Instantons, Large N_c, and Holographic Models of QCD

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Outline

- Introduction: Why Instantons?

  $U(1)_A$ puzzle, topology, and instantons

- Instantons and the large N_c limit

  smooth large N_c limit? Witten-Veneziano relation?

- $U(1)_A$ problem in a holographic model of QCD

  Relation to Instantons?

$U(1)_A$ Puzzle

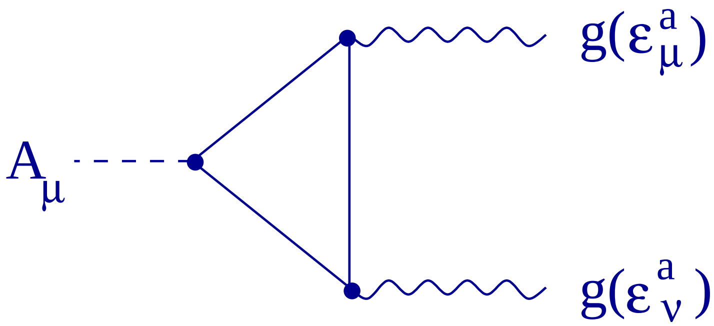
QCD has a $U(1)_A$ symmetry $\psi_L \to e^{i\phi}\psi_L,\ \psi_R \to e^{-i\phi}\psi_R$

spontaneously broken $\langle \bar{\psi}_L \psi_R \rangle = \Sigma \neq 0$

Goldstone Theorem: massless Goldstone boson $m_{\eta'} \to 0$ $(m_q \to 0)$

But: η' is heavy, $m_{\eta'}^2 \gg m_q \Lambda_{QCD}$

Resolution: axial anomaly



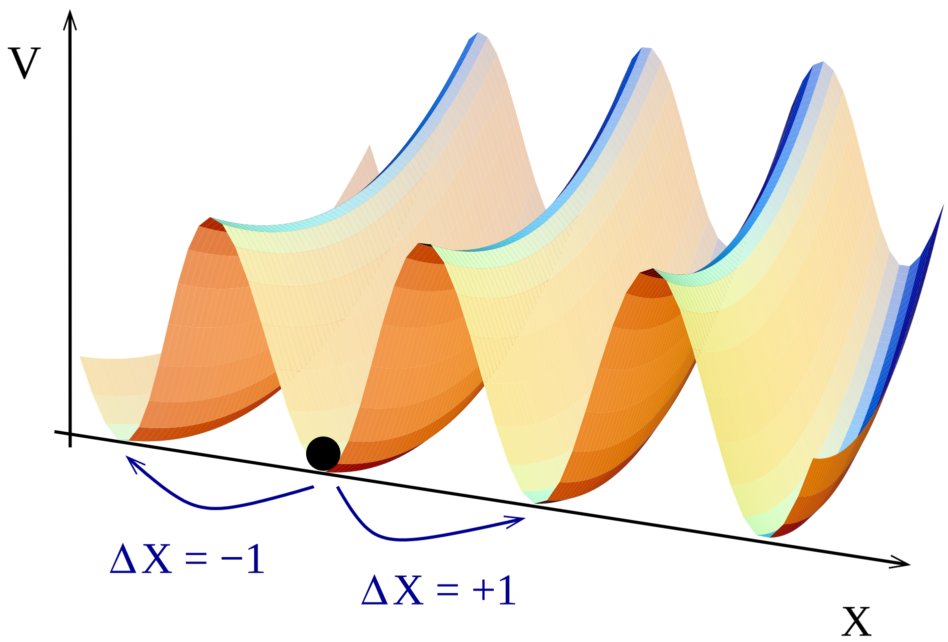
$$\partial_\mu A^\mu = \frac{N_f}{16\pi^2} G^a_{\mu\nu} \tilde{G}^a_{\mu\nu}$$

But: RHS is a total divergence $G\tilde{G} \sim \partial^\mu K_\mu$

$\Rightarrow$ Topology is important

Topology in QCD

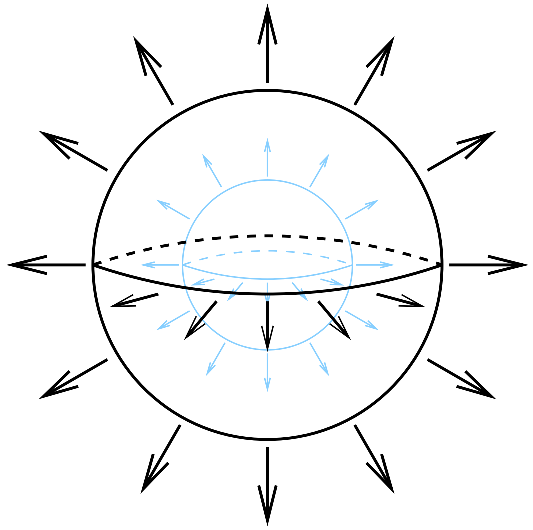
classical potential is periodic in variable X



$$X = \int d^3x\, K_0(x,t)$$

$$\partial^\mu K_\mu = \frac{1}{32\pi^2} G^a_{\mu\nu} \tilde{G}^a_{\mu\nu}$$

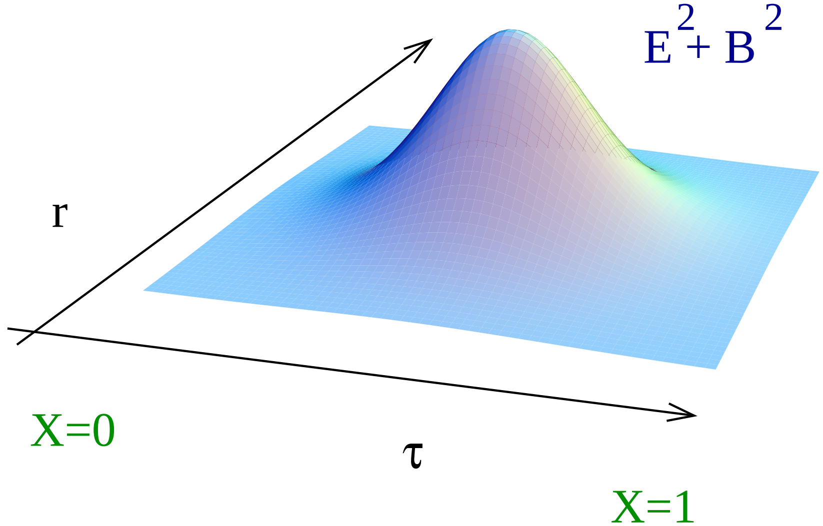
classical minima correspond to pure gauge configurations



$$A_i(x) = iU^\dagger(x)\partial_i U(x)$$

$$E^2 = B^2 = 0$$

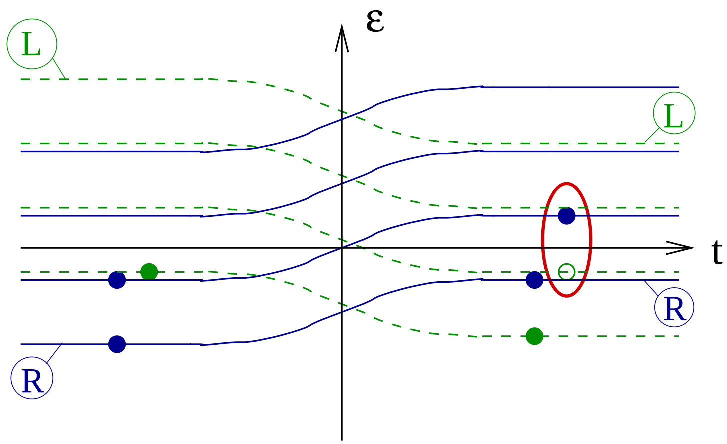
classical tunneling paths: Instantons

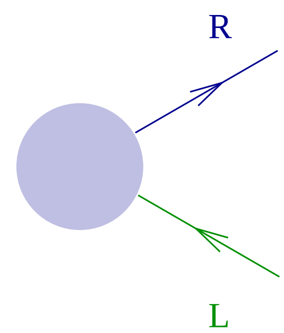


$$A^a_\mu(x) = 2\frac{\eta_{a\mu\nu}x_\nu}{x^2+\rho^2},$$

$$G^a_{\mu\nu}\tilde{G}^a_{\mu\nu} = \frac{192\rho^4}{(x^2+\rho^2)^4}.$$

Dirac spectrum in the field of an instanton

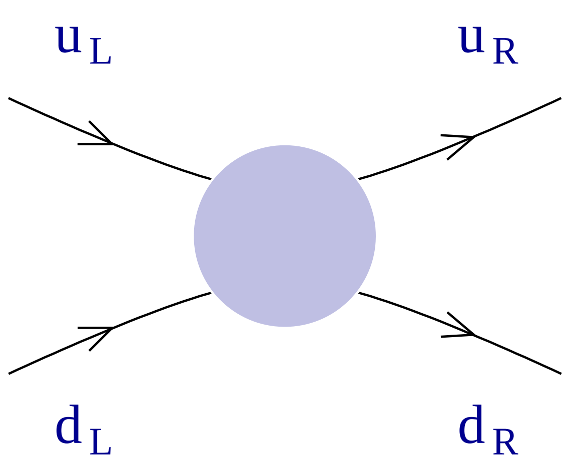




axial charge violation:

$$\Delta Q_A = 2$$

Instanton induced quark interaction ($N_f = 2$)



