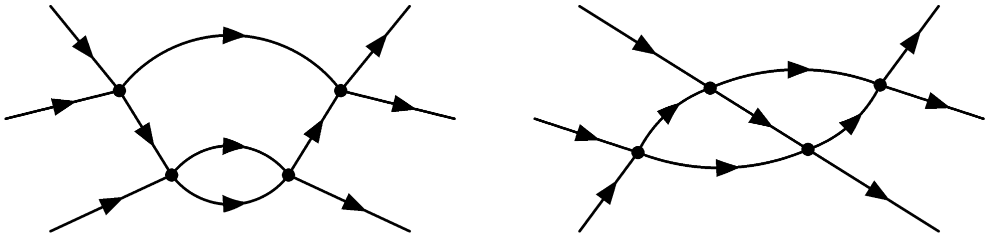


$$\frac{E}{A} = \frac{k_F^2}{2M}\frac{1}{10\pi}(k_F a)^2 (k_F r)$$

Logarithmic terms

$$\frac{E}{A} = \frac{k_F^2}{2M}(g-1)(g-2)\frac{16}{27\pi^3}(4\pi - 3\sqrt{3})(k_F a)^4 \log(k_F a)$$

related to log divergence in $3 \to 3$ scattering amplitude



local counterterm $D(\psi^\dagger\psi)^3$ exists if $g \geq 3$

Lattice Calculation

Free fermion action

$$S^{free} = \sum_{\vec{n},i} \left[e^{(m_N-\mu)\alpha_t} c_i^*(\vec{n}) c_i(\vec{n}+\hat{0}) - (1-6h) c_i^*(\vec{n}) c_i(\vec{n}) \right]$$
$$- h \sum_{\vec{n},l_s,i} \left[c_i^*(\vec{n}) c_i(\vec{n}+\hat{l}_s) + c_i^*(\vec{n}) c_i(\vec{n}-\hat{l}_s) \right]$$

Contact interaction: Hubbard-Stratonovich

$$\exp\left[-C\alpha_t a_\uparrow^\dagger a_\uparrow a_\downarrow^\dagger a_\downarrow\right] = \int \frac{ds}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}s^2\right)$$
$$\exp\left[\left(s\sqrt{-C\alpha} + \frac{C\alpha_t}{2}\right)(a_\uparrow^\dagger a_\uparrow + a_\downarrow^\dagger a_\downarrow)\right]$$

Path Integral

$$\mathrm{Tr}\exp\left[-\beta(H-\mu N)\right] = \int DsDcDc^* \exp\left[-S\right]$$

Lattice Fermions

Introduce pseudo fermions: $S = \psi_i^* Q_{ij} \psi_j + V(s)$

$$Z = \int DsD\phi D\phi^* \exp\left[-S'\right], \qquad S' = \phi_i^* Q_{ij}^{-1} \phi_j + V(s)$$

$C < 0$ (attractive): $\det(Q) \geq 0$

Hybrid Monte Carlo method

(4+1)-d Hamiltonian $\quad H(\phi, s, p) = \frac{1}{2} p_\alpha^2 + S'(\phi, s)$

Molecular Dynamics $\quad \dot{s}_\alpha = p_\alpha \qquad \dot{p}_\alpha = -\frac{\partial H}{\partial s_\alpha}$

Metropolis acc/rej $\quad P([s_\alpha, p_\alpha] \to [s'_\alpha, p'_\alpha]) = \exp(-\Delta H)$

Continuum Limit

Fix coupling constant at finite lattice spacing

$$\frac{M}{4\pi a} = \frac{1}{C_0} + \frac{1}{2}\sum_{\vec{p}} \frac{1}{E_{\vec{p}}}$$

