Instantons, the η' Mass, and the Large N_c Limit

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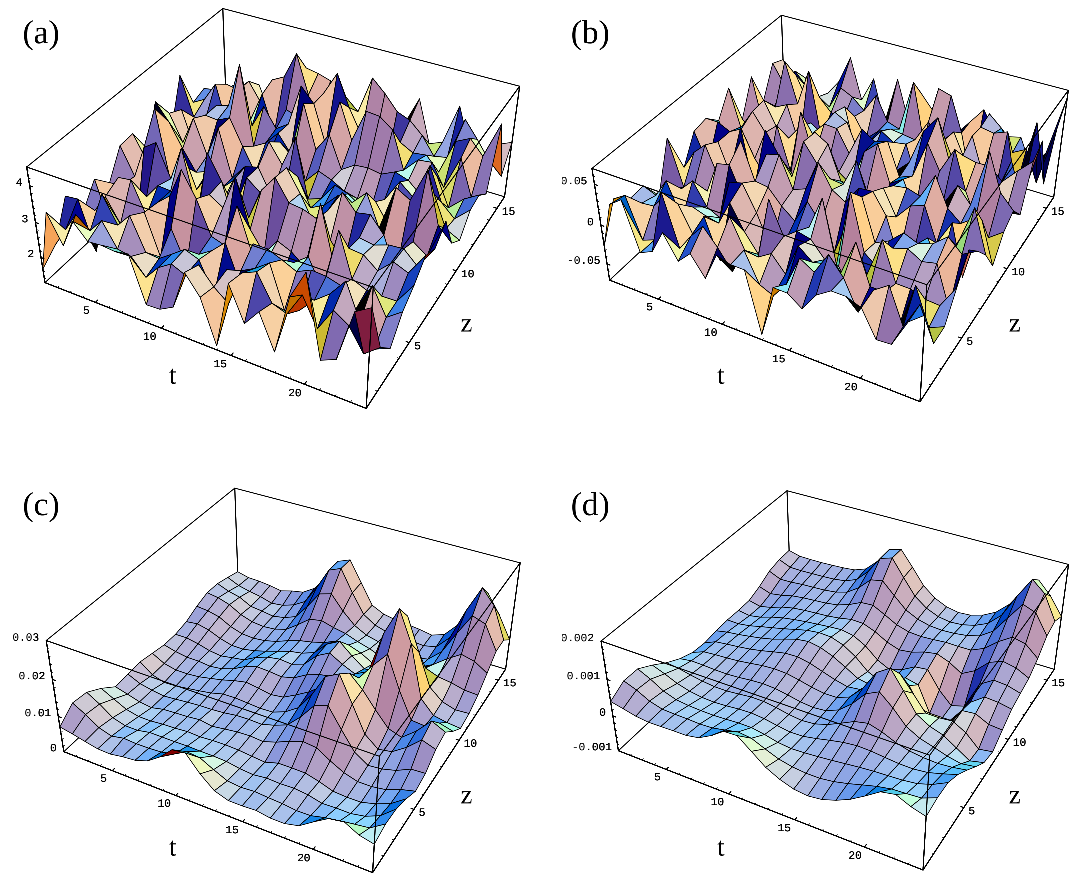


Outline

- Introduction: Why Instantons?

  $U(1)_A$ puzzle, topology, and instantons

- Instantons, the η', and the Witten-Veneziano relation: large baryon density

  $\rho \ll \Lambda_{QCD}^{-1}$: semi-classical approximation under control

- Instantons and the large N_c limit

  smooth large N_c limit? Witten-Veneziano relation?