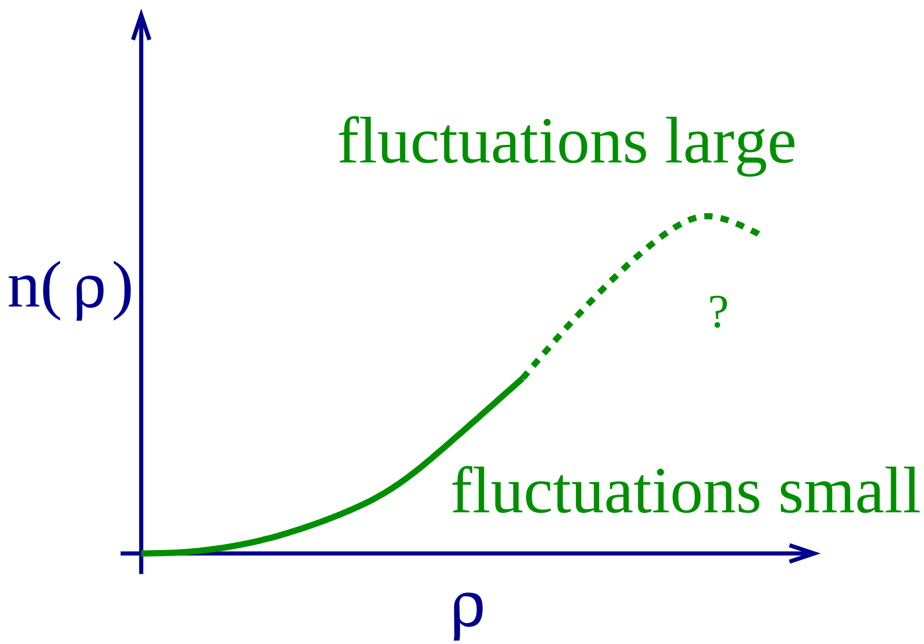
$$\mathcal{L} = G \det_f (\bar{\psi}_{L,f} \psi_{R,g})$$

$$G = \int d\rho\, n(\rho)$$

violates $U(1)_A$ but preserves $SU(2)_{L,R}$

… and contributes to the η' mass

Tunneling rate (barrier penetration factor)



$$n(\rho) \sim \exp\left[-\frac{8\pi^2}{g^2(\rho)}\right] \sim \rho^{b-5}$$

QCD at large N_c

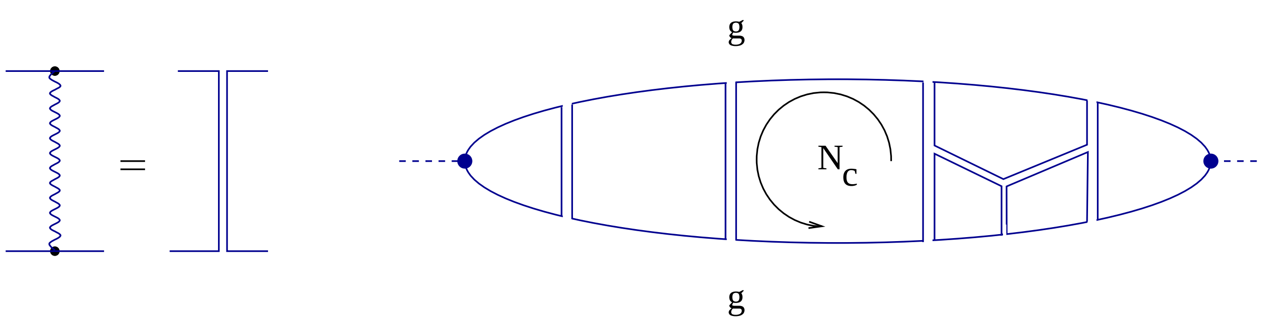
QCD $(m = 0)$ is a parameter free theory. Very beautiful.

But: No expansion parameter

't Hooft: Consider $N_c \to \infty$ and use $1/N_c$ as a small parameter

$$N_c \to \infty \qquad \Rightarrow \qquad \text{classical master field}$$

keep Λ_{QCD} fixed $\Rightarrow g^2 N_c = const$



Could the master field be a multi-instanton?

$$\text{Witten : No!} \qquad dn \sim \exp\left(-\frac{1}{g^2}\right) \sim \exp\left(-N_c\right)$$

$U(1)_A$ anomaly at large N_c

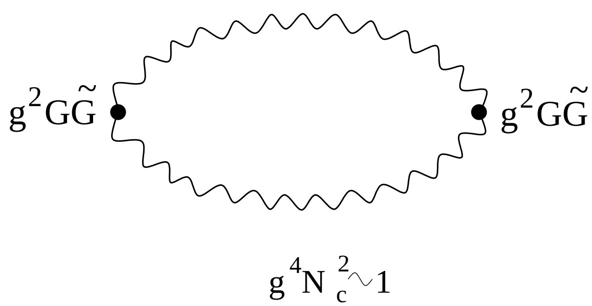
consider θ term $\mathcal{L} = \frac{ig^2\theta}{32\pi^2} G^a_{\mu\nu}\tilde{G}^a_{\mu\nu}$

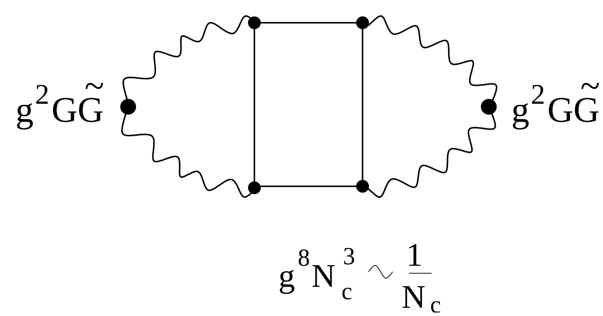
no θ dependence in perturbation theory.

Witten: *non-perturbative θ dependence* $\chi_{top} = \left.\frac{d^2E}{d\theta^2}\right|_{\theta=0} \sim O(1)$

massless quarks: topological charge screening $\lim_{m\to 0}\chi_{top} = 0$

How can that happen? Fermion loops are suppressed!





Witten: $f_\pi^2 m_{\eta'}^2 = 2N_f\,\chi_{top}(no\ quarks)$

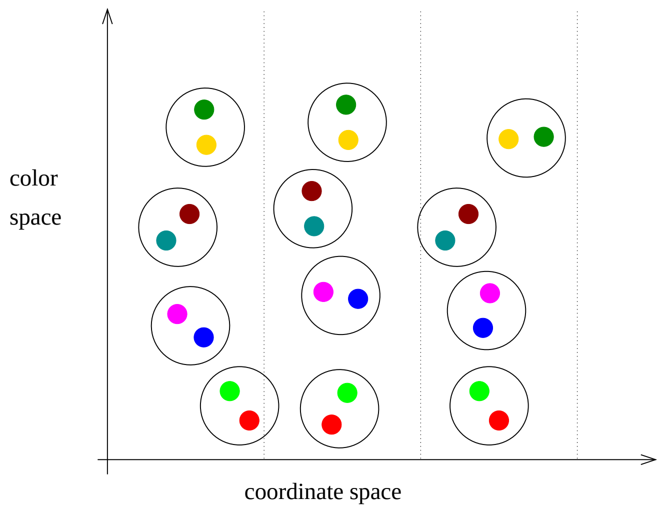
η' has to become light $\Rightarrow \quad m_{\eta'}^2 = O(1/N_c)$

First Part

The instanton ensemble at large N_c

Instantons at large N_c

semi-classical ensemble of instantons at large N_c



instantons are $N_c = 2$ configurations

$$\left(\frac{N}{V}\right) = O(N_c)$$

$$\Rightarrow \epsilon_{vac} = O(bN_c) = O(N_c^2)$$

instantons are semi-classical

$$\rho \simeq \rho^* = O(1) \qquad S_{inst} = O(N_c)$$

density $dn \sim \exp(-S_{inst}) = O(\exp(-N_c))$?

