Instantons in QCD

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Outline

- Introduction: Why Instantons?

  correlators, effective interactions, zero modes

- Instantons, the η', and the Witten-Veneziano relation: large baryon density

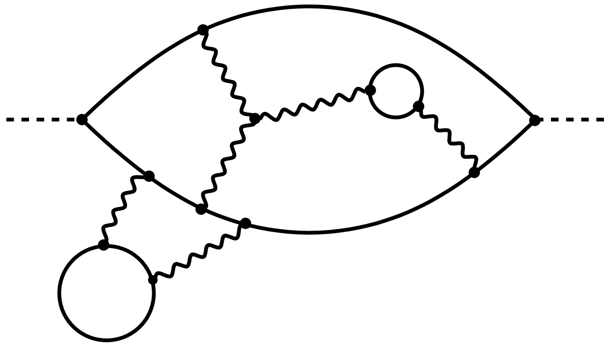
  $\rho \ll \Lambda_{QCD}^{-1}$: semi-classical approximation under control

- Instantons and the large N_c limit

  smooth large N_c limit? Witten-Veneziano relation?

Hadronic Correlation Functions

- hadronic current $j_M(x) = \bar{q}(x)\Gamma q(x)$



$$\Pi(x) = \langle j(x)j(0)\rangle$$

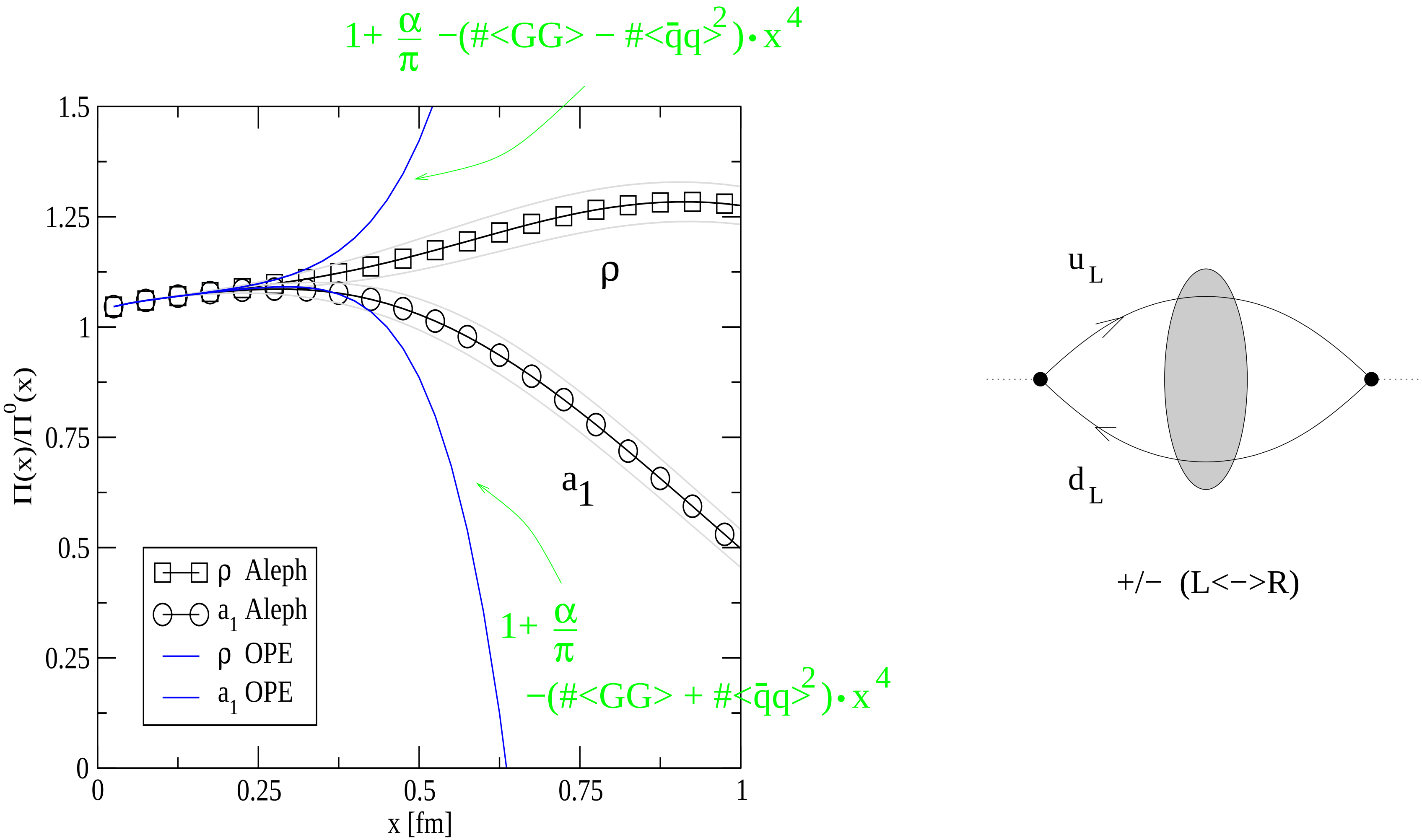
- short distance behavior: OPE

$$\Pi(Q) = c_0 \log(Q^2) + c_4 \frac{\langle \mathcal{O}_4\rangle}{Q^4} + c_6 \frac{\langle \mathcal{O}_6\rangle}{Q^6} + \ldots$$

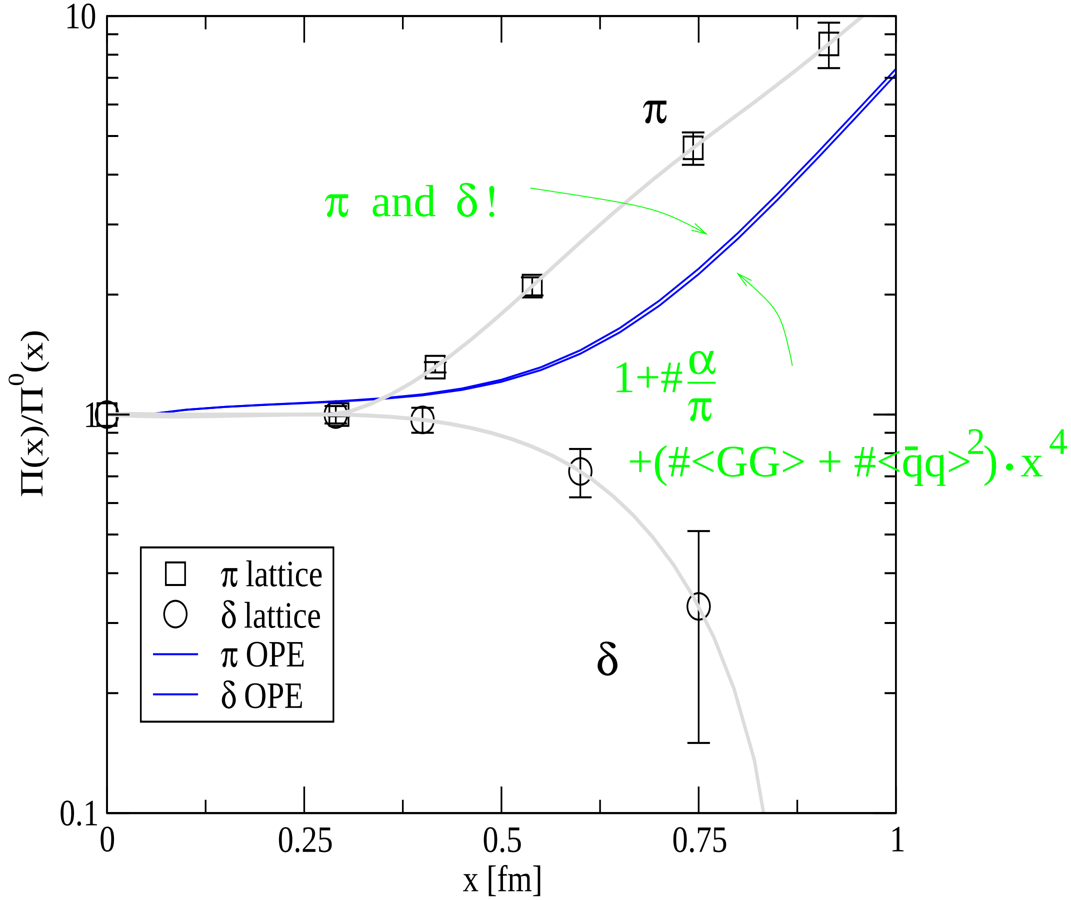
- experimental information

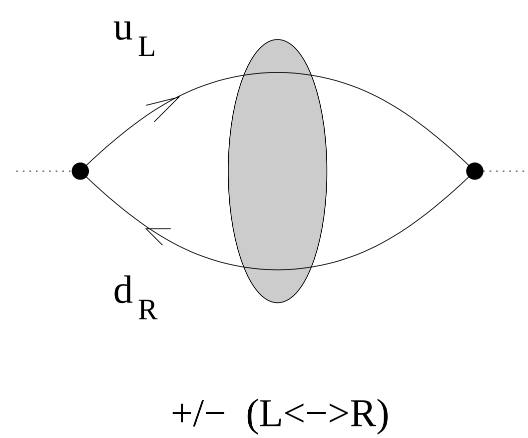
$$\Pi(Q) = \int ds \frac{\rho(s)}{s+Q^2}$$