$U(1)_A$ Puzzle

- QCD has a $U(1)_A$ symmetry

$$\psi_L \to e^{i\phi}\psi_L, \quad \psi_R \to e^{-i\phi}\psi_R$$

- This symmetry is spontaneously broken

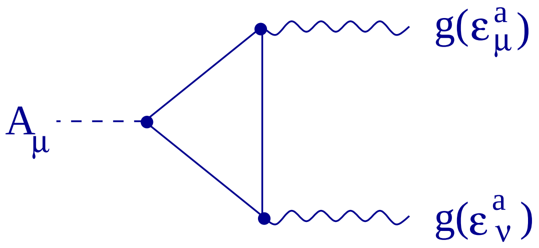
$$\langle \bar{\psi}_L \psi_R \rangle = \Sigma \neq 0$$

- Goldstone Theorem: massless Goldstone boson

$$\boxed{m_{\eta'} \to 0 \quad (m_q \to 0)}$$

But: η' is heavy, $m_{\eta'}^2 \gg m_q \Lambda_{QCD}$

- Resolution: axial anomaly



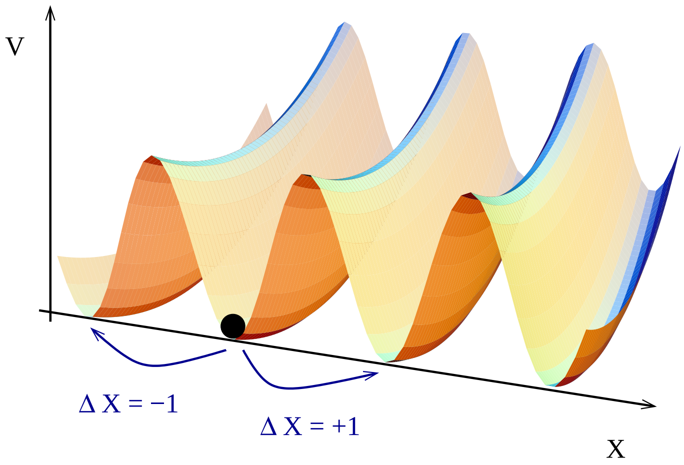
$$\partial_\mu A^\mu = \frac{N_f}{16\pi^2} G^a_{\mu\nu} \tilde{G}^a_{\mu\nu}$$

But: RHS is a total divergence $G\tilde{G} \sim \partial^\mu K_\mu$

$\Rightarrow$ Topology is important

Topology in QCD

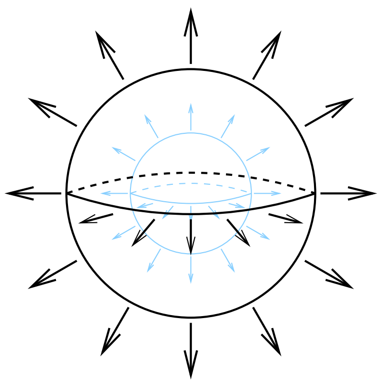
- Classical potential is periodic in variable X



$$X = \int d^3x\, K_0(x,t)$$

$$\partial^\mu K_\mu = \frac{1}{32\pi^2} G^a_{\mu\nu} \tilde{G}^a_{\mu\nu}$$

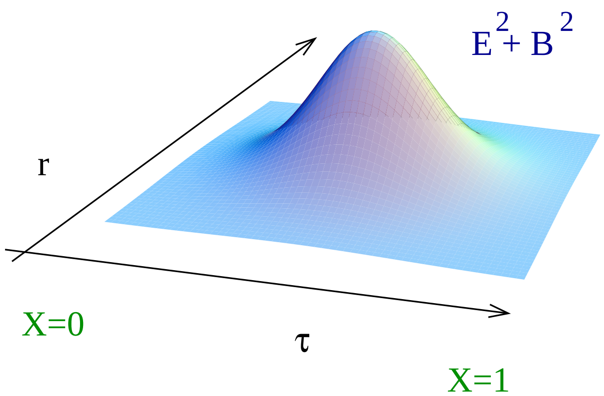
- classical minima correspond to pure gauge configurations