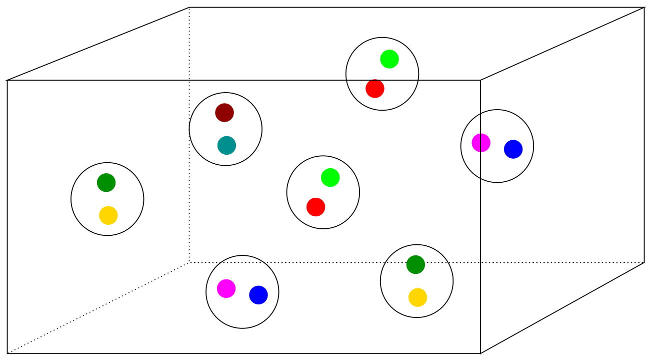
$NO!$ *large entropy* $dn \sim \exp(+N_c)$

topological susceptibility $\chi_{top} \simeq (N/V) = O(N_c)$?

$NO!$ *fluctuations suppressed* $\chi_{top} = O(1)$

Instanton ensemble

instanton ensemble



partition function

$$Z = \frac{1}{N_I! N_A!} \prod_I^{N_I+N_A} \int [d\Omega_I \, n(\rho_I)] e^{-S_{int}}$$

complicated N_c dependence

$$n(\rho) = C_{N_c} \left(\frac{8\pi^2}{g^2}\right)^{2N_c} \rho^{-5} \exp\left[-\frac{8\pi^2}{g(\rho)^2}\right]$$

$$C_{N_c} = \frac{0.47\exp(-1.68N_c)}{(N_c-1)!(N_c-2)!} \qquad \frac{8\pi^2}{g^2(\rho)} = -b\log(\rho\Lambda), \qquad b = \frac{11}{3}N_c$$

$$S_{int} = -\frac{32\pi^2}{g^2}|u|^2 \left\{ \frac{\rho_I^2\rho_A^2}{R_{IA}^4} \left(1 - 4\cos^2\theta\right) + S_{core} \right\}$$

complicated ensemble, size distribution

$$n(\rho) \sim \begin{cases} \exp(-N_c) & \rho < \rho^* \\ const & \rho \sim \rho^* \end{cases} \qquad \rho^* \sim O(1)$$

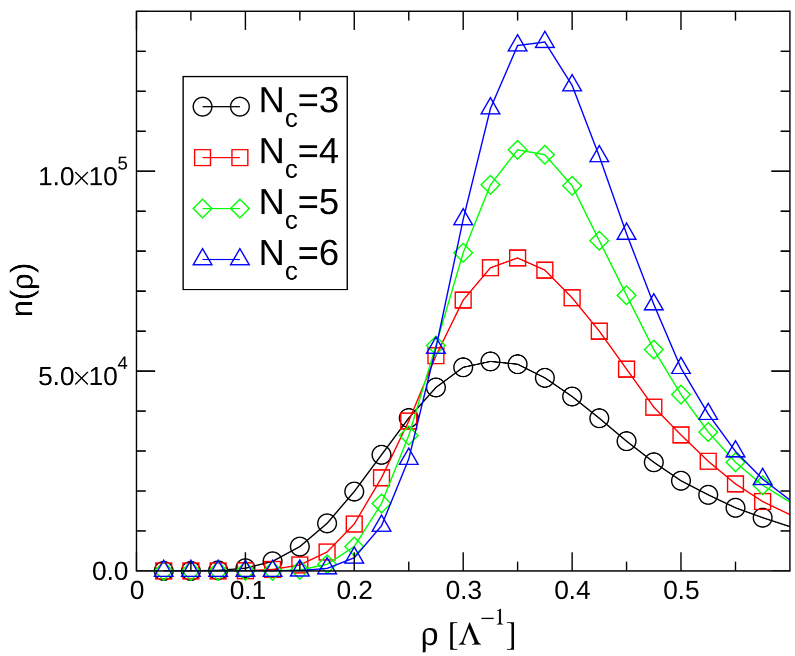
total density determined by interactions

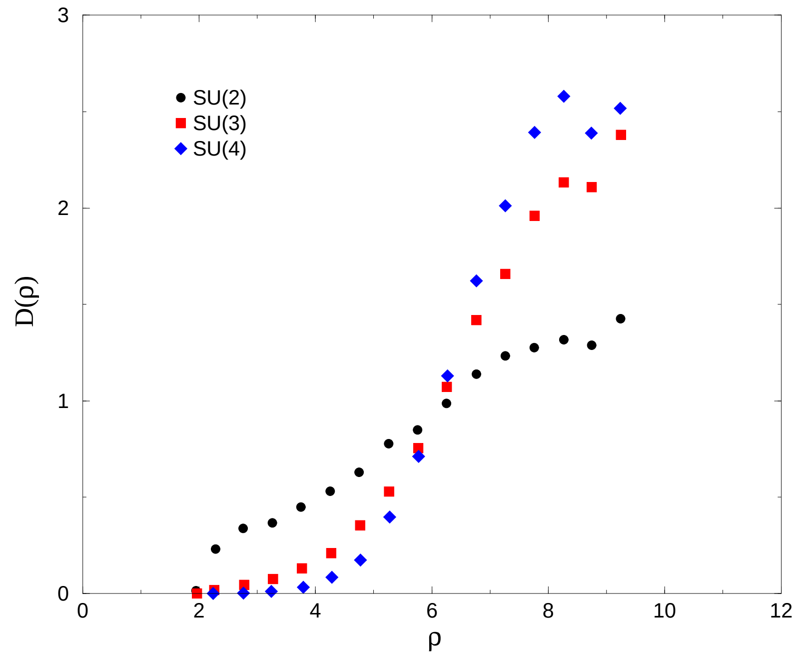
$$\begin{array}{ccccccccc} S(1-body) & \sim & & & S(2-body) & & \\ N_c & \sim & N_c & \times & \frac{1}{N_c} & \times & \left(\frac{N}{V}\right) \\ classical & \sim & classical & \times & color\ overlap & \times & density \end{array}$$