Vector Channels: ρ and a_1

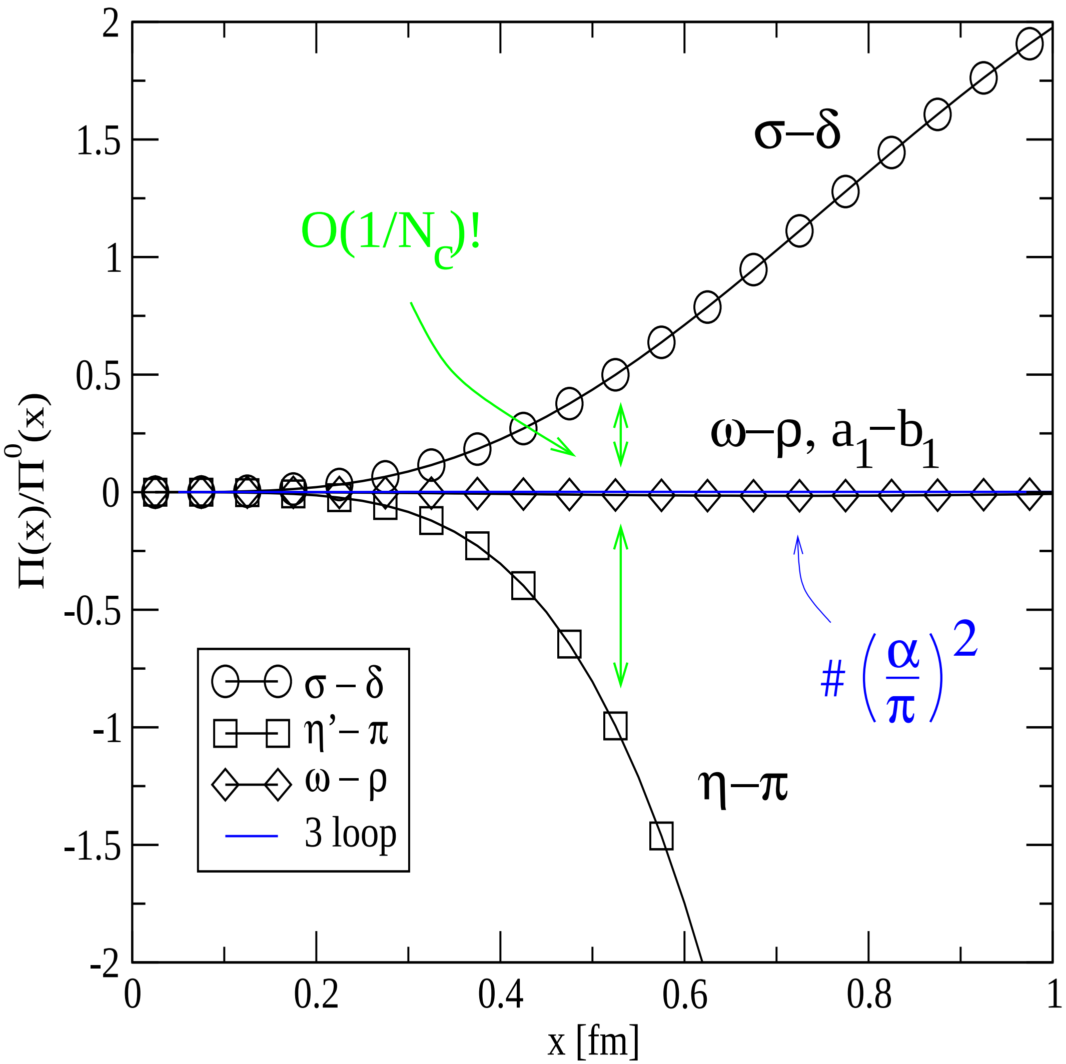


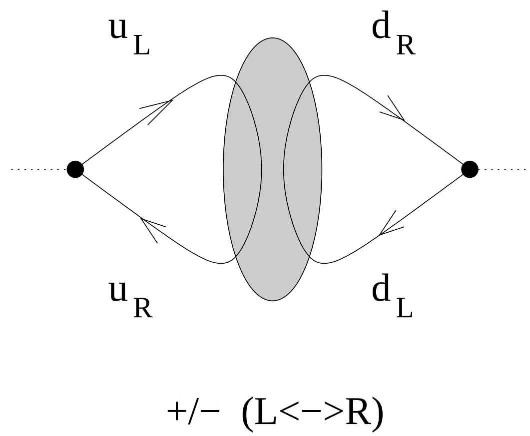
Scalar Channels: π and δ





OZI violation: $\eta' - \pi$, $\sigma - \delta$, $\omega - \rho$, $a_1 - b_1$





Summary

- Only small effects in $(\bar{L}L \pm \bar{R}R)^2$.
- Sign changes for $\bar{L}R \leftrightarrow \bar{R}L$.
- Sign changes for $(\bar{u}d)(\bar{u}d) \leftrightarrow (\bar{u}u)(\bar{d}d)$.

$$\mathcal{L} = G \det_f (\bar{\psi}_L \psi_R) + (L \leftrightarrow R)$$

Glueballs

- currents
  $O_S = g^2G^2, \quad O_P = g^2G\tilde{G}, \quad O_T = \frac{1}{4}g^2(G_{\mu\nu})^2 - g^2G_{0\alpha}G_{0\alpha}$
- OPE: power corrections small

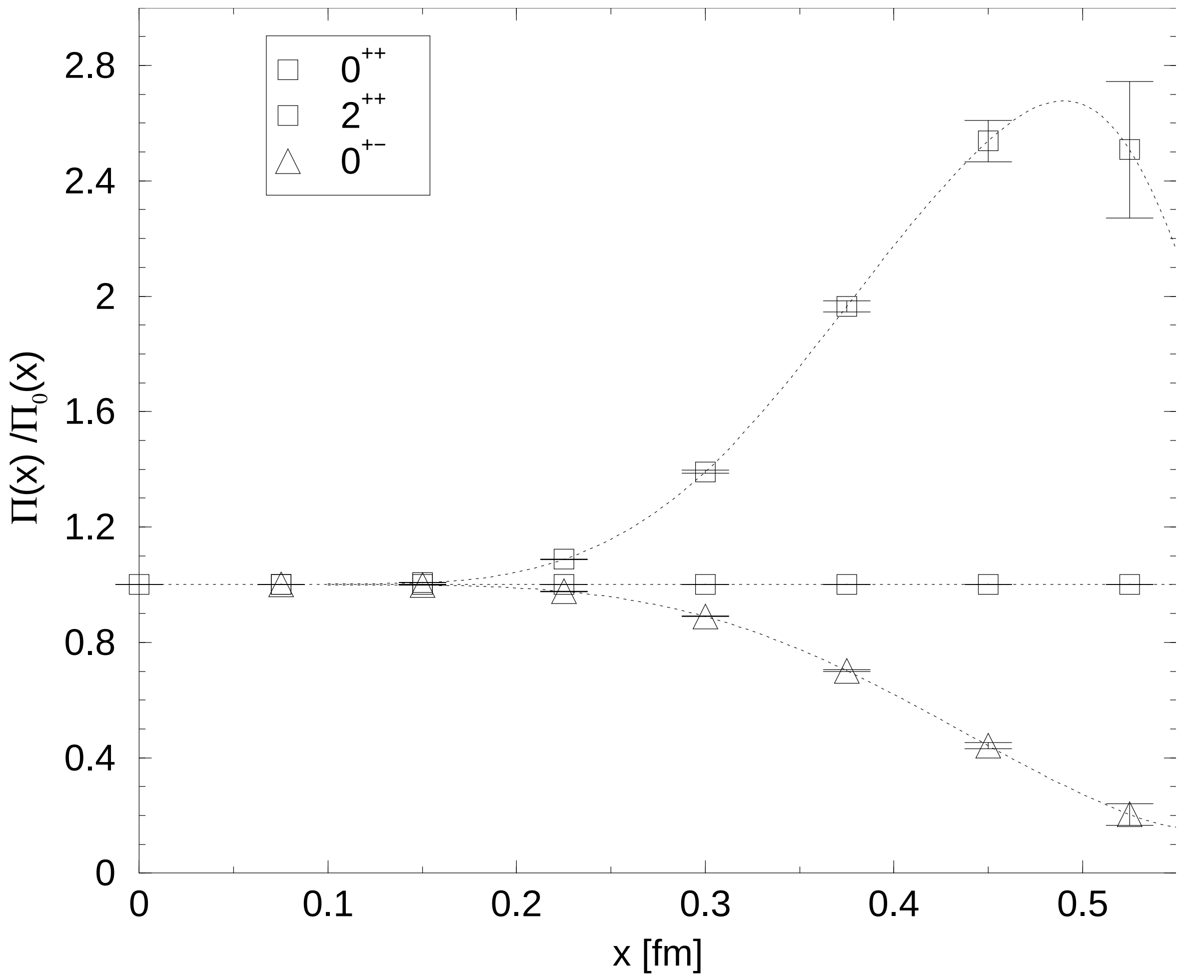
$$\Pi_{S,P}(x) = \Pi^0_{S,P}\left(1 \pm \frac{\pi^2 g}{192}\langle f^{abc}G^a_{\mu\nu}G^b_{\nu\beta}G^c_{\beta\mu}\rangle x^6 + \ldots\right)$$

$$\Pi_T(x) = \Pi^0_T\left(1 + \frac{25\pi^2 g^2}{9216}\langle 2\mathcal{O}_1 - \mathcal{O}_2\rangle \log(x^2)x^8 + \ldots\right)$$

- Sum Rules

$$\int d^4x\, \Pi_P(x) = \chi_{top}, \qquad\qquad \int d^4x\, \Pi_S(x) = \frac{4}{b}\langle g^2G^2\rangle$$

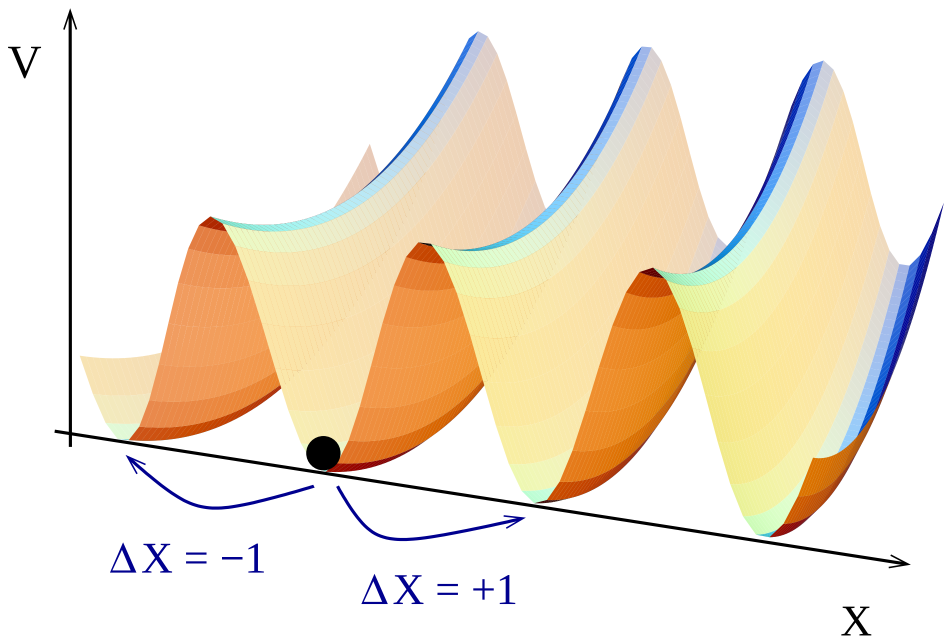
Glueball Correlation Functions



pattern explained by self-dual fields $G^2 = \pm G\tilde{G}$

Topology in QCD

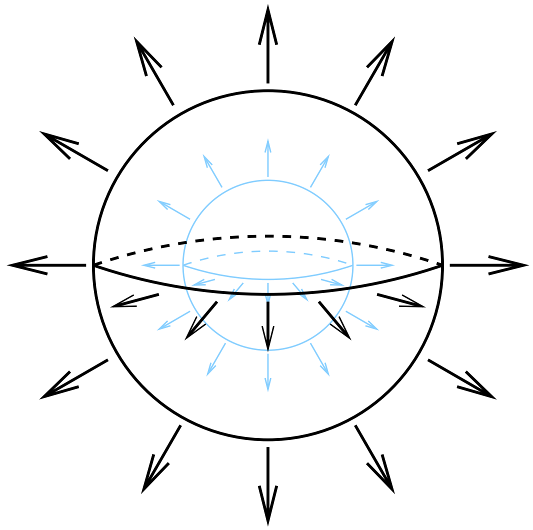
- classical potential is periodic in variable X