$$A_i(x) = i U^\dagger(x) \partial_i U(x)$$

$$E^2 = B^2 = 0$$

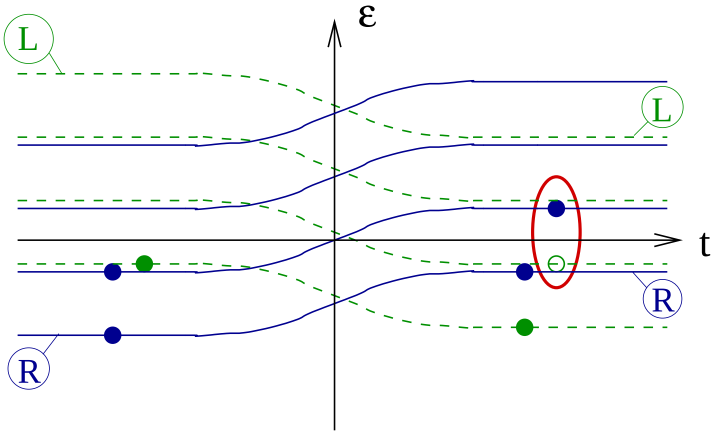
- classical tunneling paths: Instantons

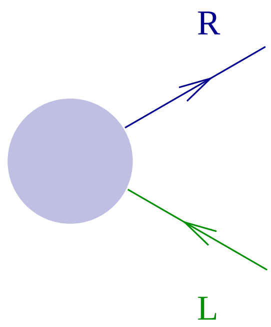


$$A^a_\mu(x) = 2\frac{\eta_{a\mu\nu} x_\nu}{x^2+\rho^2},$$

$$G^a_{\mu\nu}\tilde{G}^a_{\mu\nu} = \frac{192\rho^4}{(x^2+\rho^2)^4}.$$

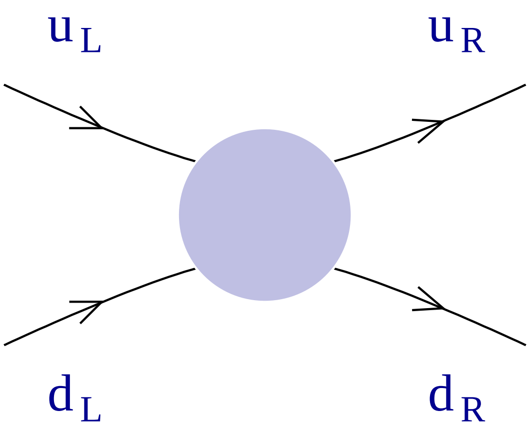
- Dirac spectrum in the field of an instanton





axial charge violation: $\Delta Q_A = 2$

- Instanton induced quark interaction ($N_f = 2$)



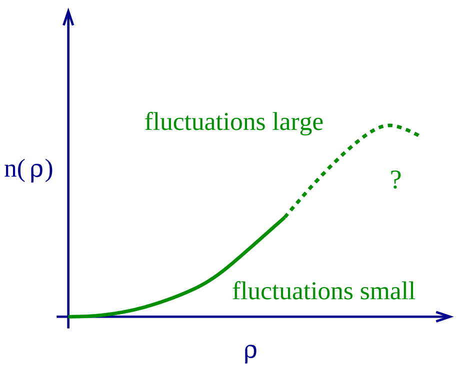
$$\mathcal{L} = G \det_f (\bar\psi_{L,f} \psi_{R,g})$$

$$G = \int d\rho\, n(\rho)$$

violates $U(1)_A$ symmetry (preserves $SU(2)_A$

... and contributes to the η' mass

- Tunneling rate (barrier penetration factor)