Take lattice spacing b, b_τ to zero

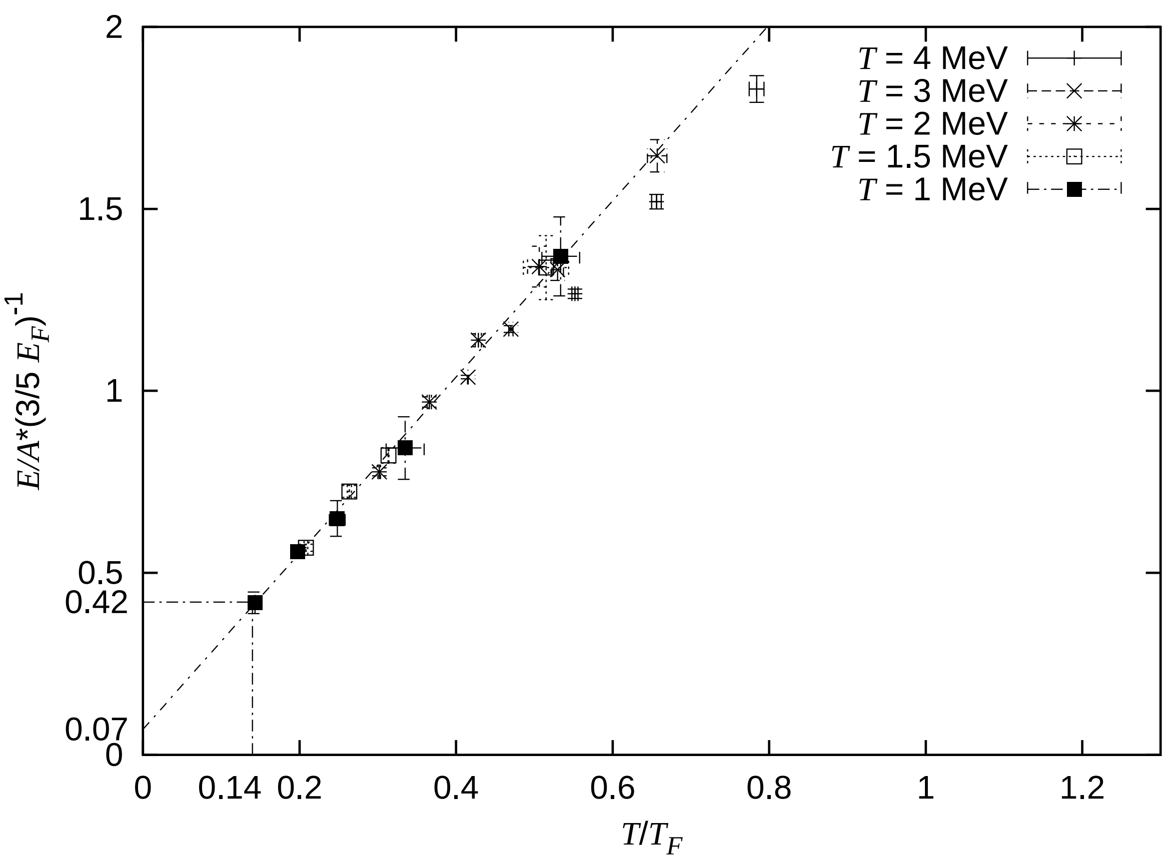
$$\mu b_\tau \to 0 \qquad n^{1/3} b \to 0 \qquad n^{1/3} a = const$$