conclude

$$\boxed{\left(\frac{N}{V}\right) = O(N_c)}$$

instanton size distribution





B. Lucini, M. Teper

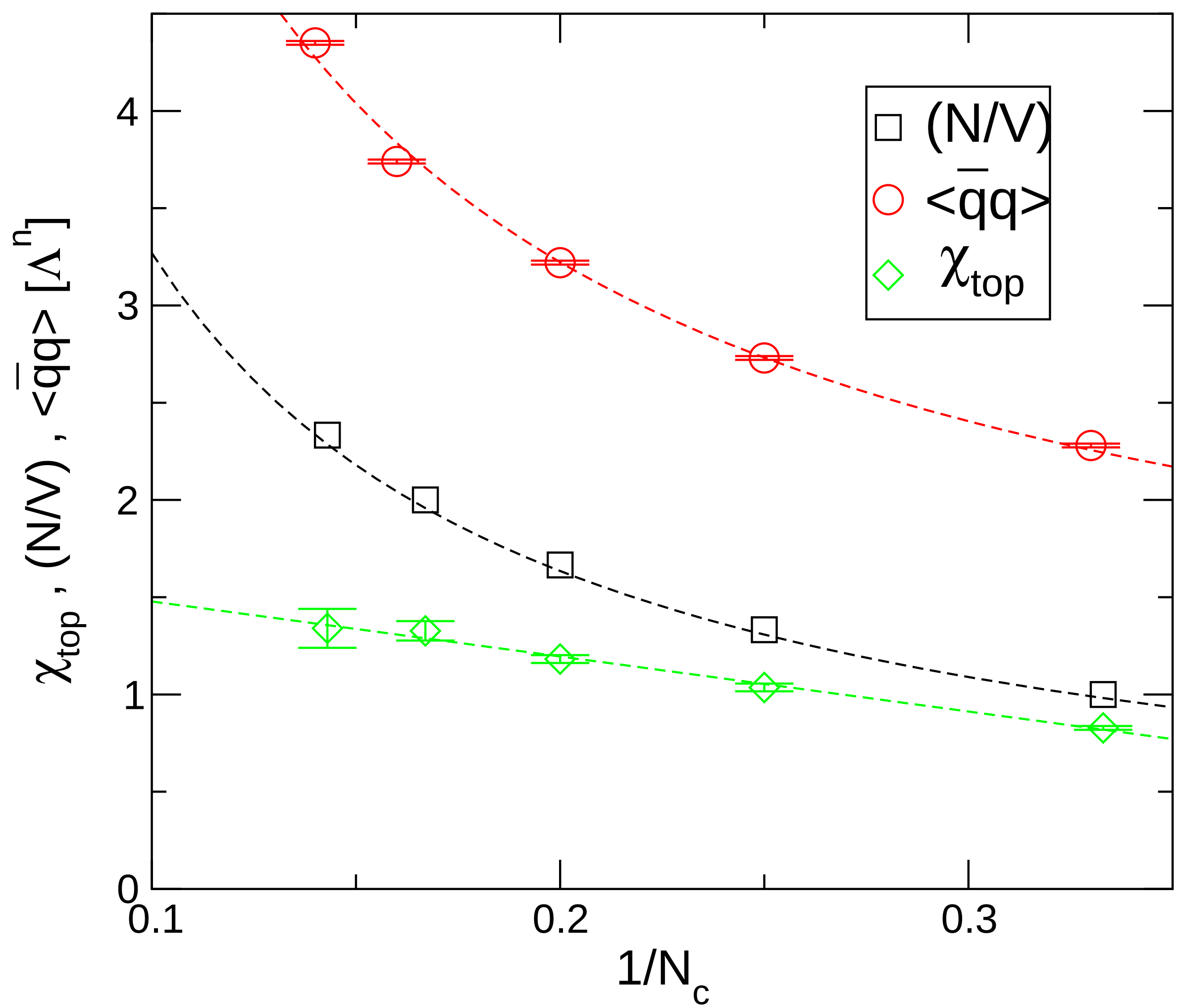
fluctuations in N are $1/N_c$ suppressed

$$\langle N^2\rangle - \langle N\rangle^2 = \frac{4}{b}\langle N\rangle \sim O(1) \qquad (\text{not } O(N_c)!)$$

also true for topological charge

$$\langle Q^2\rangle = \frac{4}{b-r(b-4)}\langle N\rangle \sim O(1)$$

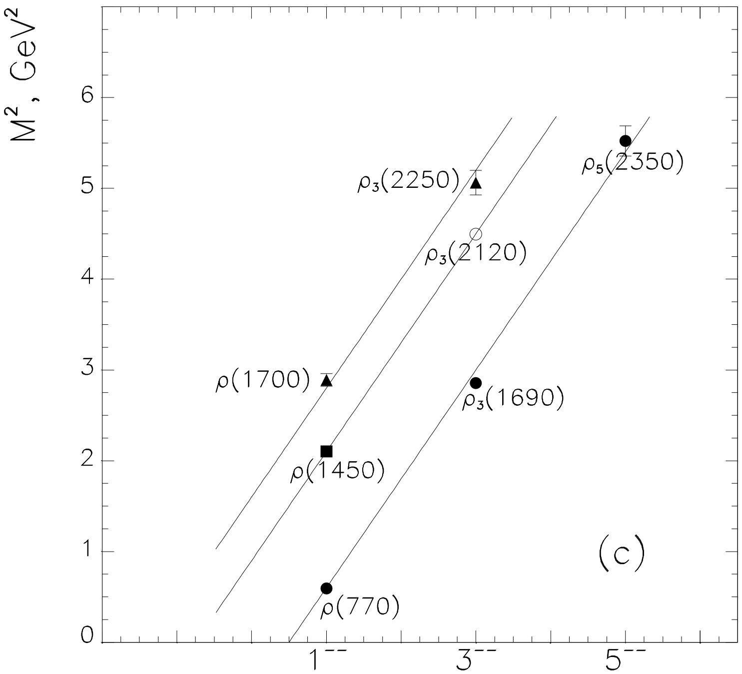
global observables: $(N/V), \langle \bar{q}q \rangle, \chi_{top}$



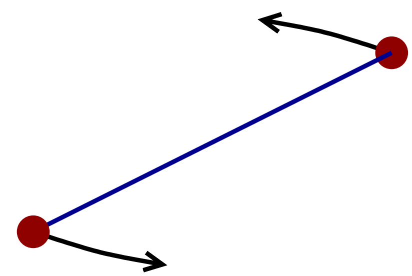
Second Part

Holographic QCD

QCD and Strings: Pre-History



Anisovich et al. (2000)



$$J \sim \alpha' M^2$$

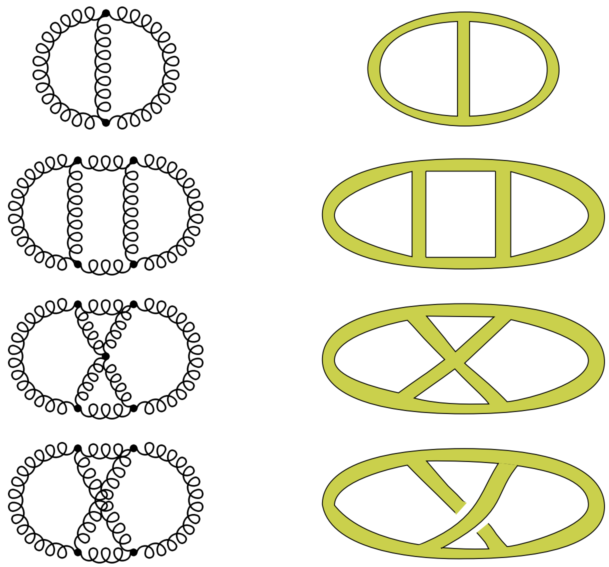
$$T_{str} = \frac{1}{2\pi\alpha'} = 1 \text{ GeV/fm}$$

Regge trajectories, pomerons, dual models, Veneziano amplitude, ...

faded away with the advent of QCD

QCD and Strings: Pre-History

'*t* Hooft: large N_c expansion ($\lambda = g^2 N_c = const$)