$$n(\rho) \sim \exp\left[-\frac{8\pi^2}{g^2(\rho)}\right]$$

QCD at large N_c

- QCD $(m = 0)$ is a parameter free theory

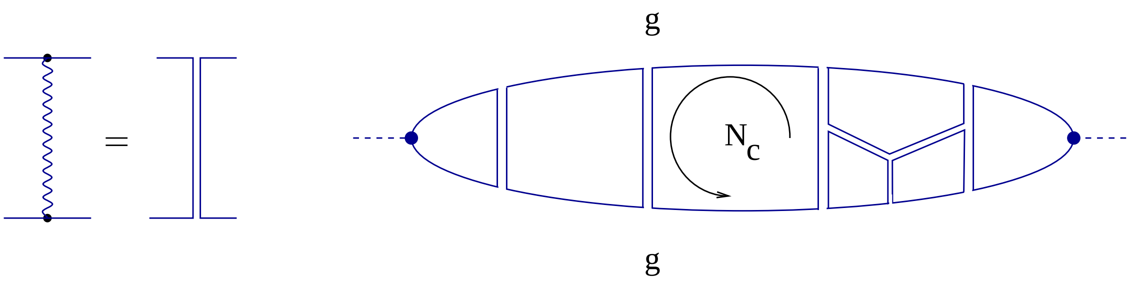
Very beautiful   But: No expansion parameter

- 't Hooft: Consider $N_c \to \infty$ and use $1/N_c$ as a small parameter

$$N_c \to \infty \quad \Rightarrow \quad \text{classical master field}$$

- keep Λ_{QCD} fixed

$$\Rightarrow g^2 N_c = const$$



- Could the master field be a multi-instanton configuration?

$$\text{Witten (1979)}: \ \text{No} \qquad dn \sim \exp\left(-\frac{1}{g^2}\right) \sim \exp\left(-N_c\right)$$

$U(1)_A$ **anomaly at large** N_c

- consider θ term

$$\mathcal{L} = \frac{ig^2\theta}{32\pi^2} G^a_{\mu\nu}\tilde{G}^a_{\mu\nu},$$

- no θ dependence in perturbation theory.

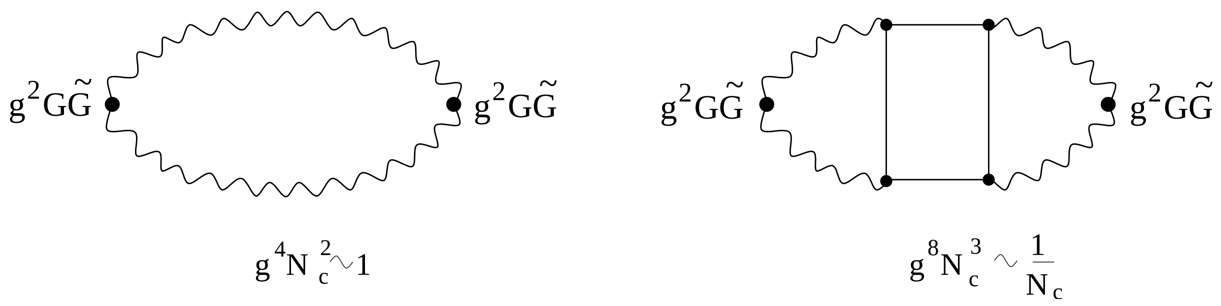
Witten: *non-perturbative θ dependence*

$$\chi_{top} = \left.\frac{d^2E}{d\theta^2}\right|_{\theta=0} \sim O(1)$$

- massless quarks: topological charge screening

$$\lim_{m\to 0} \chi_{top} = 0$$

- How can that happen? Fermion loops are suppressed!



Witten: *η' has to become light*

$$f_\pi^2 m_{\eta'}^2 = 2N_f\chi_{top}(no\ quarks) \quad \Rightarrow \quad m_{\eta'}^2 = O(1/N_c)$$

Questions

Witten-Veneziano relation "works"

$$\chi_{top} \simeq (200\,\mathrm{MeV})^4 \;\; (\textit{quenched lattice})$$

$$\Rightarrow m_{\eta'} \simeq 900\,\mathrm{MeV}$$

Where does $\chi_{top} \neq 0$ come from?