$$X = \int d^3x \, K_0(x,t)$$

$$\partial^\mu K_\mu = \frac{1}{32\pi^2} G^a_{\mu\nu} \tilde{G}^a_{\mu\nu}$$

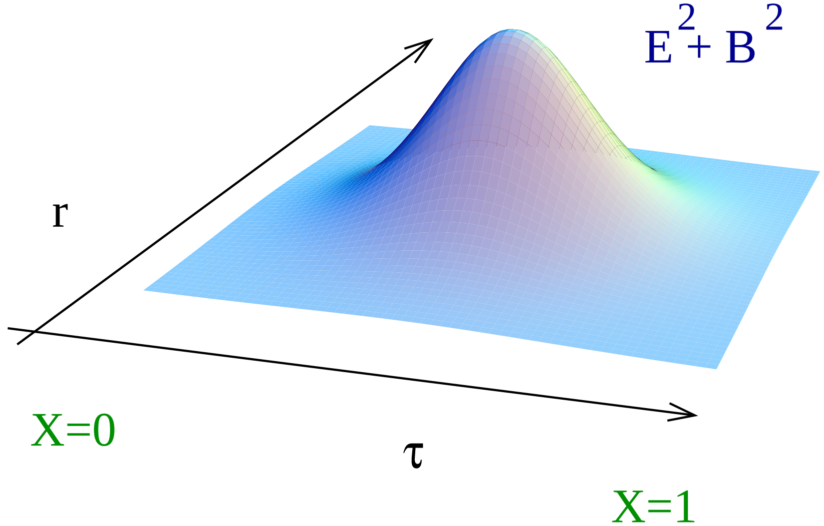
- classical minima correspond to pure gauge configurations



$$A_i(x) = iU^\dagger(x)\partial_i U(x)$$

$$E^2 = B^2 = 0$$

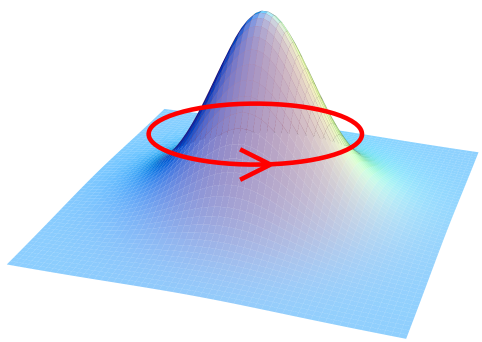
- classical tunneling paths: Instantons



$$A^a_\mu(x) = 2\frac{\eta_{a\mu\nu}x_\nu}{x^2+\rho^2},$$

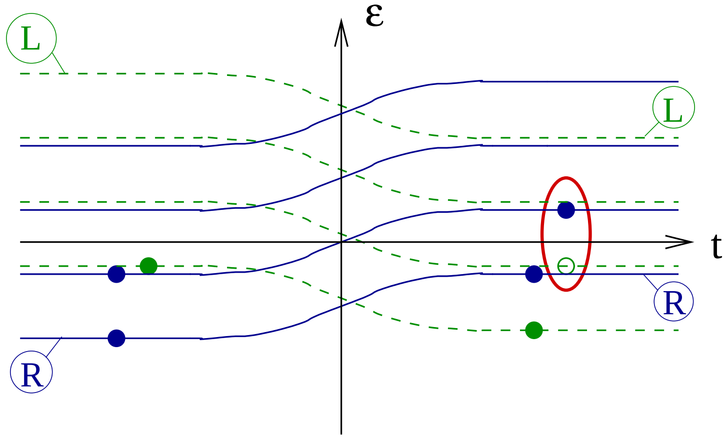
$$G^a_{\mu\nu}\tilde{G}^a_{\mu\nu} = \frac{192\rho^4}{(x^2+\rho^2)^4}.$$

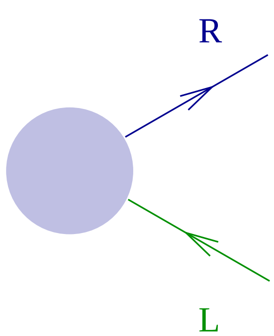
- (Anti)Instantons: Dirac operator has a L/R zero mode.



$$\gamma \cdot (\partial + A_{I,A})\, \psi^0_{L,R} = 0$$

- spectrum of Hamiltonian

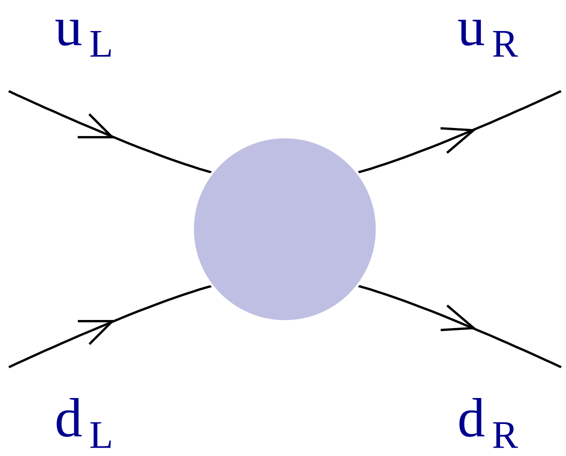




axial charge violation:

$$\Delta Q_A = 2$$

- instanton induced quark interaction ($N_f = 2$)



$$\mathcal{L} = G \det_f(\bar{\psi}_{L,f}\psi_{R,g})$$

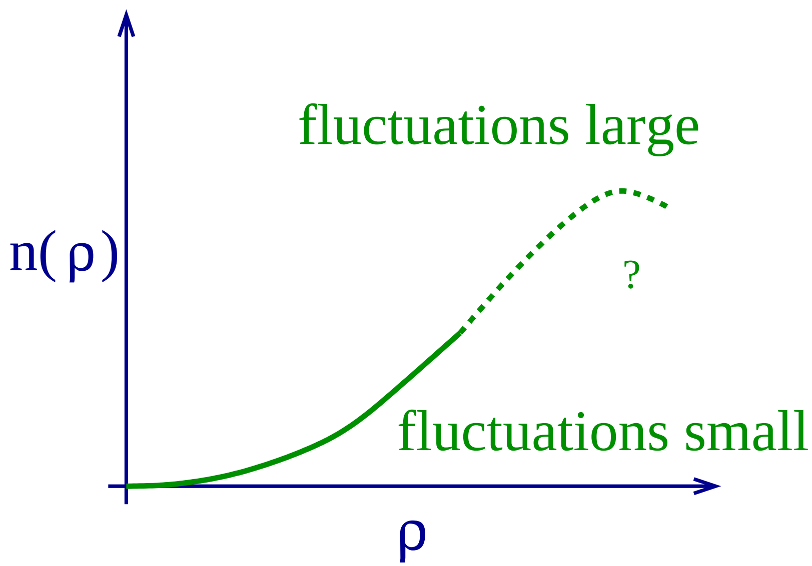
$$G = \int d\rho\, n(\rho)$$

violates $U(1)_A$ but

preserves $SU(2)_{L,R}$

... and contributes to the η' mass

- tunneling rate (barrier penetration factor)



$$n(\rho) \sim \exp\left[-\frac{8\pi^2}{g^2(\rho)}\right] \sim \rho^{b-5}$$

Chiral Symmetry Breaking

- spectrum of Dirac operator in gluon background

$$iD\!\!\!/\,\psi_\lambda = \lambda\psi_\lambda, \qquad S(x,y) = -\sum_\lambda \frac{\psi_\lambda(x)\psi_\lambda^\dagger(y)}{\lambda + im}$$

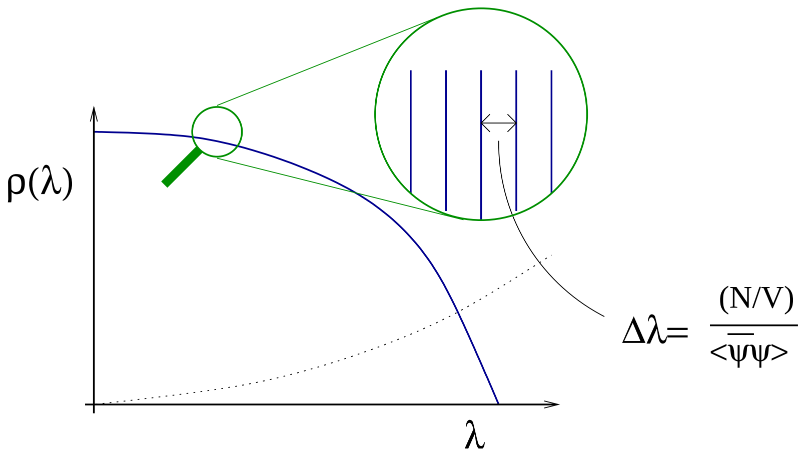
- quark condensate

$$\langle\bar{q}q\rangle = i\int d^4x\,\langle\mathrm{tr}\,(S(x,x))\rangle = -\int_0^\infty d\lambda\,\rho(\lambda)\frac{2m}{\lambda^2+m^2}$$

$$\langle\bar{q}q\rangle = -\pi\rho(\lambda=0) \qquad (Banks - Casher)$$

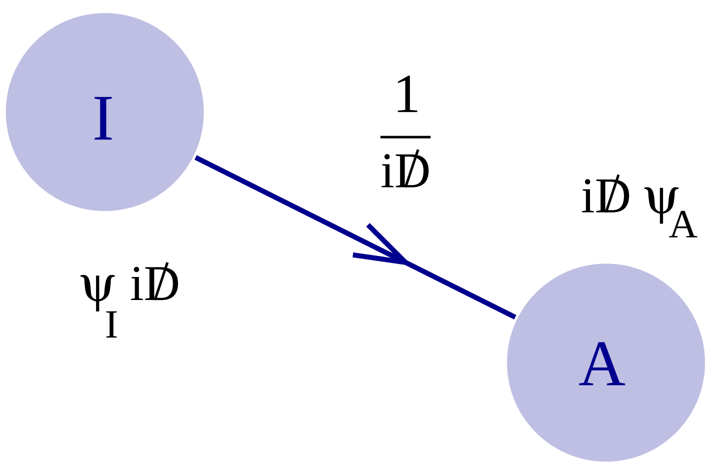
$\Rightarrow \chi SB$ connected with "almost" zero modes

- ensemble of instantons: zero modes mix



$$i\not{D} = \begin{pmatrix} 0 & T_{IA} \\ T_{AI} & 0 \end{pmatrix},$$

- overlap matrix elements T_{IA}



$$T_{IA} = \int d^4x\, \psi_I^\dagger(x - z_I) i\not{D}\, \psi_A(x - z_A)$$

$$\sim i(u \cdot \hat{R}) \frac{(\rho_I \rho_A)^3}{R^4}$$

- width of the zero mode zone $\sim$ average matrix element

$$\langle |T_{IA}|^2 \rangle = \frac{2\pi^2}{3N_c}\frac{N\rho^2}{V}.$$

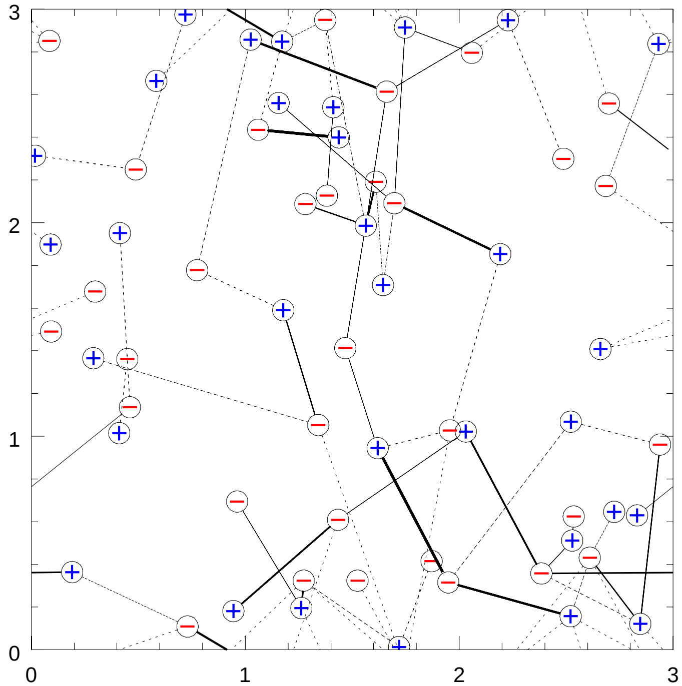
- Gaussian ensemble $\Rightarrow$ spectral density is a semi-circle