Physical density fixed, lattice filling $\to 0$

Consider universal (unitary) limit

$$n^{1/3} a \to \infty$$

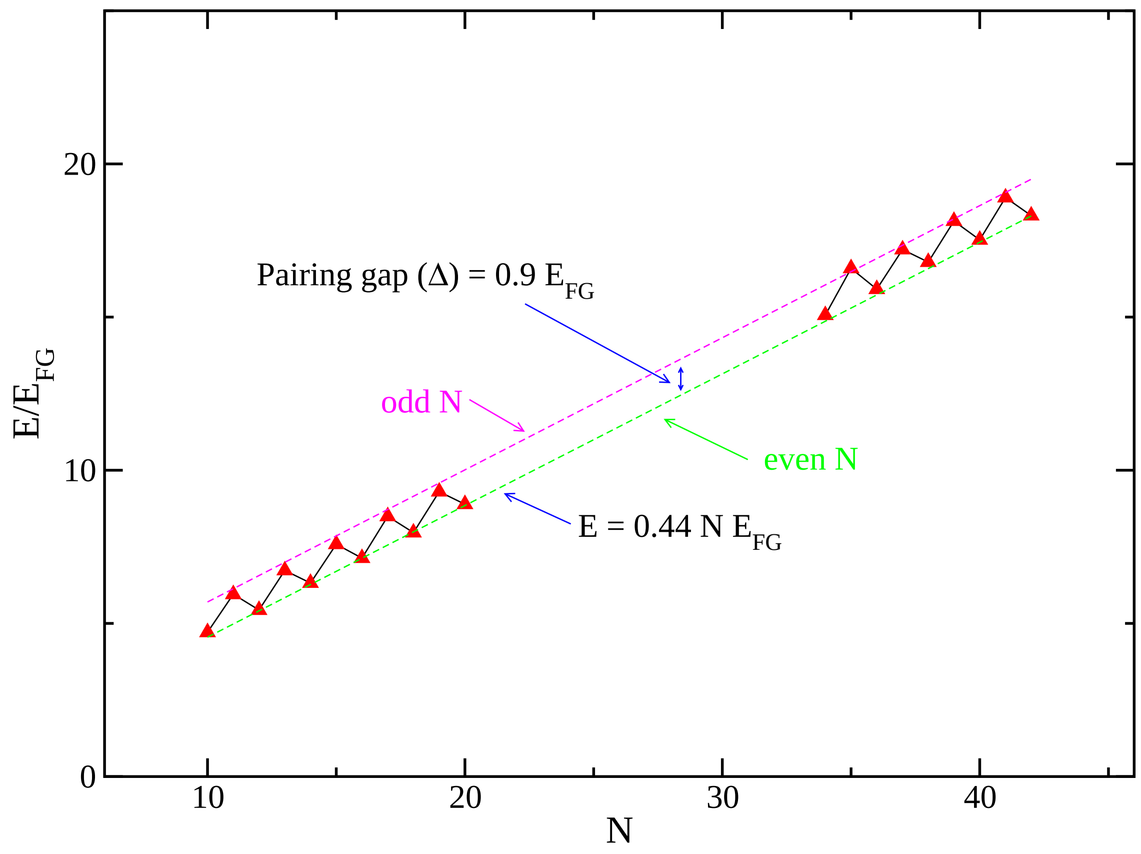
Lattice Results



Canonical $T = 0$ calculation: $\xi = 0.25(3)$ (D. Lee)

Not extrapolated to zero lattice spacing

Green Function Monte Carlo



Other lattice results: $\xi = 0.42$ (Bulgac et al. ,UMass)

Experiment: $\xi = 0.27^{+0.12}_{-0.09}\,[1],\ 0.51 \pm 0.04\,[2],\ 0.74 \pm 0.07\,[3]$

[1] Bartenstein et al., [2] Kinast et al., [3] Gehm et al.

Other Lattice Calculations

Neutron matter with realistic interactions (pions)

Sign problem returns; can be handled at $T \neq 0$

Neutron matter with finite polarization

Sign problem returns

Nuclear Matter (neutrons and protons)

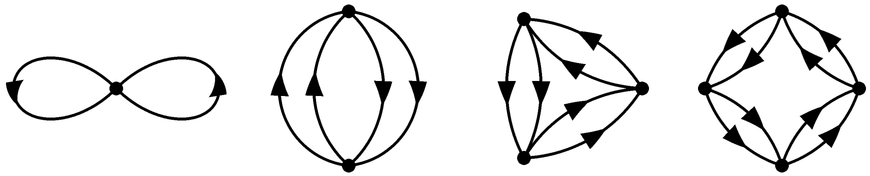
No sign problem in $SU(4)$ limit (Wigner symmetry)

Need a three body force (can be handled with HS)