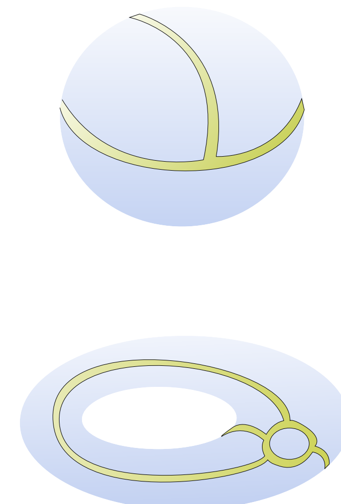


$\sim (g^2_{\mathrm{YM}})^{3-2}\, N_c^3 = \lambda\, N_c^2$

$\sim (g^2_{\mathrm{YM}})^{6-4}\, N_c^4 = \lambda^2\, N_c^2$

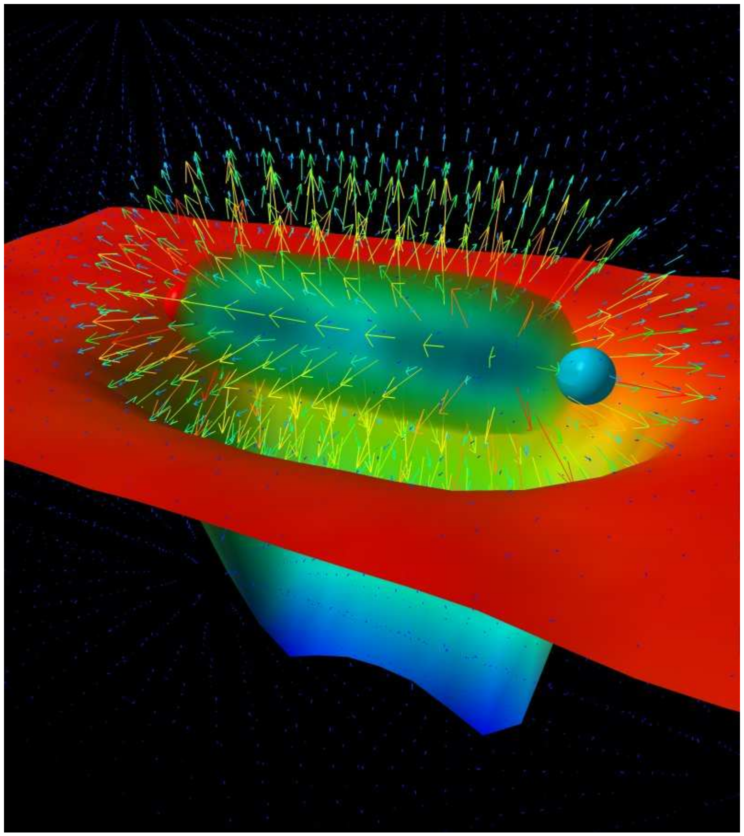
$\sim (g^2_{\mathrm{YM}})^{8-5}\, N_c^5 = \lambda^3\, N_c^2$

$\sim (g^2_{\mathrm{YM}})^{6-4}\, N_c^2 = \lambda^2$

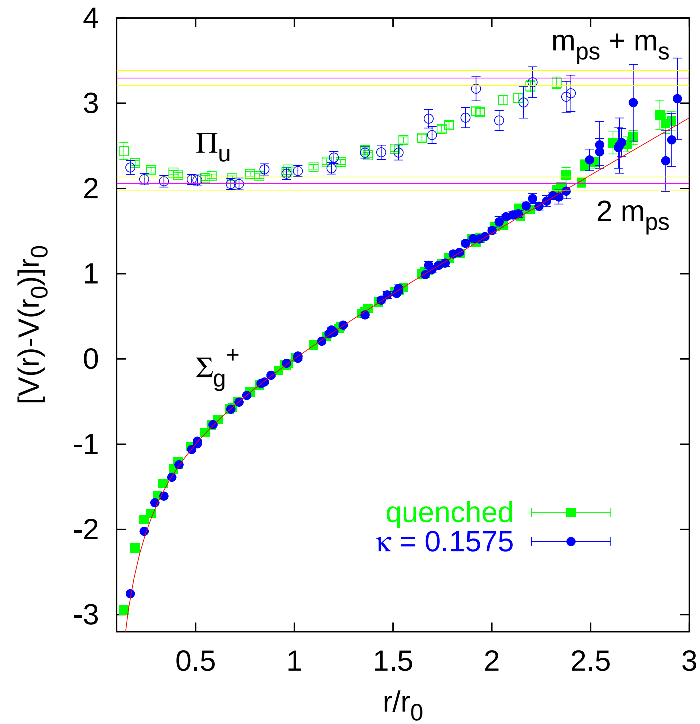


Large N_c limit: topological expansion (string theory?)

QCD and Strings: Pre-History



Leinweber (2001)



Bali (2001)

QCD: flux tubes and string potentials

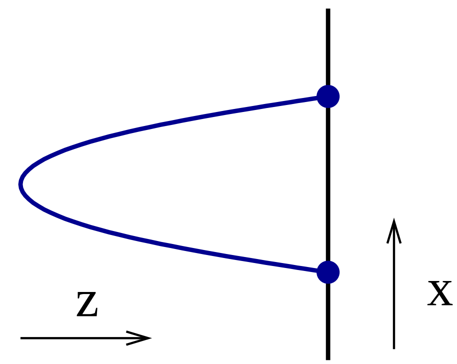
QCD and Strings: Holography

The AdS/CFT duality relates

$\mathcal{N} = 4$ large N_c gauge theory in 4 dimensions	$\Leftrightarrow$	type IIb string theory on $AdS_5 \times S_5$
correlation fcts of gauge invariant operators	$\Leftrightarrow$	boundary correlation fcts of AdS fields

$$\langle \exp \int dx \; \phi_0 \mathcal{O} \rangle =$$

$$Z_{string}[\phi(\partial AdS) = \phi_0]$$



The correspondence is simplest at strong coupling $g^2 N_c$

strongly coupled gauge theory $\Leftrightarrow$ classical string theory

Maldacena (1997)

$\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory

Fields: Gluons, Gluinos, Higgses; all in the adjoint of $SU(N_c)$

$$\mathcal{L} = \frac{1}{4}(F^a_{\mu\nu})^2 + \bar{\lambda}^a_A \sigma^\mu (D_\mu \lambda^A)^a + (D_\mu \Phi_{AB})^a (D_\mu \Phi^{AB})^a + \ldots$$

$$A^a_\mu \qquad \lambda^a_A \ (\bar{4}_R) \qquad \Phi^a_{AB} \ (6_R)$$

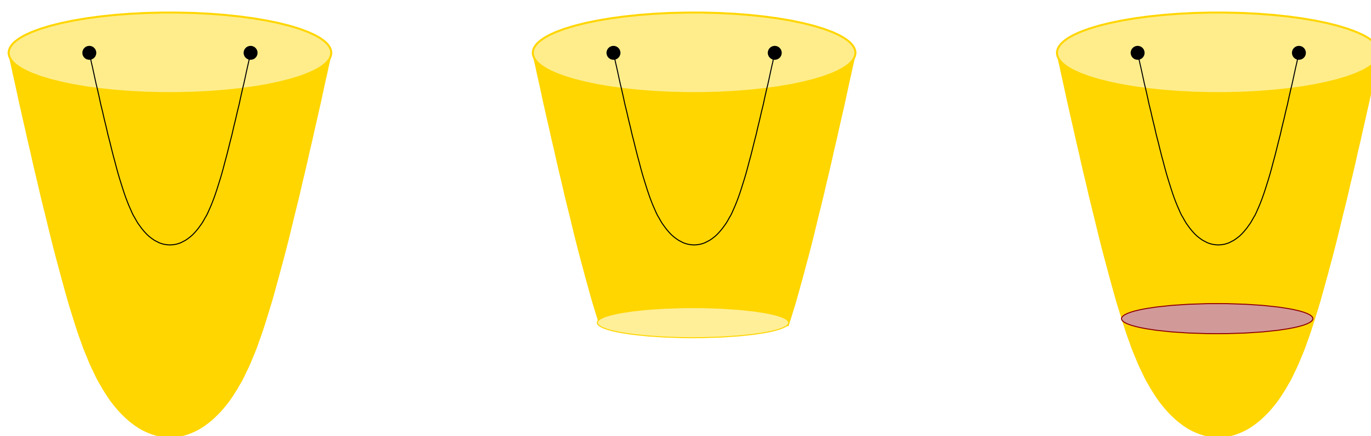
Global symmetries: Conformal and $SU(4)_R$

$$SO(4,2) \times SU(4)_R$$

Properties: Conformal $\beta(g) = 0$, extra scalars, no fundamental fermions, no chiral symmetry breaking, no confinement

QCD and Strings: Towards QCD

non-AdS/non-CFT correspondence, a.k.a "AdS/QCD"



AdS: conformal | cutoff AdS | AdS black hole

Example: 5d Gauge Field

Consider five dimensional action ($F_{\mu\nu} = \partial_\mu V_\nu - \partial_\mu V_\nu - i[V_\mu, V_\nu]$)

$$S_5 = -\frac{1}{4g_5^2} \int d^5x \,\sqrt{g} F^a_{\mu\nu} F^{a\,\mu\nu}$$

$$ds^2 = \frac{1}{z^2}(-dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu) \qquad\qquad 0 \leq z \leq z_m$$

Equ. of motion: linearize, F-trafo in x^μ, $V_5 = 0$ gauge

$$z\,\partial_z \left(\frac{1}{z} \partial_z V^a_\mu \right) + q^2 V^a_\mu = 0$$

Using equ. of motion

$$S_5 = -\frac{1}{2g_5^2} \int d^4x \,\frac{1}{z}\, V^a_\mu \partial_z V^{a\,\mu} \Big|_{z=0}$$