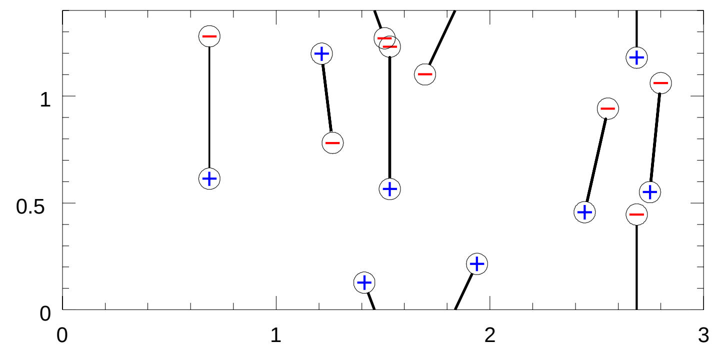
$$\rho(\lambda) = \frac{N}{\pi\sigma V}\left(1 - \frac{\lambda^2}{4\sigma^2}\right)^{1/2}.$$

- Casher-Banks gives

$$\langle \bar{q}q \rangle = -\frac{1}{\pi\rho}\left(\frac{3N_c}{2}\frac{N}{V}\right)^{1/2} \simeq -(240\,\mathrm{MeV})^3,$$

Chiral Phase Transition ∼ Metal-Insulator Transition





$\langle \bar{q} q \rangle \neq 0$

hadronic phase

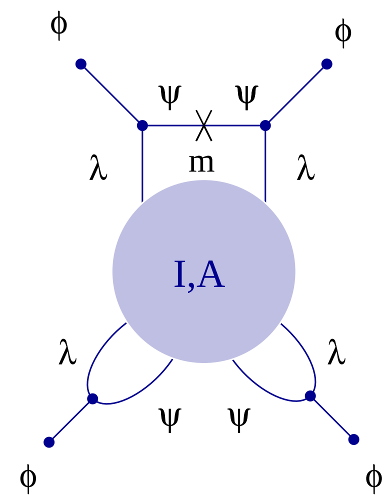
$\langle \bar{q} q \rangle = 0$

quark gluon plasma

Digression: Exact Results

- weak coupling result for gluino condensate in $\mathcal{N}=1$ SUSY

$$\langle \lambda^a_\alpha \lambda^{a\,\alpha} \rangle = -6 N_c \Lambda^3_{PV}$$

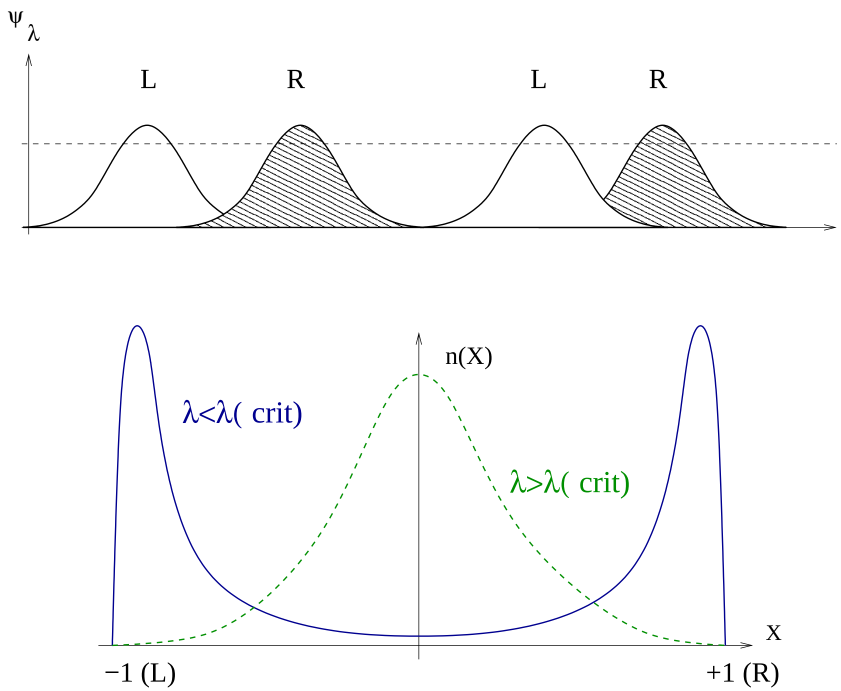


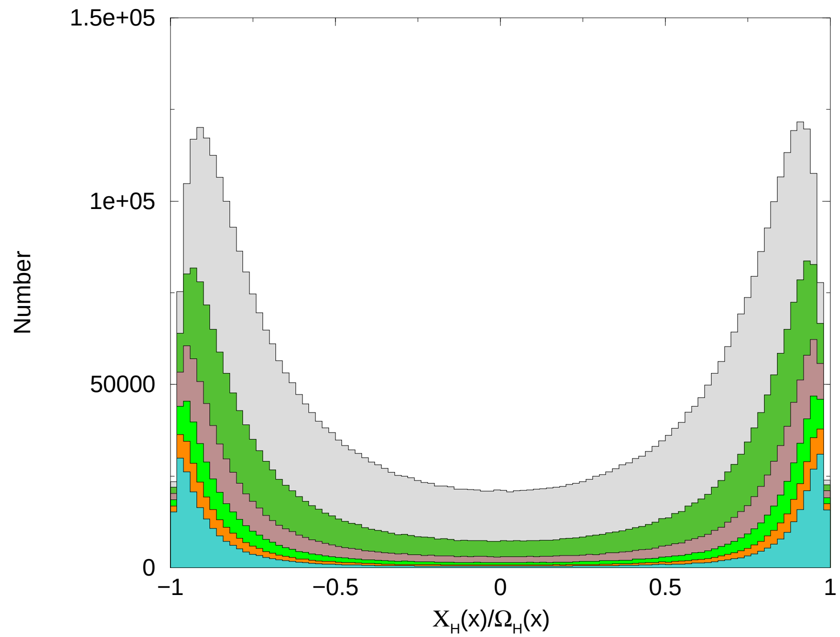
(alternative: use calorons with non-trivial holonomy at $\beta \to 0$)

- large N_c: planar equivalence ($\mathcal{N}=1$ SUSY $\leftrightarrow$ $N_f=1$ QCD)

$$\langle \bar\psi \psi \rangle = \langle \lambda^a_\alpha \lambda^{a\,\alpha} \rangle \left(1 + O(1/N_c)\right)$$

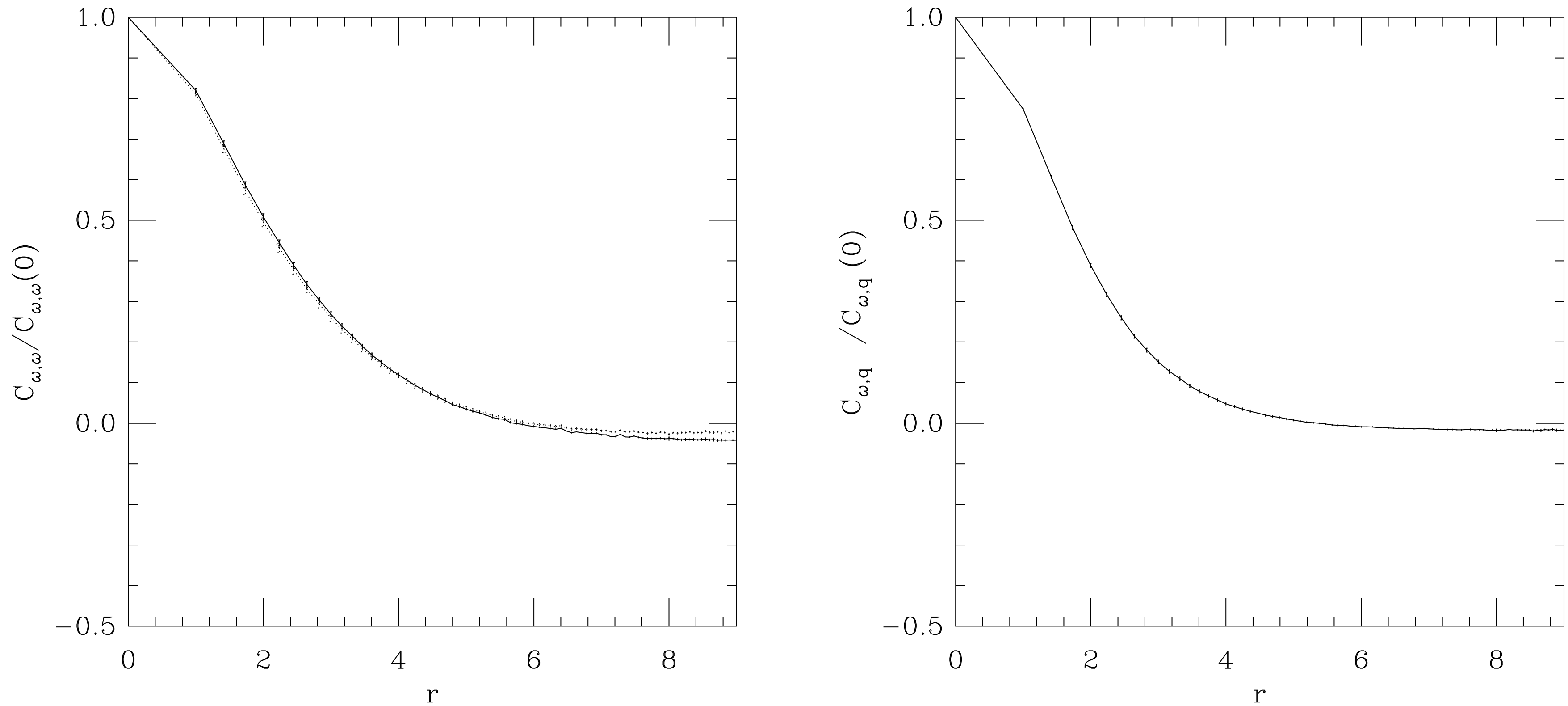
Chiral Symmetry Breaking on the Lattice





chirality dsitribution from T. Blum et al., [hep-lat/0105006]