Isospin asymmetry possible

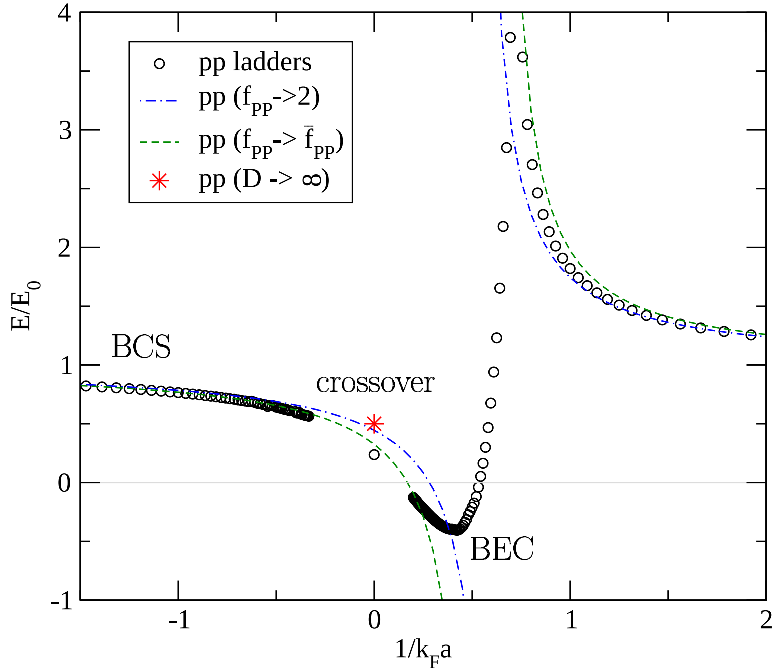
Analytic Approaches: Ladder Diagrams

Sum ladder diagrams at finite density (Brueckner theory)



$$\frac{E}{A} = \frac{k_F^2}{2M} \times$$

$$\left[\frac{3}{5} + \frac{2(k_F a)/(3\pi)}{1 - \frac{6}{35\pi}(11 - 2\log(2))(k_F a)}\right]$$



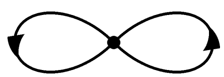
Independent of renormalization scale μ_{PDS}

Unitary Limit $(k_F a) \to \infty$: $\xi = 0.32$

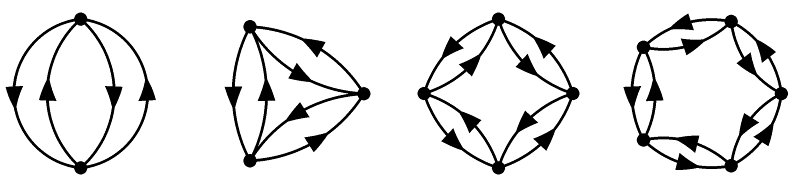
Large N approximation: Ring Diagrams

Consider N fermion species. Define $x \equiv N k_F a/\pi$

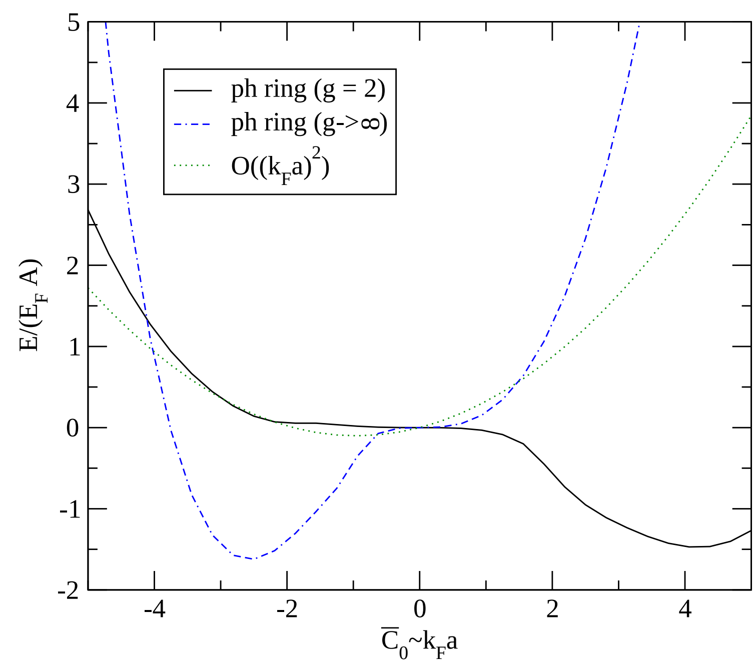
$$\frac{E}{A} = \frac{k_F^2}{2M} \times \left[\left(\frac{3}{5} + \frac{2x}{3} \right) + \frac{1}{N} \left(\frac{3}{\pi} H(x) - \frac{2x}{3} + \frac{4}{35}(22 - 2\log(2))x^2 \right) \right]$$