1. Find solution with $V(z \to 0, x) = V_0(x)$
2. Compute action $S_5[V]$.
3. Take functional derivative $\Pi_{\mu\nu} = (\delta^2 S_5)/(\delta V_0^\mu \delta V_0^\nu)$

Write $V^\mu(q,z) = V_0^\mu(q) V(q,z)$ with $V(q,0) = 1$. Then

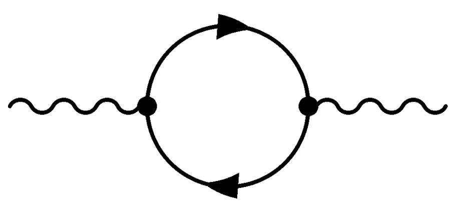
$$\Pi(Q^2) = -\frac{1}{g_5^2 Q^2} \left. \frac{\partial_z V(q,z)}{z} \right|_{z=0} \qquad Q^2 = -q^2$$

The required solution is

$$V(q,z) \simeq 1 + \frac{1}{2} Q^2 z^2 \log(Qz) + \ldots$$

$$\Pi(Q^2) = -\frac{1}{2g_5^2} \log(Q^2)$$

match to QCD:



$$g_5^2 = \frac{12\pi^2}{N_c}$$

AdS/CFT Dictionary

4d field theory	$\leftrightarrow$	5d gravitational theory
generating functional $W[\phi_0]$	$\leftrightarrow$	boundary action $S[\phi_0]$
operator $O(x)$ coupled to $\phi_0(x)$	$\leftrightarrow$	field $\phi(z, x)$ (boundary val $\phi_0(x)$)
dimension, spin of O	$\leftrightarrow$	5-d mass of ϕ
symmetry breaking:	$\leftrightarrow$	non-normalizable mode:
$\langle O \rangle \neq 0$ as $\phi_0 \to 0$		$\phi \sim \phi_0 z^{d_\phi} + A z^{d_O}$
large N_c	$\leftrightarrow$	weak coupling g_5
large Q	$\leftrightarrow$	small z

The model: Chiral Symmetry Breaking

5-d action with vector and scalar fields

$$S = \int d^5x \sqrt{g} \left\{ -\frac{1}{4g_5^2} \mathrm{Tr} \left(F_L^2 + F_R^2 \right) + \mathrm{Tr} \left(|DX|^2 + 3|X|^2 \right) \right\}$$

Erlich et al. (2005), DaRold et al (2005)

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ (for L/R)

$X \to LXR$ and $D_\mu X = \partial_\mu X - iA_{L\mu}X + iXA_{R\mu}$

Chiral symmetry breaking

$$\langle X_{ij} \rangle = \sigma_{ij} z^3 + M_{ij} z,$$

Pseudoscalar fields

$$X_{ij} = \langle X_{ij} \rangle \exp(i\pi^a t^a),$$

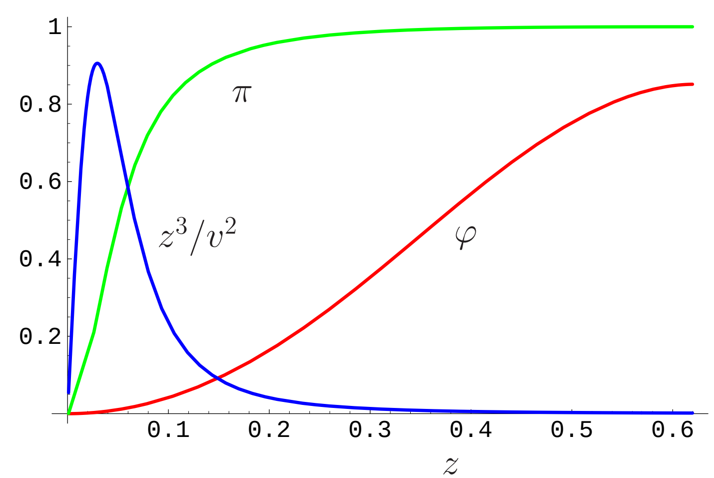
Chiral Symmetry Breaking and the Pion

Mixing between axial and pseudoscalars: $A_\mu = A_{\mu\perp} + \partial_\mu \varphi$

$$\begin{aligned}
\partial_z\left(\frac{1}{z}\partial_z A_\perp^a\right) + \frac{q^2}{z}A_\perp^a - \frac{g_5^2 v^2}{z^3}A_\perp^a &= 0 \\
\partial_z\left(\frac{1}{z}\partial_z \varphi^a\right) + \frac{g_5^2 v^2}{z^3}(\pi^a - \varphi^a) &= 0\,. \quad [v(z) = mz + \sigma z^3] \\
-q^2\partial_z\varphi^a + \frac{g_5^2 v^2}{z^2}\partial_z \pi^a &= 0\,.
\end{aligned}$$

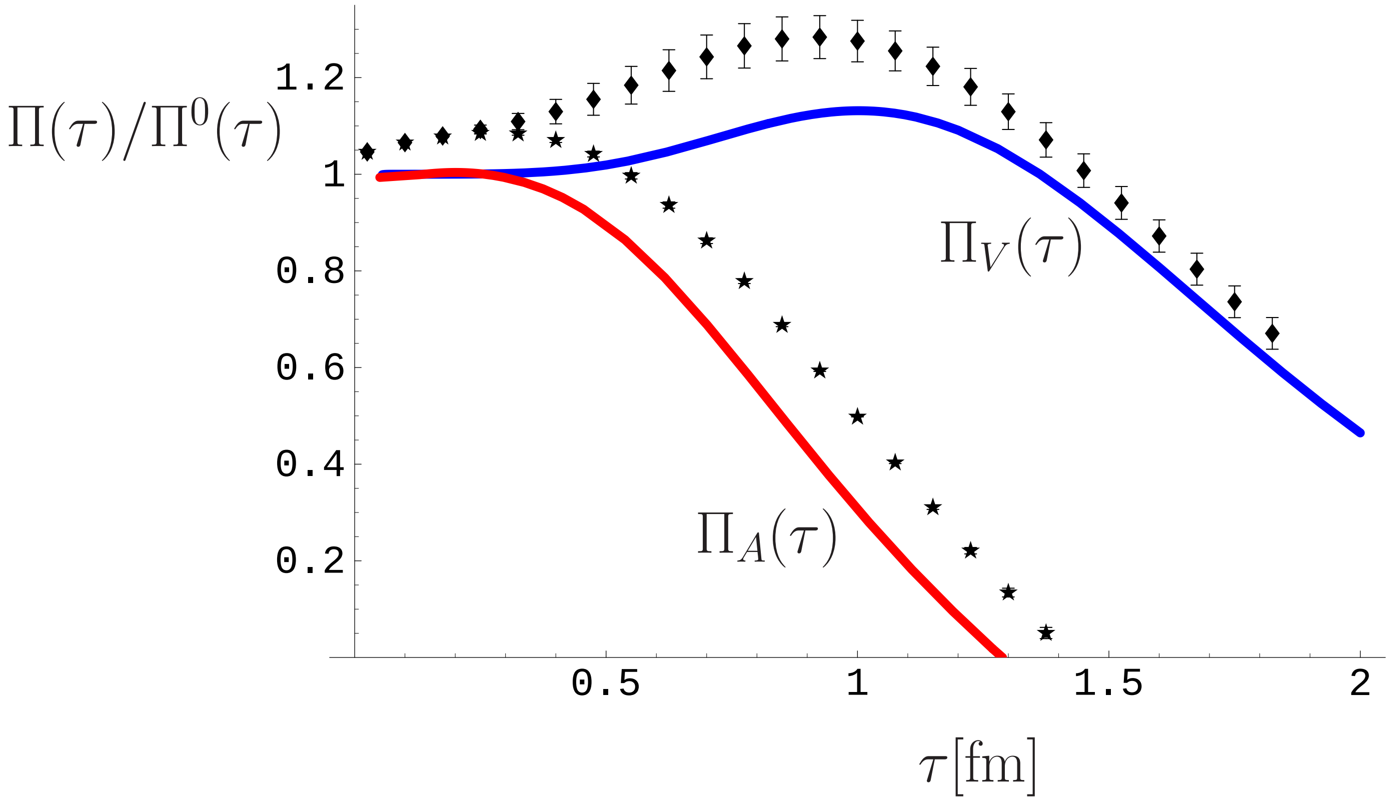
Goldstone mode: Define $f_\pi^2 = -\frac{1}{g_5^2}\left.\frac{\partial_z A(z,0)}{z}\right|_{z=0}$ (b.c. $A(0,q) = 1$)

$$\begin{aligned}
\phi(z) &= A(0,z) - 1 \;\; (\pi(z) = 1) \\
\pi(z) &= q^2 \int_0^z d\bar{z}\, \frac{\bar{z}^3}{v(\bar{z})^2}\frac{\partial_{\bar{z}} A(0,\bar{z})}{g_5^2\,\bar{z}}
\end{aligned}$$



$$m_\pi^2 f_\pi^2 = 2m\sigma$$

Vector/Axialvector Correlation Funtions



Data: V/A spectral functions from $\tau \longrightarrow \nu_\tau + hadrons$ (Aleph)