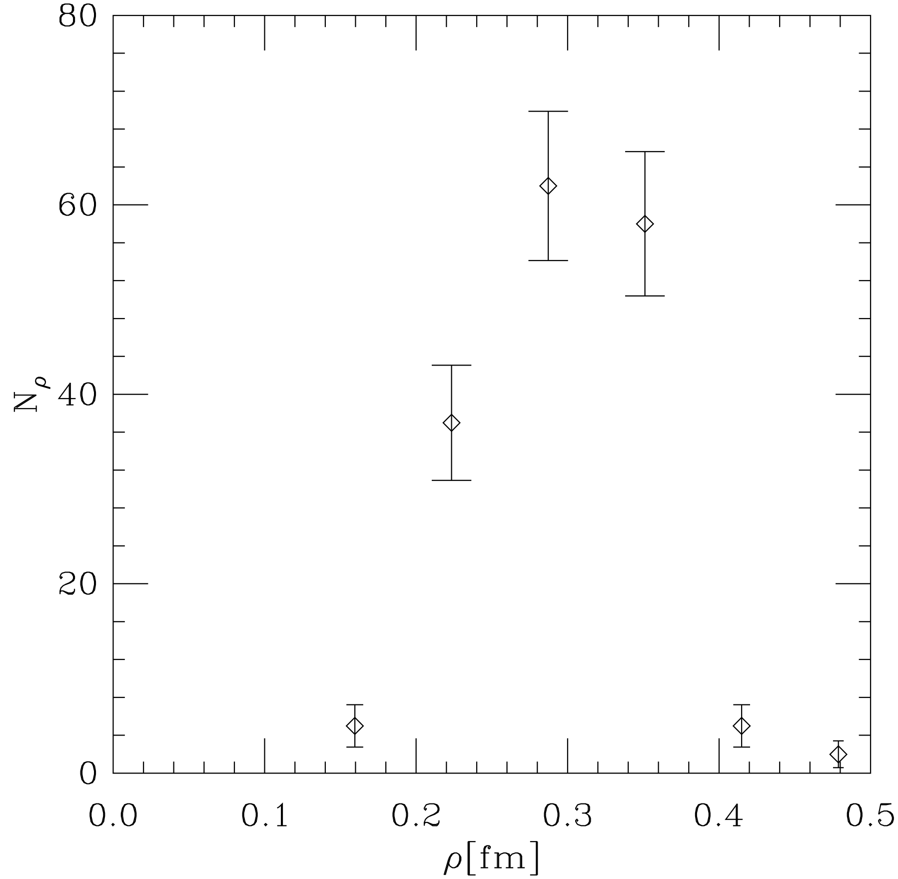
Chirality-Chirality and Chirality-Topology Correlation Functions



$\langle q_L(0)q_L(x)\rangle$ $\langle q_L(0)G\tilde{G}(x)\rangle$

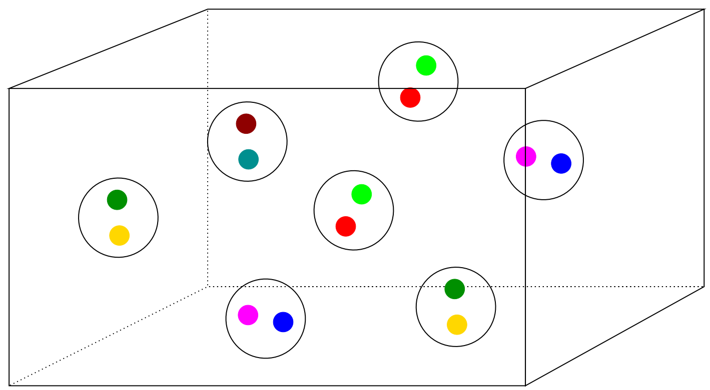
T. DeGrand, A. Hasenfratz, Phys.Rev.D64:034512,2001

Size Distribution



Instanton Ensemble

- instanton liquid described by partition function



$$Z = \frac{1}{N_I! N_A!} \prod_I^{N_I+N_A} \int [d\Omega_I \, n(\rho_I)] \\ \times \det(\not{D}) \exp(-S_{int})$$

- quark propagator

$$S(x,y) = \sum_{IJ} \psi_I(x) \left(\frac{1}{T + im} \right)_{IJ} \psi_J^\dagger(y) + S_{NZM}(x,y)$$

Example: Scalar Glueball

- short distance: dilute gas approximation

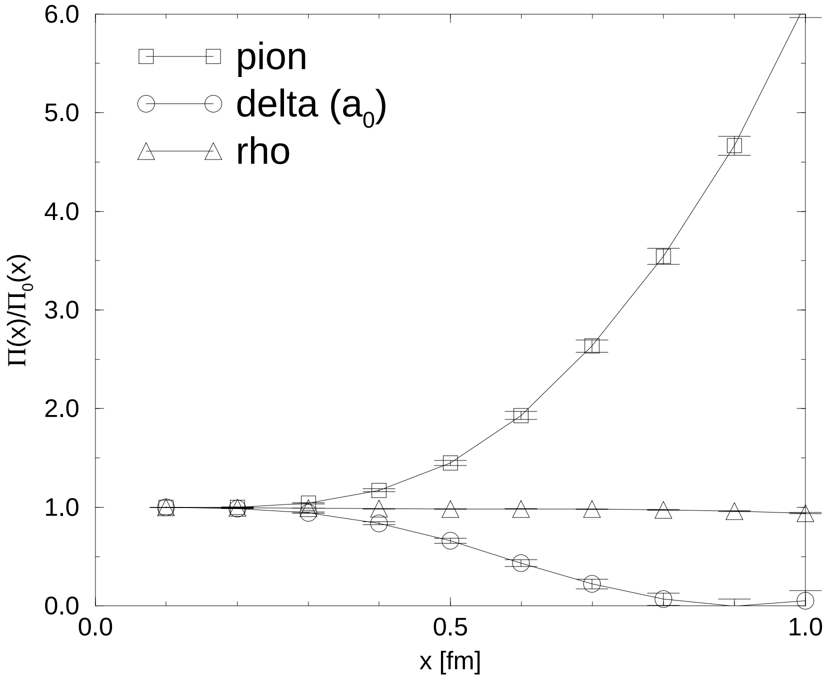
$$\Pi_S(x) = \frac{2^9 3^2}{\pi^2} \int d\rho n(\rho) \int d^4z \, \frac{\rho^8}{[(x-z)^2+\rho^2]^4[z^2+\rho^2]^4}$$

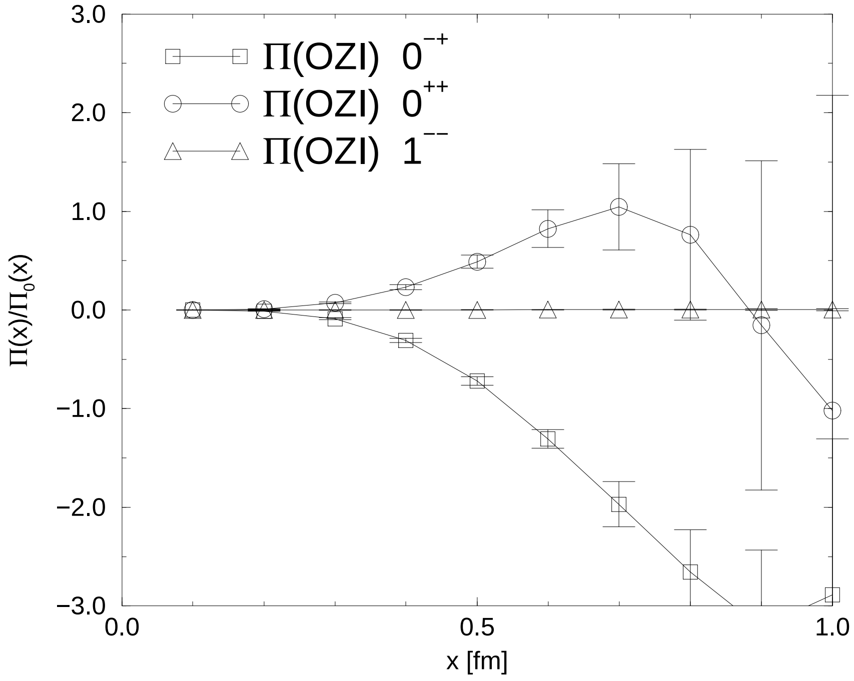
$$\Pi_S(Q^2) = 2^5\pi^2 \int d\rho n(\rho) \, (Q\rho)^4 \left[K_2(Q\rho)\right]^2$$

- long distance: multi-instanton effects

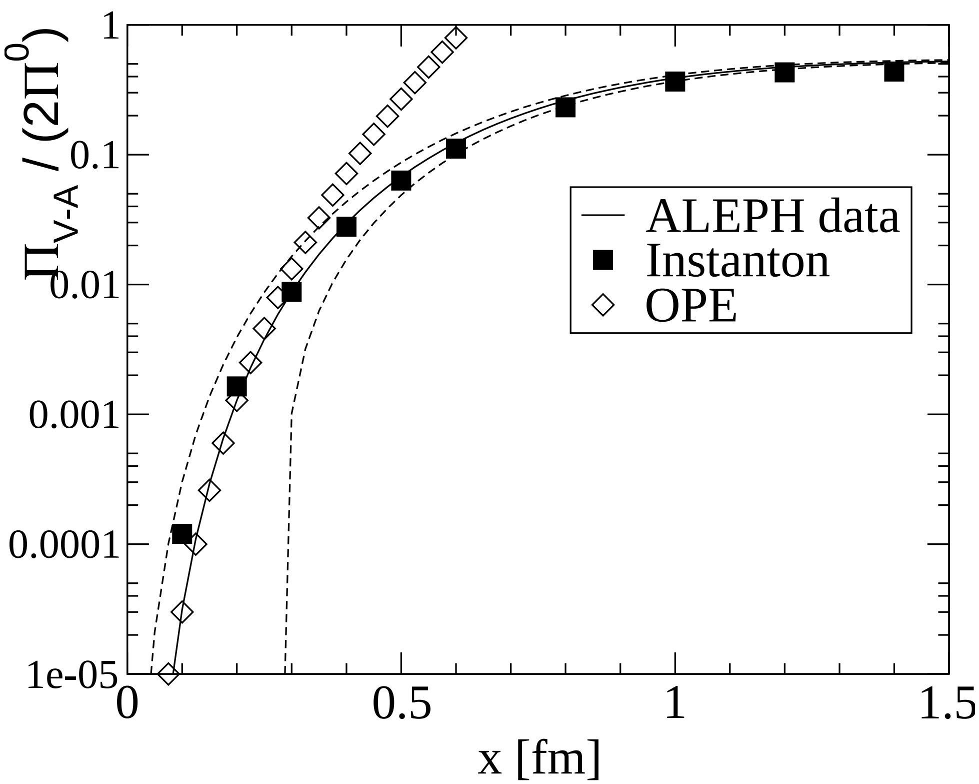
$$\int d^4x \, \Pi_S(x) = \frac{4}{b}\left(\frac{N}{V}\right)$$