$N(C_0 N)$



$(C_0 N)^k$

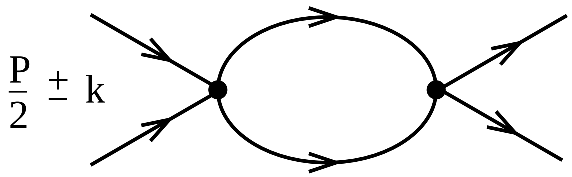


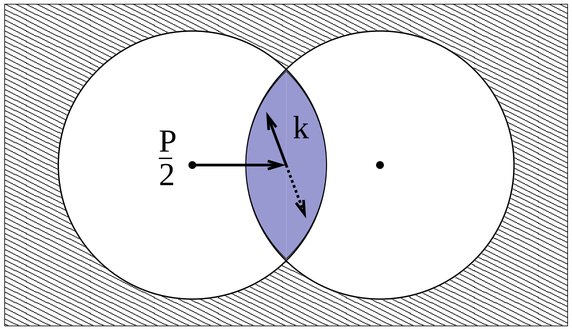
depends on PDS scale parameter μ_{PDS}

not suitable for $(k_F a) \to \infty$

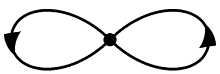
Large d Limit

In medium scattering strongly restricted by phase space



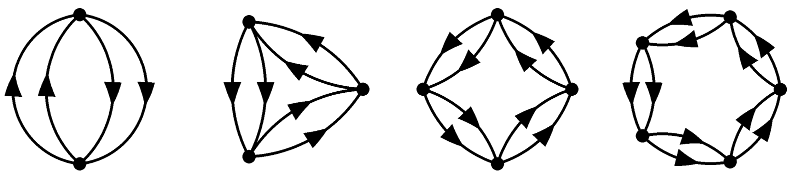


Find limit in which ladders are leading order



$(C_0/d) \cdot 1/d$

$$\lambda \equiv \left[\frac{\Omega_d C_0 k_F^{d-2} M}{d(2\pi)^d} \right]$$

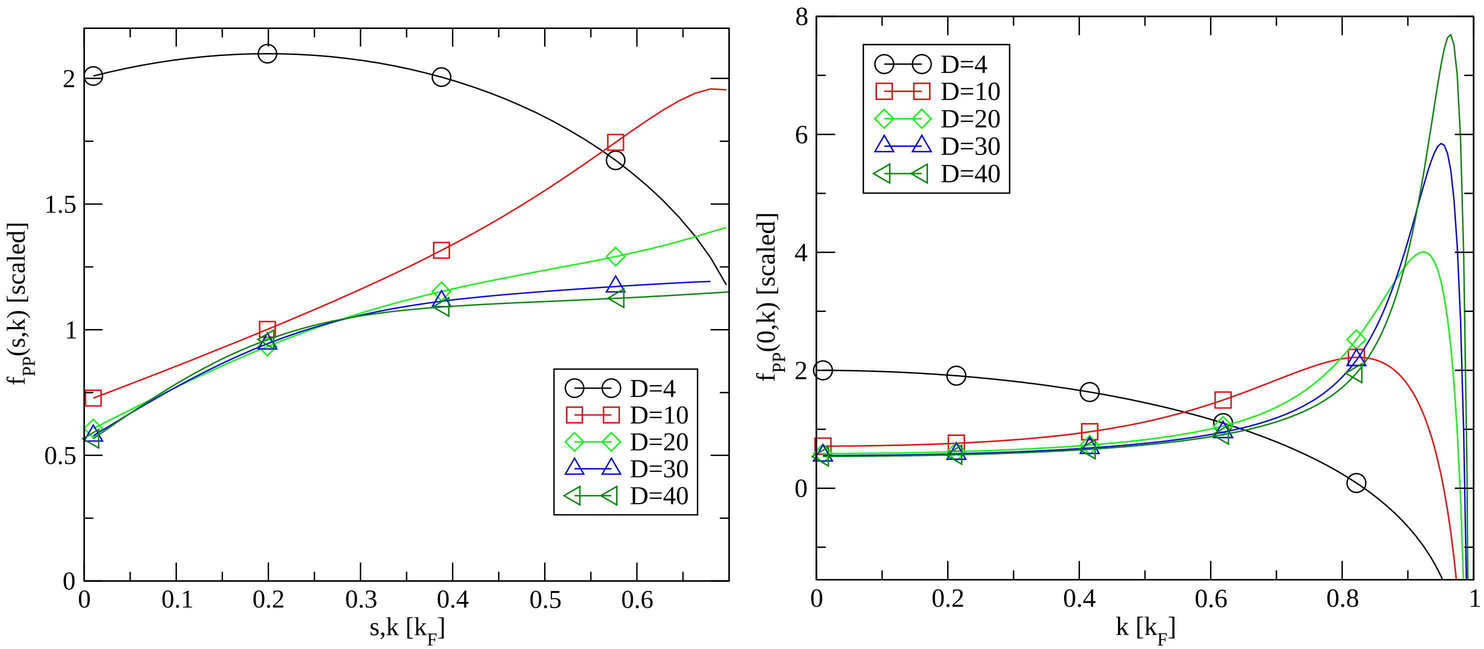


$(C_0/d)^k \cdot 1/d$

$$\lambda = const \quad (d \to \infty)$$

Particle-Particle Scattering Amplitude

$$\int \frac{d^{D-1}q}{(2\pi)^{D-1}} \; \frac{\theta_q^+}{k^2 - q^2 + i\epsilon} = f_{vac}(k) + \frac{k_F^{d-2}\Omega_d}{2(2\pi)^d} f_{PP}^d(\kappa, s),$$