Instantons

- large N_c limit?
- Witten-Veneziano relation?
- ...

???

- topology?
- semi-classical limit?
- $\chi_{top}(m \to 0) = 0$?

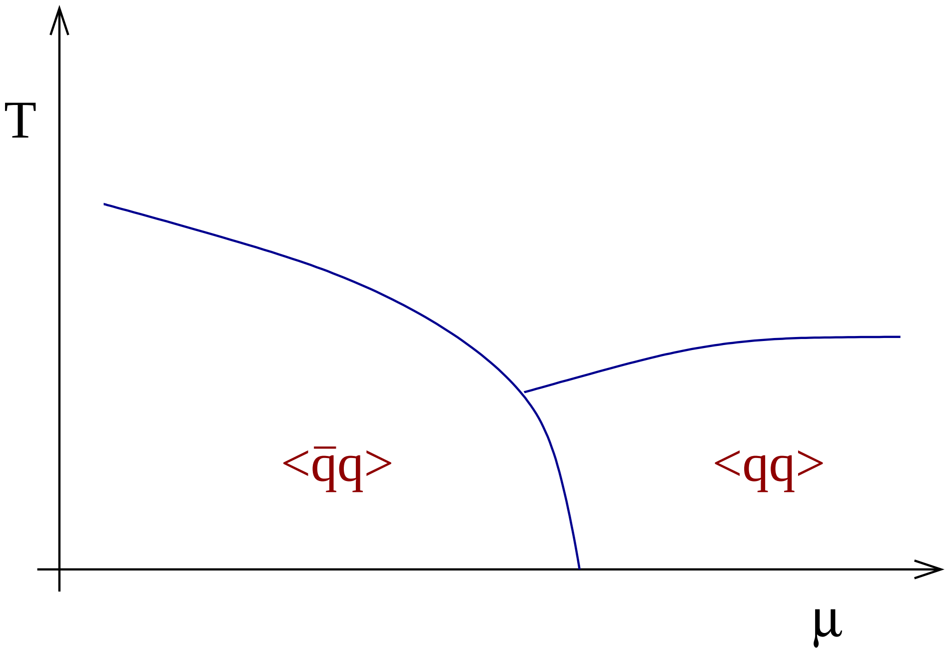
ACT I

Instantons and the mass of the η':

Large Baryon Density

QCD at large density ($N_f = 2$)

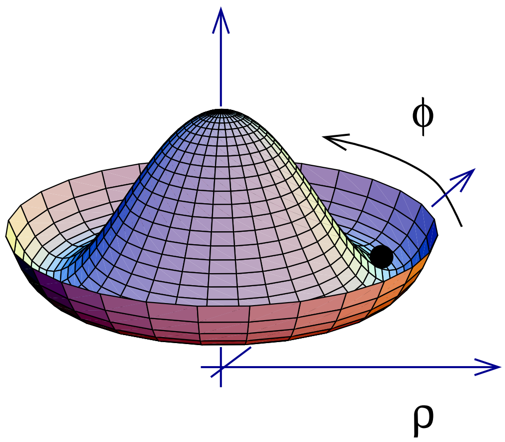
- schematic phase diagram



- diquark condensate breaks $U(1)_B$ and $U(1)_A$

$$\langle q_L q_L \rangle = \rho\, e^{i(\chi+\phi)/2}, \qquad \langle q_R q_R \rangle = \rho\, e^{i(\chi-\phi)/2}$$

- effective lagrangian for $U(1)_A$ Goldstone boson

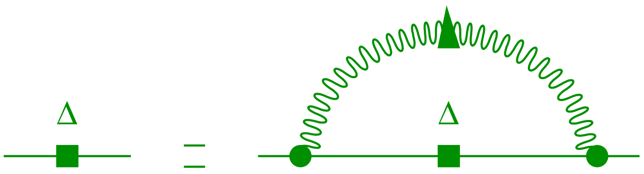


$$\mathcal{L} = \frac{f^2}{2}\left[(\partial_0\phi)^2 - v^2(\partial_i\phi)^2\right]$$
$$-V(\phi+\theta) + \mathcal{L}(\rho,\chi)$$

$U(1)_A$ problem: Where does $V(\phi)$ come from?