Flavor Singlet Axialvector

Add singlet field $Y = \langle Y \rangle e^{ia}$ (pseudoscalar glueball, "axion")

$$S = \int d^5x \sqrt{g} \left\{ \frac{1}{2}|DY|^2 + \frac{\kappa_0}{2} \left(Y^{N_f} \det(X) + h.c. \right) \right\}$$

Katz & Schwartz (2007)

$$Y = \langle Y \rangle = c + \Xi z^4 \qquad c \sim g^2, \;\; \Xi \sim G^2$$

QCD axial anomaly: $\partial^\mu j^5_\mu = 2N_f \, \frac{g^2}{32\pi^2} G\tilde{G}$

$$\int d^4x \, e^{iqx} \, \langle \partial^\mu j^5_\mu(x) \partial^\mu j^5_\mu(0) \rangle = (2N_f)^2 \frac{\alpha_s^2}{8\pi^4} Q^4 \log(Q^2) + \ldots$$

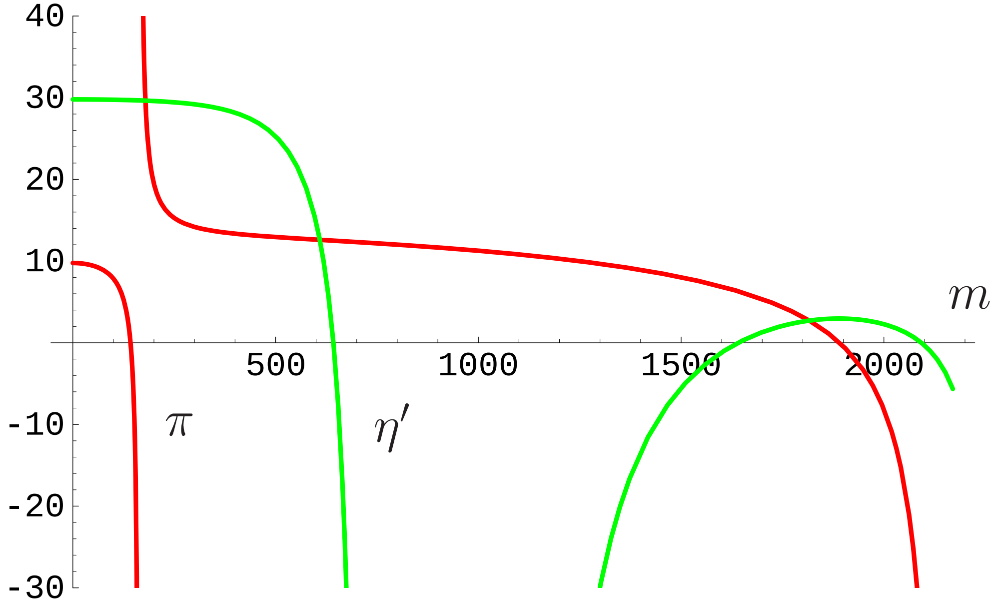
Matching: axial fields $A^0_\mu = A^0_{\mu\perp} + \partial_\mu \varphi^0$ and a

$$\int d^4x \, e^{iqx} \, \langle \partial^\mu j^5_\mu(x) \partial^\mu j^5_\mu(0) \rangle = -\frac{Q^2}{g_5^2} \left. \frac{\partial_z \phi^0(z)}{z} \right|_{z=0}$$

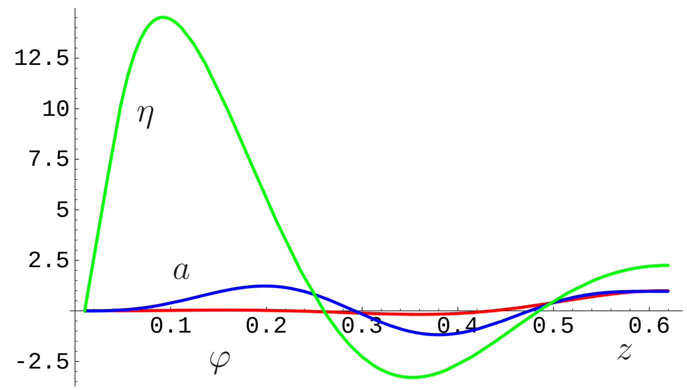
$$c = \sqrt{2N_f} \, \frac{\alpha_s}{2\pi^2} \qquad \Xi \to \text{OPE} \qquad \kappa \text{ free}$$

Spectrum: Pseudoscalar Singlets

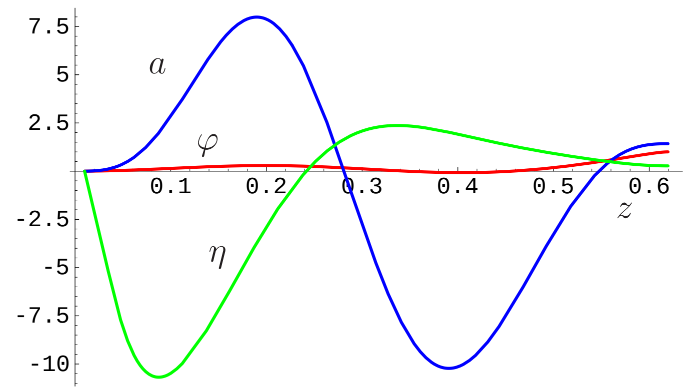
eigenvalues of (φ^0, η^0, a) and (φ, π) system



excited state (mostly $\bar{q}q$)

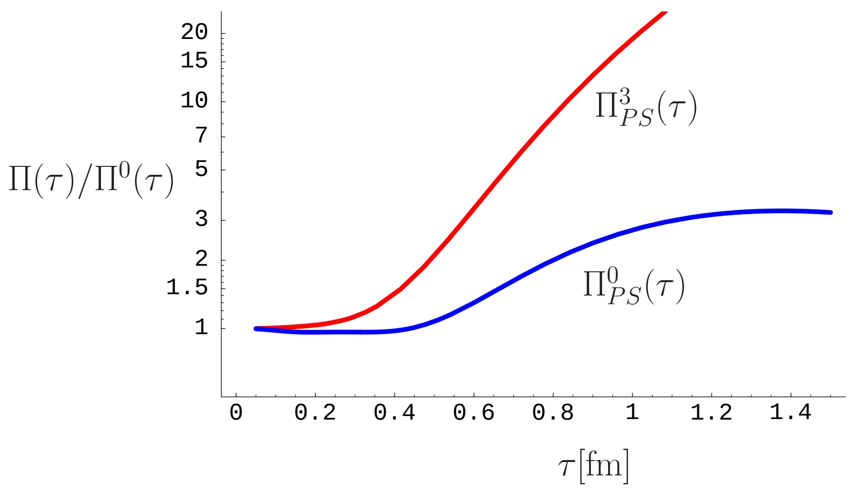


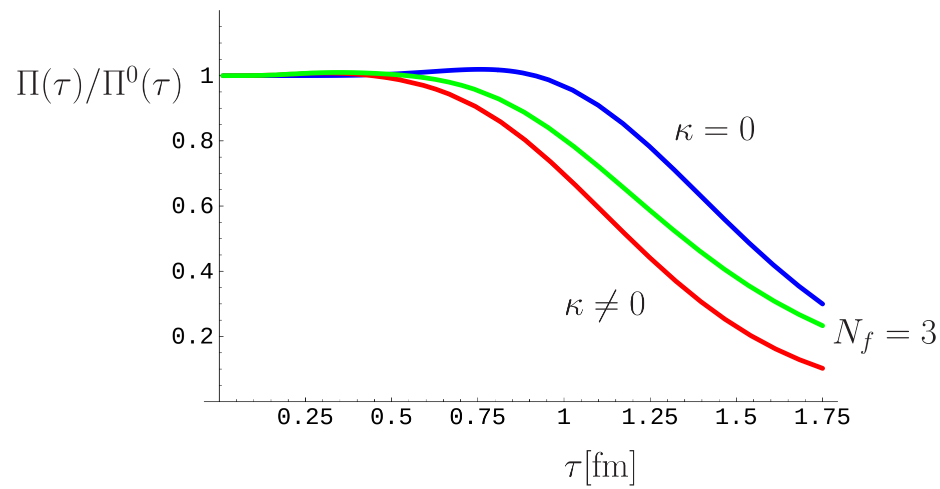
excited state (mostly $G\tilde{G}$)