$$f_{PP}^{(d)}(s,\kappa) = \frac{1}{d} f_{PP}^{(0)}(s,\kappa)\Big(1 + O\Big(\frac{1}{d}\Big)\Big).$$

Example: 2nd order diagram

$$\int \frac{d^d P}{(2\pi)^d} \int \frac{d^d k}{(2\pi)^d} \theta_k^- f_{PP}^{(d)}(\kappa, s) = \frac{k_F^{2d}}{(d+1)^2} \left[\frac{\Omega_d}{(2\pi)^d}\right]^2 \frac{4}{d+1} + \ldots$$

Energy per particle is given by

$$\frac{E_2}{A} = 2 \left[\frac{\Omega_d C_0 k_F^{d-2} M}{(d+1)(2\pi)^d}\right]^2 \left(\frac{k_F^2}{2M}\right).$$

Ladder diagrams form geometric series

$$\frac{E}{A} = \left\{1 + \frac{\lambda}{1-2\lambda} + O\left(\frac{1}{d}\right)\right\} \left(\frac{k_F^2}{2M}\right)$$

$\lambda \to \infty$: $\xi = \frac{1}{2} + O(1/d)$

Pairing in the Large d Limit

BCS gap equation

$$\Delta = \frac{|C_0|}{2} \int \frac{d^d p}{(2\pi)^d} \frac{\Delta}{\sqrt{\epsilon_p^2 + \Delta^2}}$$