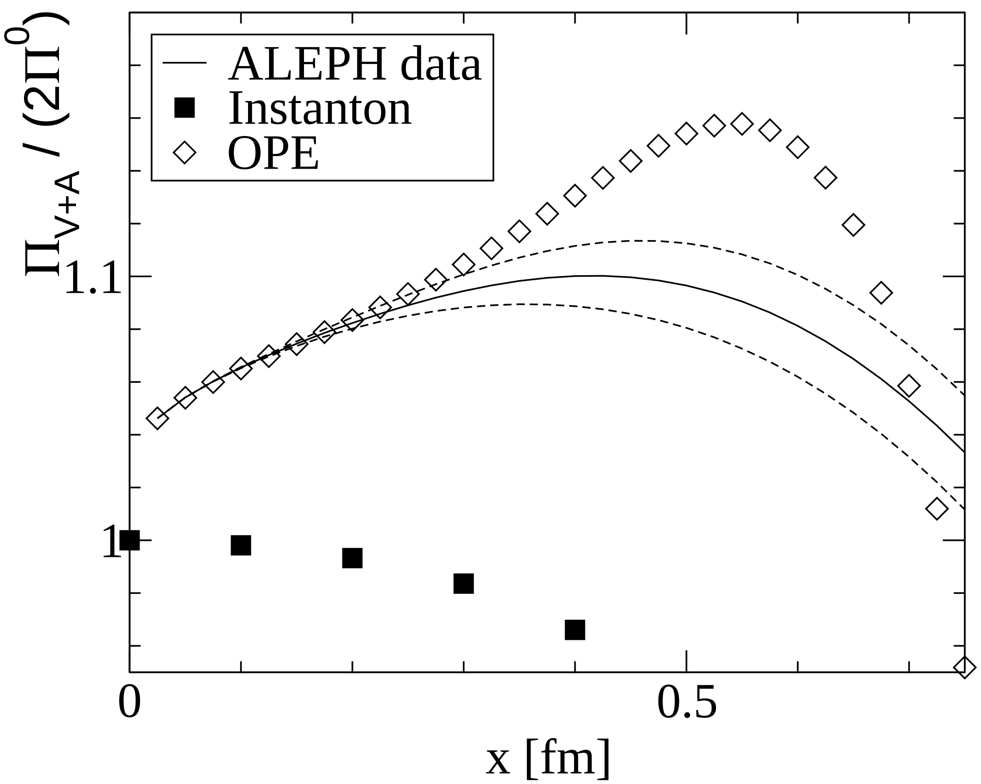
Meson Correlation Functions





V±A Correlation Functions





$V - A$

chiral symmetry breaking

$V + A$

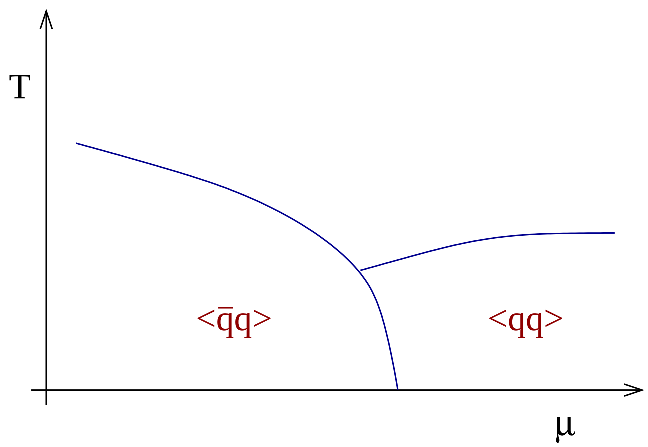
perturbation theory, renormalons, etc

Instantons and the Mass of the η':

Large Baryon Density

QCD at Large Density ($N_f = 2$)

- schematic phase diagram



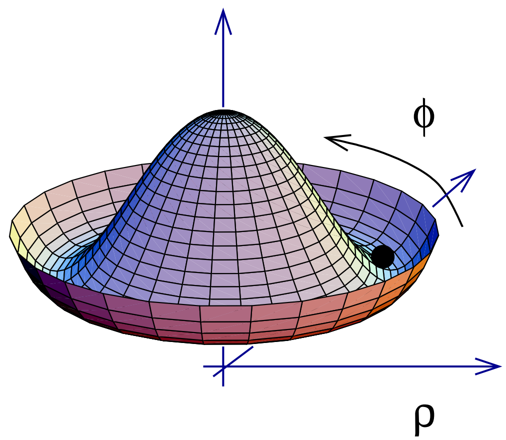
diquark condensate breaks

$U(1)_B$ and $U(1)_A$

$$\langle q_L q_L \rangle = \rho\, e^{i(\chi+\phi)/2}$$

$$\langle q_R q_R \rangle = \rho\, e^{i(\chi-\phi)/2}$$

- effective lagrangian for $U(1)_A$ Goldstone boson

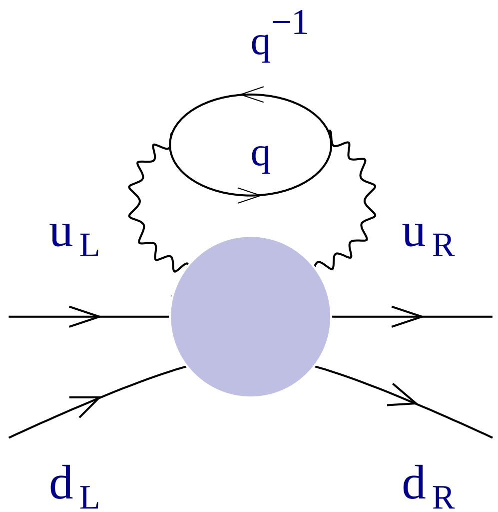


$$\mathcal{L} = \frac{f^2}{2}\left[(\partial_0\phi)^2 - v^2(\partial_i\phi)^2\right] - V(\phi+\theta) + \mathcal{L}(\rho,\chi)$$

$V(\phi+\theta)$ vanishes in perturbation theory

η' Mass at Large Baryon Density ($N_c = N_f = 2$)

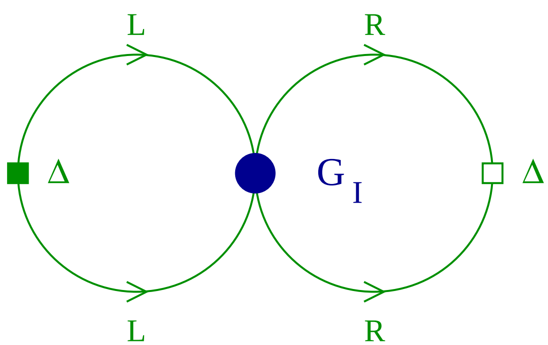
- instanton induced effective interaction for quarks with $p \sim p_F$



$$n(\rho, \mu) = n(\rho, 0) \exp\left[-N_f \rho^2 \mu^2\right]$$

$$\rho \sim \mu^{-1} \ll \Lambda_{QCD}^{-1}$$

- instanton contribution to vacuum energy

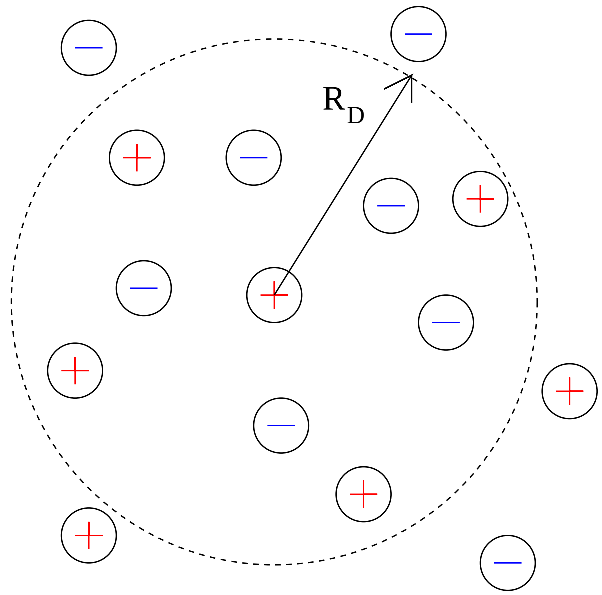


$$\langle \mathcal{L} \rangle = A \cos(\phi + \theta)$$

$$A = C_N \Phi^2 \left[\log\left(\frac{\mu}{\Lambda}\right)\right]^4 \left(\frac{\Lambda}{\mu}\right)^8 \Lambda^{-2}$$

- η' mass satisfies "Witten-Veneziano" relation $\boxed{f^2 m_\phi^2 = A}$

- very dilute instanton gas



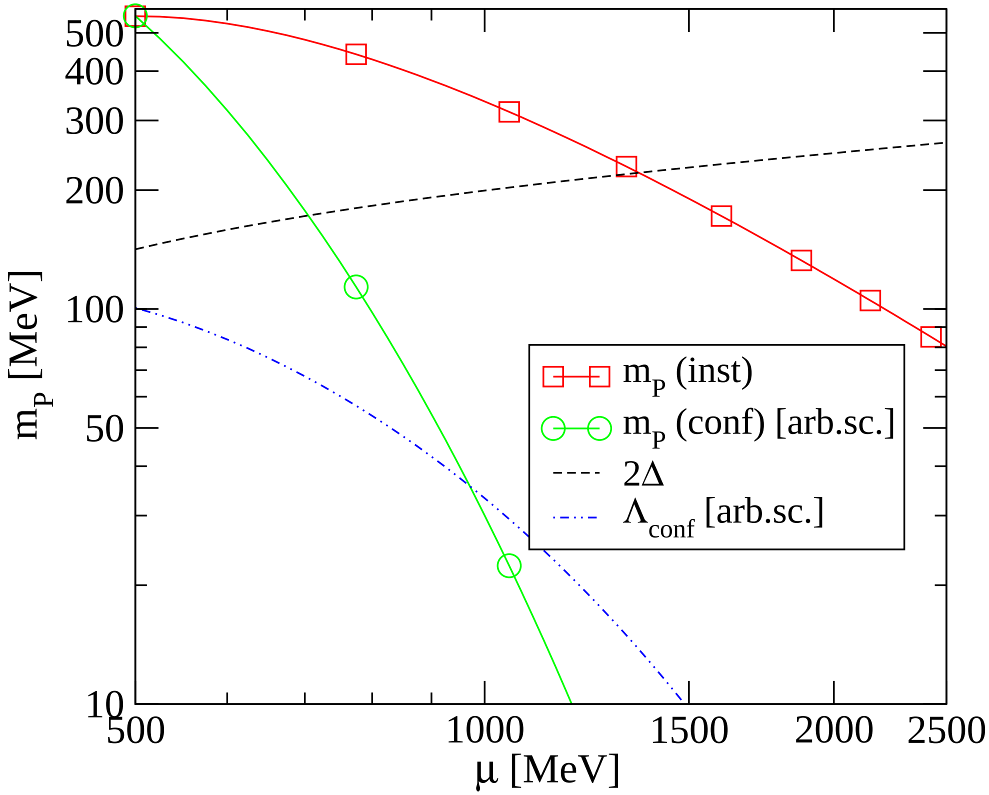
$$\rho \ll r_{IA} \ll R_D$$
$$\rho \sim \mu^{-1}$$
$$r_{IA} = A^{1/4}$$
$$R_D = m_\phi^{-1}$$

- A is the local topological susceptibility

$$A = \chi_{top}(V) = \frac{\langle Q_{top}^2 \rangle_V}{V} \qquad r_{IA}^4 \ll V \ll R_D^4$$

- Global topological susceptibility vanishes

$$\chi_{top} = \lim_{V\to\infty} \frac{\langle Q_{top}^2 \rangle_V}{V} = 0 \qquad (m = 0)$$



- extrapolate to zero density

$$\left(\frac{N}{V}\right) \sim 1\,\mathrm{fm}^{-4} \qquad m'_\eta \sim 800\,\mathrm{MeV}$$

- Instantons predict density dependence of $m_{\eta'}$

can be checked on the lattice

Instantons and the Mass of the η':

Large Number of Colors

QCD at Large N_c

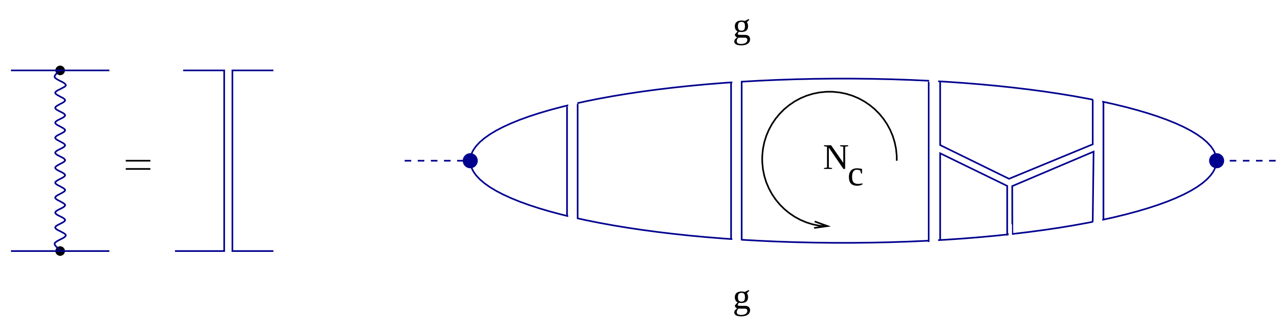
- QCD $(m = 0)$ is a parameter free theory. Very beautiful.