η' mass at large baryon density $(N_c = N_f = 2)$

- $\mu \gg \Lambda_{QCD}$: superconductivity driven by perturbative gluon exchange



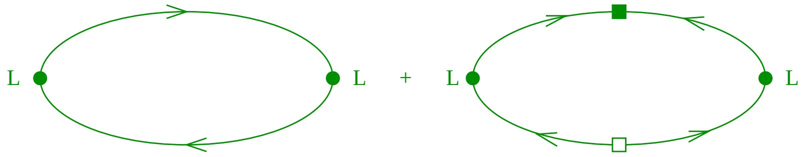
$$\Delta = 512\pi^4 b_0' \mu g^{-5} \exp\left(-\frac{3\pi^2}{\sqrt{2}g}\right)$$

$$\epsilon^{ab}\epsilon_{ij}\langle q_i^a C q_j^b\rangle = \frac{4\pi}{g}\Delta\left(\frac{\mu^2}{2\pi^2}\right)$$

- effective lagrangian for $U(1)_A$ Goldstone boson

$$\mathcal{L} = \frac{f^2}{2}\left[(\partial_0\phi)^2 - v^2(\partial_i\phi)^2\right] - V(\phi+\theta)$$

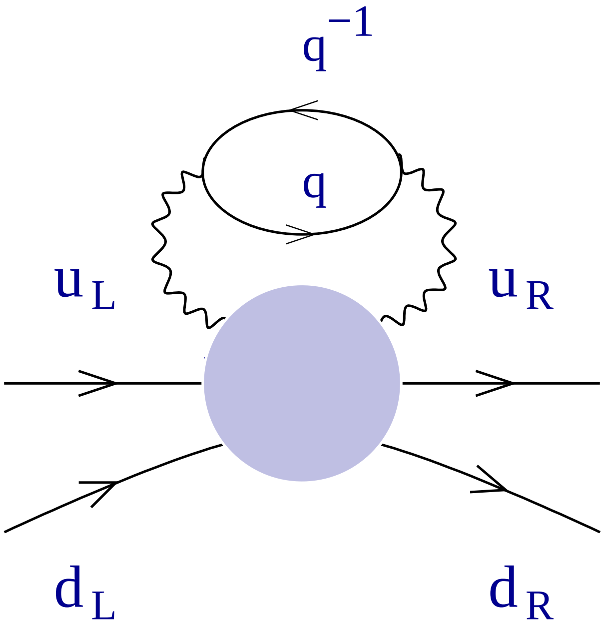
- decay constant calculable in perturbation theory