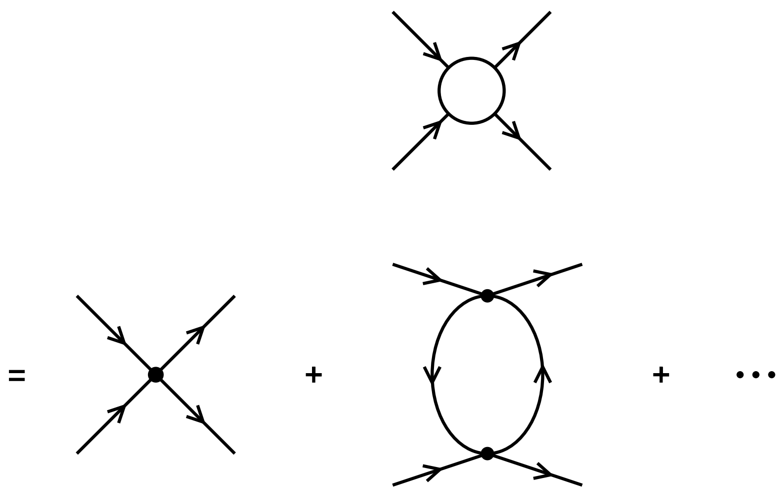
Solution

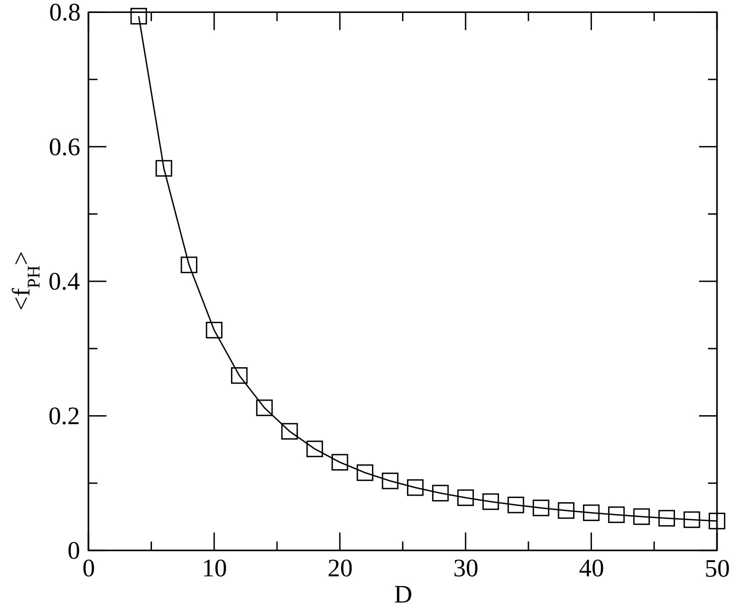
$$\Delta = \frac{2e^{-\gamma} E_F}{d} \exp\left(-\frac{1}{d\lambda}\right) \left(1 + O\left(\frac{1}{d}\right)\right),$$

Pairing Energy

$$\frac{E}{A} = -\frac{d}{4} E_F \left(\frac{\Delta}{E_F}\right)^2 \sim \frac{1}{d}.$$

Screening Corrections





$$\Delta = \frac{2e^{-\gamma}E_F}{d} \exp\left(-\bar{f}_{PH}\right) \exp\left(-\frac{1}{d\lambda}\right)$$

Screening corrections suppressed as $d \to \infty$

Shallow Bound States For Arbitary d

Upper and lower critical dimension (Nussinov & Nussinov)

$d = 2:$ Arbitrarily weak attractive potential has a bound state

$$\xi(d = 2) = 1$$

$d = 4:$ Bound state wave function $\psi \sim 1/r^{d-4}$. Pairs do not overlap

$$\xi(d = 4) = 0$$

Conclude $\xi(d = 3) \sim 1/2$? Try expansion around $d = 4$?

Summary

Numerical Approaches

No sign problem, can compute EOS, T_c, $\ldots$

Extensions: pions, range corrections, two flavors

Analytical Approaches

Large d: $\xi = 1/2 + O(1/d)$. Higher orders?

Other ideas? ($\epsilon = 4 - d$?, $\ldots$)