But: No expansion parameter

- 't Hooft: Consider $N_c \to \infty$ and use $1/N_c$ as a small parameter

$$N_c \to \infty \qquad \Rightarrow \qquad \text{classical master field}$$

- keep Λ_{QCD} fixed $\Rightarrow g^2 N_c = const$



- Could the master field be a multi-instanton?

$$\text{Witten}: \ \text{No!} \qquad dn \sim \exp\left(-\frac{1}{g^2}\right) \sim \exp\left(-N_c\right)$$

$U(1)_A$ Anomaly at Large N_c

- consider θ term

$$\mathcal{L} = \frac{ig^2\theta}{32\pi^2} G^a_{\mu\nu}\tilde{G}^a_{\mu\nu}$$

- no θ dependence in perturbation theory.

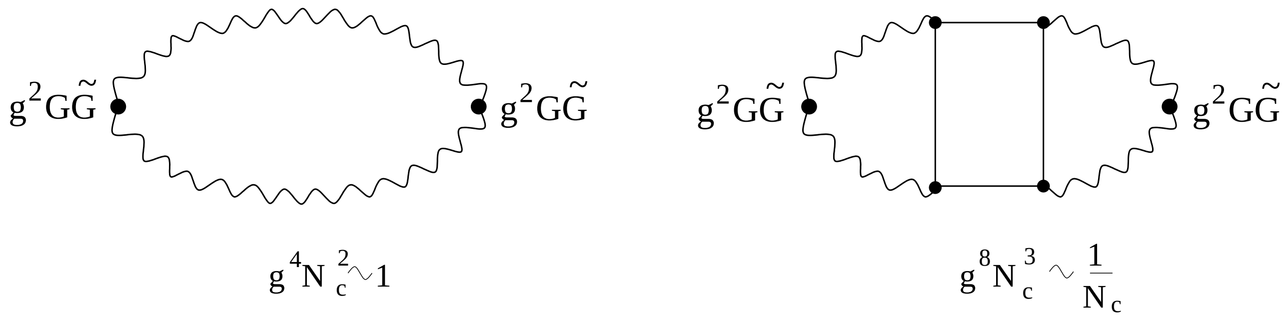
Witten: *non-perturbative θ dependence*

$$\chi_{top} = \left.\frac{d^2E}{d\theta^2}\right|_{\theta=0} \sim O(1)$$

- massless quarks: topological charge screening

$$\lim_{m\to 0} \chi_{top} = 0$$

- How can that happen? Fermion loops are suppressed!



Witten: *η' has to become light*

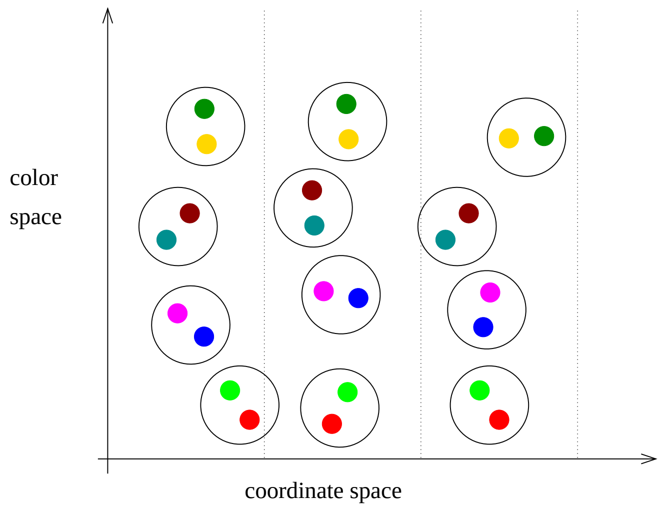
$$f_\pi^2 m_{\eta'}^2 = 2N_f\,\chi_{top}(no\ quarks) \quad \Rightarrow \quad m_{\eta'}^2 = O(1/N_c)$$

- Witten-Veneziano relation “works”

$$\chi_{top} \simeq (200\,\mathrm{MeV})^4 \quad (quenched\ lattice) \ \Rightarrow m_{\eta'} \simeq 900\,\mathrm{MeV}$$

Instantons at Large N_c

- semi-classical ensemble of instantons at large N_c



instantons are $N_c = 2$ configurations

$$\left(\frac{N}{V}\right) = O(N_c)$$

$$\Rightarrow \epsilon_{vac} = O(bN_c) = O(N_c^2)$$

- instantons are semi-classical

$$\rho \simeq \rho^* = O(1) \qquad S_{inst} = O(N_c)$$

- density $dn \sim \exp(-S_{inst}) = O(\exp(-N_c))$?

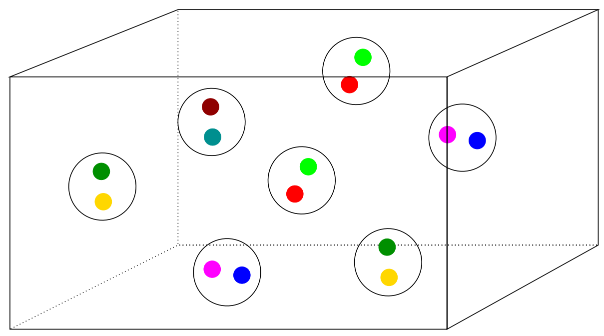
$NO!$ *large entropy* $dn \sim \exp(+N_c)$

- topological susceptibility $\chi_{top} \simeq (N/V) = O(N_c)$?

$NO!$ *fluctuations suppressed* $\chi_{top} = O(1)$

Instanton Ensemble

instanton ensemble



described by partition function

$$Z = \frac{1}{N_I! N_A!} \prod_I^{N_I+N_A} \int [d\Omega_I \, n(\rho_I)] \times \exp(-S_{int})$$

$$n(\rho) = C_{N_c} \left(\frac{8\pi^2}{g^2}\right)^{2N_c} \rho^{-5} \exp\left[-\frac{8\pi^2}{g(\rho)^2}\right]$$

$$C_{N_c} = \frac{0.47\exp(-1.68N_c)}{(N_c-1)!(N_c-2)!} \qquad \frac{8\pi^2}{g^2(\rho)} = -b\log(\rho\Lambda), \qquad b = \frac{11}{3}N_c$$

$$S_{int} = -\frac{32\pi^2}{g^2}|u|^2 \left\{ \frac{\rho_I^2 \rho_A^2}{R_{IA}^4} \left(1 - 4\cos^2\theta\right) + S_{core} \right\}$$

- complicated ensemble, size distribution

$$n(\rho) \sim \begin{cases} \exp(-N_c) & \rho < \rho^* \\ const & \rho \sim \rho^* \end{cases} \qquad \rho^* \sim O(1)$$

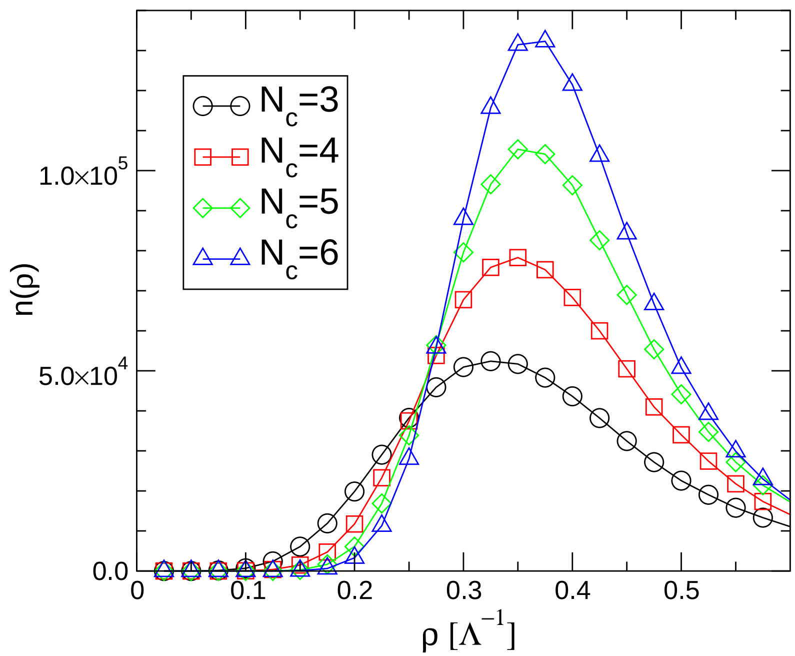
- total density determined by interactions

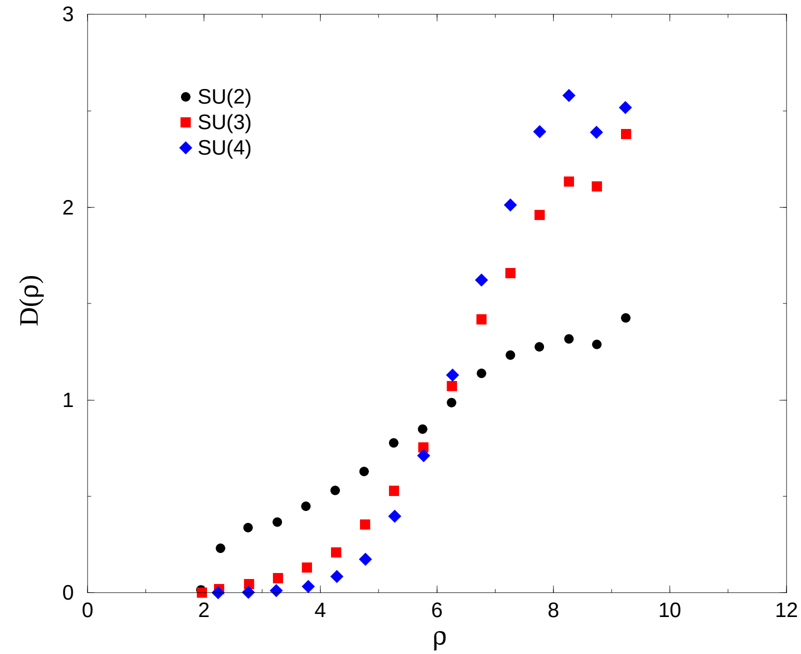
$$\begin{array}{ccccccc} S(1-body) & \sim & & & S(2-body) & & \\ N_c & \sim & N_c & \times & \frac{1}{N_c} & \times & \left(\frac{N}{V}\right) \\ classical & \sim & classical & \times & color\ overlap & \times & density \end{array}$$