Pseudoscalar Correlation Functions

$$\Pi(x) = \left\langle \bar{q}t^a\gamma_5 q(x)\bar{q}t^b\gamma_5 q(0)\right\rangle \qquad \Pi(x) = \left\langle g^2 G\tilde{G}(x) g^2 G\tilde{G}(0)\right\rangle$$





$$m_{\eta'} \simeq 660 \text{ MeV}$$

$$m_{0^{+-}} \simeq 1400 \text{ MeV}$$

$$\langle 0|g^2 G\tilde{G}|\eta'\rangle \neq 0$$

Toplogical Suscetibility

Topological susceptibility

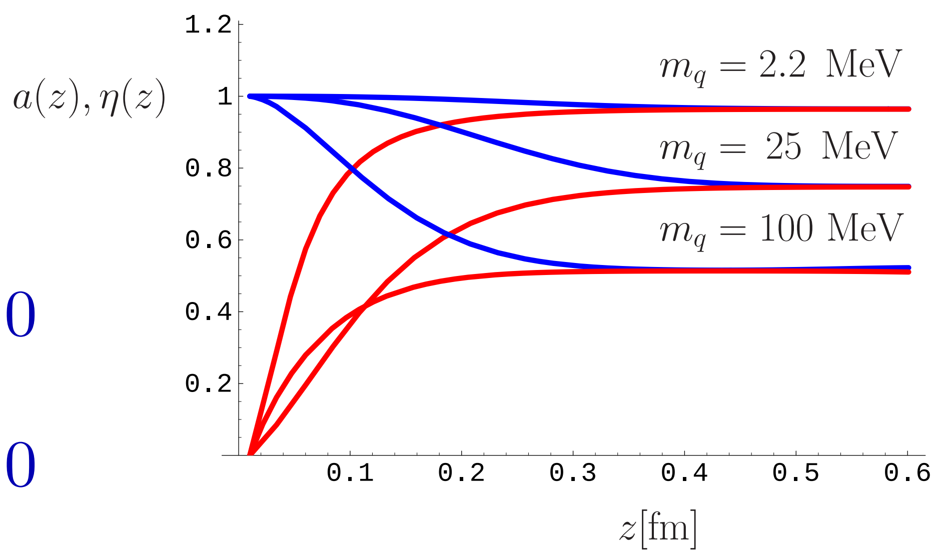
$$\chi_{top} = \lim_{V\to\infty} \frac{\langle Q_{top}^2 \rangle}{V} = -\int d^4x \, \Pi_P(x)$$

Holography: Find solution with $q^2 = 0$ and $a(0, q) = 1$

$$\chi_{top} = -\frac{c^2}{2N_f} \left. \frac{\partial_z a}{z^3} \right|_\epsilon$$

$$\partial_z \left(\frac{c^2}{z^3} \partial_z a \right) + \kappa \frac{v^{N_f}}{z^5} \left(\eta^0 - a \right) = 0$$

$$v^2 \partial_z \eta^0 + c^2 \partial_z a = 0$$



Note that $\chi_{top} \sim m_q \sigma$.

Witten-Veneziano

Pure gluodynamics

$$a(z) = \frac{N_f}{2c^2}\,\chi_{top} z^4 + \dots$$

Full QCD: (Pseudo) Goldstone modes $\eta - \varphi$

$$\eta^0(z) \simeq 1 \qquad\qquad \varphi^0(z) \simeq \frac{g_5^2}{2}\,f_\pi^2 z^2 + \dots$$

Study coupling, use perturbation theory in c ($\sim 1/N_c$)

$$m_{\eta'}^2 z^2 \partial_z \varphi^0 - g_5^2 v^2 \partial_z \eta^0 - g_5^2 c^2 \partial_z a = 0$$

Witten-Veneziano relation

$$f_{\eta'}^2 m_{\eta'}^2 = 4 N_f \chi_{top}$$

What about instantons?

Topological charge correlator: Trear κa^2 as a perturbation

$$\Pi_P(Q) = -\frac{1}{2N_f}\int_0^{z_m} \frac{dz}{z^5}\,\bar{\kappa}\,\left[\frac{1}{2}(Qz)^2 K_2(Qz)\right]^2,$$

AdS_5 measure $\times$ (Bulk-to-boundary prop)^2

Compare to instanton result

$$\Pi_P(Q) = -2\int \frac{d\rho}{\rho^5}\,d(\rho)\,\left[\frac{1}{2}(Q\rho)^2 K_2(\rho Q)\right]^2,$$

instanton measure $\times$ (F-trafo of $G\tilde{G}_I$)^2

- AdS cutoff provides instanton size cutoff
- Correspondence extends to other correlators

Positivity and all that

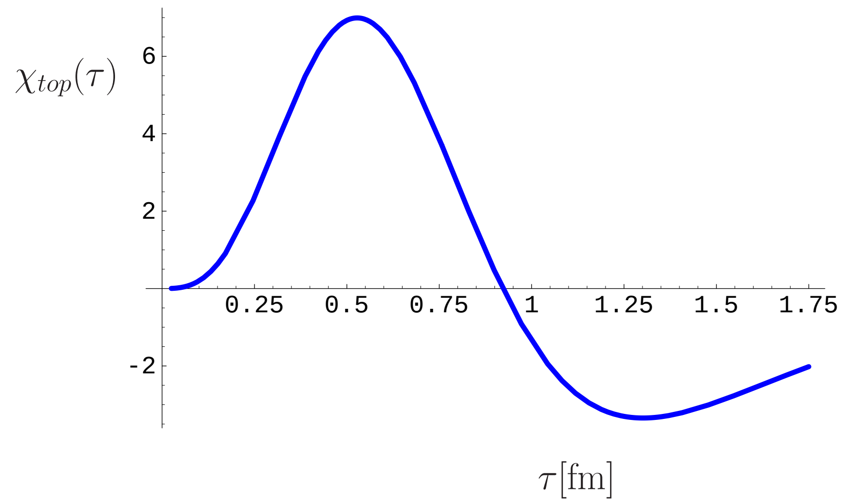
$$\chi_{top} = \lim_{V\to\infty} \frac{\langle Q_{top}^2 \rangle}{V} = -\int d^4x \, \Pi_P(x)$$

Have $\chi_{top} \geq 0$ and $\Pi_P(x) \geq 0$ (spectral positivity)

How can that be? $\Pi_P(x) \sim \alpha_s^2/x^8$ singular $\Rightarrow$ need regulator

$$\Pi_P^{reg}(x) = \Pi_P^{AdS|}(x) - \Pi_P^{AdS}(x)$$

$$\int d^4x \, \Pi_P^{reg}(x) = -\frac{c^2}{2N_f} \left. \frac{\partial_z a}{z^3} \right|_\epsilon$$



Anomaly term: $\delta\Pi_P(x) \leq 0$ $(\chi_{top} \geq 0)$

Outlook

Improved models: Asymptotic freedom? OPE?

Evans (2004), Kiritsis (2007), . . .

Top-down approach: Origin of anomaly term?

Witten (1998), Barbon, Mateos, Myers (2004), Armoni (2004)

Large N_c limit: Lattice/instanton calculations suggest that $d(\rho) \to \delta(\rho - \rho^*)$.

Teper (2003), Schäfer (2003), Shuryak (2007)

Non-zero T, μ: Better perturbative control. Holographic duals?