$$f^2 = \left(\frac{\mu^2}{4\pi^2}\right)$$

... but $V(\phi)$ vanishes in perturbative QCD

• instanton induced effective interaction for quarks with $p \sim p_F$

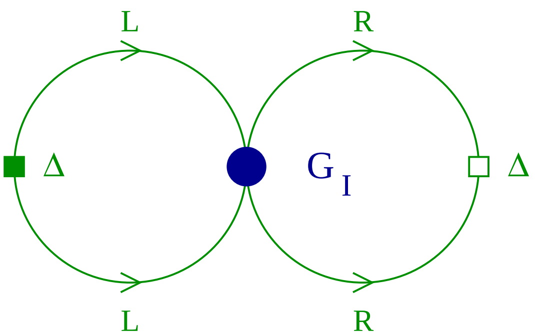


$$n(\rho,\mu) = \\ n(\rho,0)\exp\left[-N_f\rho^2\mu^2\right]$$

$$\boxed{\rho \sim \mu^{-1} \ll \Lambda_{QCD}^{-1}}$$

$$\mathcal{L} \;=\; \int n(\rho,\mu)d\rho\,\rho^6\epsilon_{f_1f_2}\epsilon_{g_1g_2}\left\{\frac{2N_c-1}{2N_c}(\bar\psi_{R,f_1}\psi_{L,g_1})(\bar\psi_{R,f_2}\psi_{L,g_2}) \right.$$
$$\left. -\frac{1}{8N_c}(\bar\psi_{R,f_1}\sigma_{\mu\nu}\psi_{L,g_1})(\bar\psi_{R,f_2}\sigma_{\mu\nu}\psi_{L,g_2})\right\}$$

• instanton contribution to vacuum energy



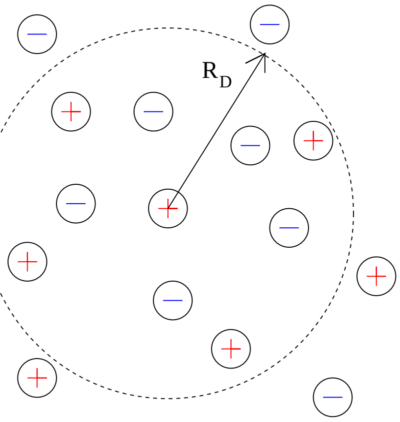
$$\langle\mathcal{L}\rangle = A\cos(\phi+\theta)$$

$$A = C_{N_c}^{N_f}\Phi^2\left[\log\left(\frac{\mu}{\Lambda}\right)\right]^4\left(\frac{\Lambda}{\mu}\right)^8\Lambda^{-2}.$$

- η' mass satisfies "Witten-Veneziano" relation

$$\boxed{f^2 m_\phi^2 = A} \qquad A \sim \chi_{top}?$$

- very dilute instanton gas



$$\rho \ll r_{IA} \ll R_D$$

$$r_{IA} = A^{1/4}, \quad R_D = m_\phi^{-1}$$

- A is the <u>local</u> topological susceptibility

$$A = \chi_{top}(V) = \frac{\langle Q_{top}^2 \rangle_V}{V} \qquad r_{IA}^4 \ll V \ll R_D^4$$

- <u>Global</u> topological susceptibility vanishes

$$\chi_{top} = \lim_{V \to \infty} \frac{\langle Q_{top}^2 \rangle_V}{V} = 0 \qquad (m = 0)$$

A. Polyakov (1977)
D. Son, M. Stephanov, A. Zhitnitsky, hep-ph/0103099.
E. Shuryak, J. Verbarrschot, hep-lat/9409020
R. Rapp, T. Schäfer, E. V. Shuryak, and M. Velkovsky, hep-ph/9904353.
T. Schäfer, hep-ph/0201189.

Comments

- Claim: same mechanism operates at zero density

$$\left(\frac{N}{V}\right) = 1\,\mathrm{fm}^{-4} \qquad \Rightarrow \chi_{top} = (200\,\mathrm{MeV})^4$$

- Instantons predict density dependence of $m_{\eta'}$

can be checked on the lattice ($N_c = 2$ or $\mu_I \neq 0$)

- High density phases ($N_c = 2$ or $\mu_I \neq 0$) continuously connected to low density phase

"Quark Hadron Continuity"