- conclude

$$\boxed{\left(\frac{N}{V}\right) = O(N_c)}$$

instanton size distribution





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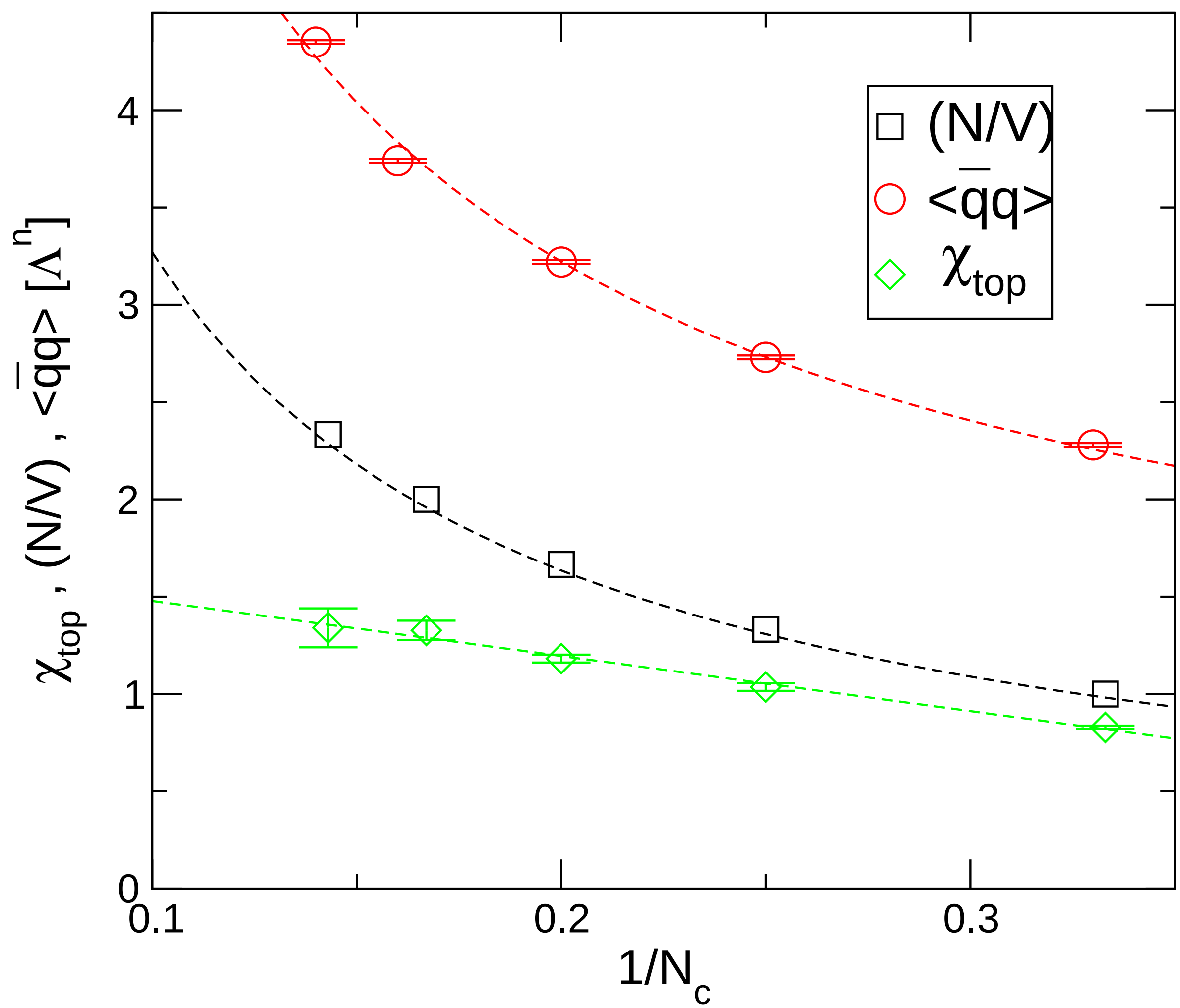
fluctuations in N are $1/N_c$ suppressed

$$\langle N^2\rangle - \langle N\rangle^2 = \frac{4}{b}\langle N\rangle \sim O(1) \qquad\qquad (\text{not } O(N_c)!)$$

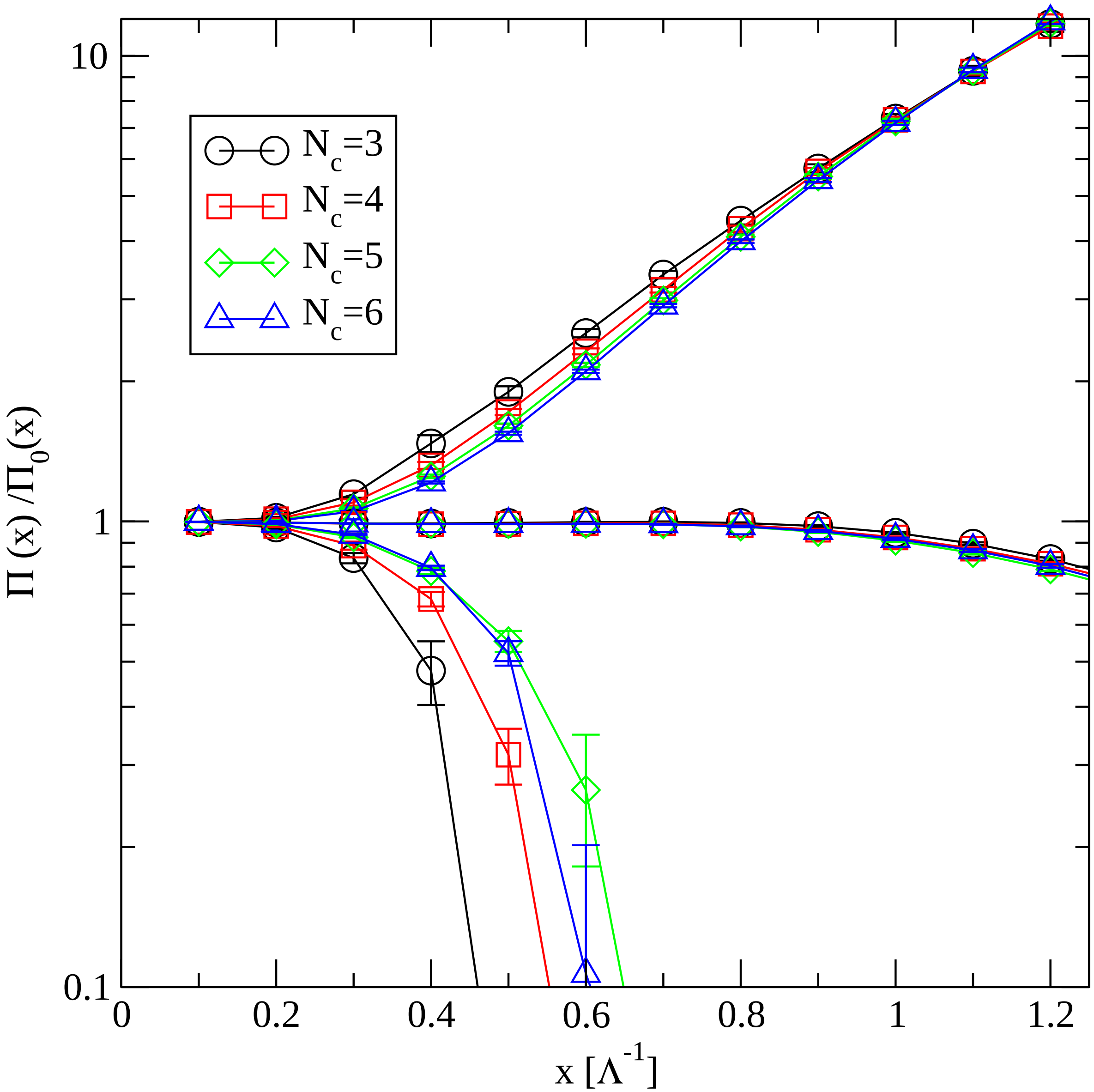
also true for topological charge

$$\langle Q^2\rangle = \frac{4}{b-r(b-4)}\langle N\rangle \sim O(1)$$

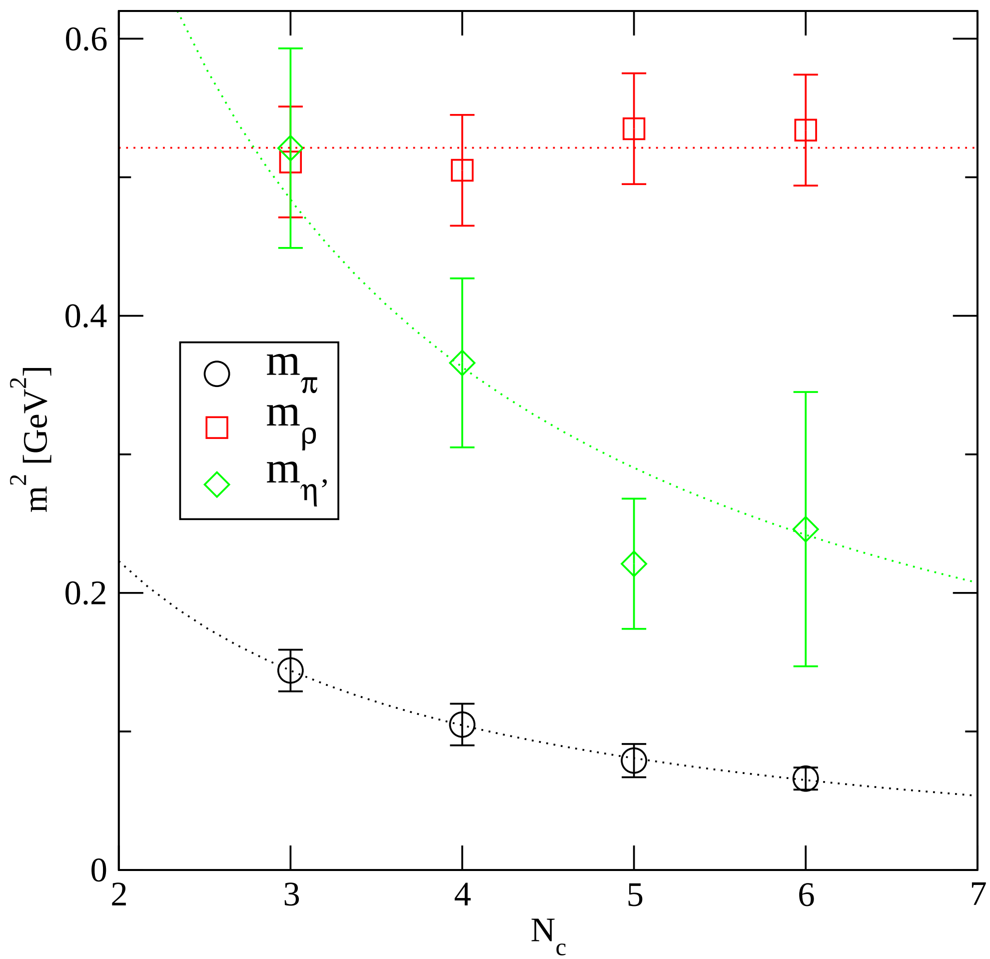
global observables: $(N/V), \langle\bar{q}q\rangle, \chi_{top}$



meson correlation functions (π, ρ, η')



meson masses: $m_\pi^2, m_\rho^2 \sim 1,\ m_{\eta'}^2 \sim 1/N_c$



Note: $m_{\eta'}^2 \sim 1/N_c$ even though $(N/V) \sim N_c$

Summary

- instantons provide i) simple picture of chiral symmetry breaking and ii) successful phenomenology of non-perturbative effects in QCD correlation functions

- nice example for semi-classical instanton liquid: QCD at large baryon density.

can be studied on the lattice

- instanton liquid can have a smooth large N_c limit

$$\left(\frac{N}{V}\right) = O(N_c), \qquad \chi_{top} = O(1), \qquad m^2_{\eta'} = O(1/N_c)$$