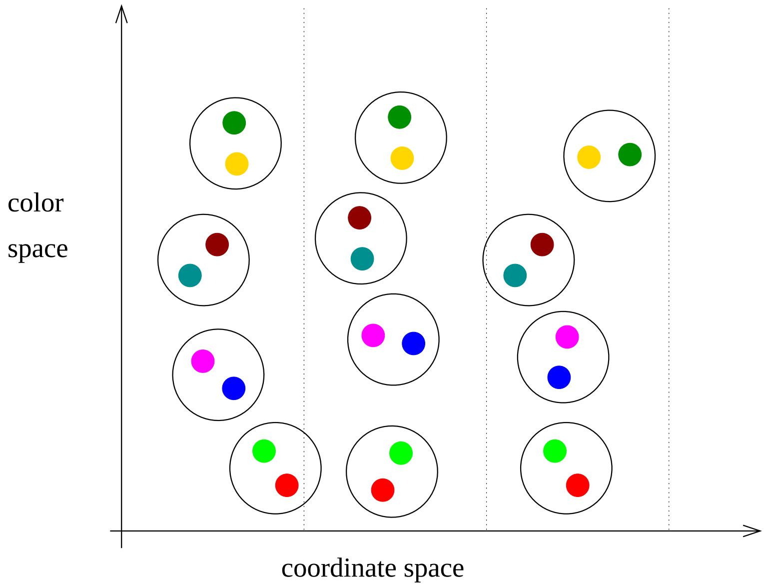
ACT II

Instantons and the mass of the η':

Large Number of Colors

Instantons at large N_c

- semi-classical ensemble of instantons at large N_c



- instantons are $N_c = 2$ configurations

$$\left(\frac{N}{V}\right) = O(N_c) \quad \Rightarrow \epsilon_{vac} = O(bN_c) = O(N_c^2)$$

- instantons are semi-classical

$$\rho \simeq \rho^* = O(1) \quad S_{inst} = O(N_c)$$

- density $dn \sim \exp(-S_{inst}) = O(\exp(-N_c))$?

$$NO! \quad \textit{large entropy} \quad dn \sim \exp(+N_c)$$

- topological susceptibility $\chi_{top} \simeq (N/V) = O(N_c)$?

$$NO! \quad \textit{fluctuations suppressed} \quad \chi_{top} = O(1)$$

Digression: $\mathcal{N}=4$ SUSY Yang Mills

- string/field theory duality (Maldacena)

$$\begin{array}{ccc} \mathcal{N}=4 \text{ SUSY YM } (d=4) & \Leftrightarrow & \text{IIB strings on } AdS_5\times S_5 \\ \lambda = g^2N\to\infty & \Leftrightarrow & (l_s/R)^4\to 0 \\ (g^2\to 0) & & (g_s\to 0) \end{array}$$

- $\mathcal{N}=4$ conformal $\Rightarrow$ g does not run
- string theory contains D-instantons characterized by location on $AdS_5\times S_5 \Leftrightarrow$ field theory instantons

T. Banks, M. Green

$$\int d^4x \int \frac{d\rho}{\rho^5} \int d\Lambda^{ab}$$

$$AdS_5\times S_5$$

- k instanton amplitudes

M. Mattis et al.

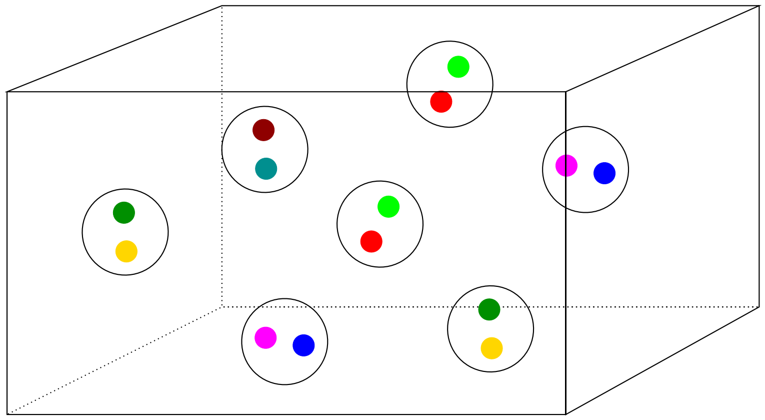
$$(AdS_5\times S_5)^k \quad\to\quad (AdS_5\times S_5)$$

k instantons in commuting $SU(2)'s$

(bound by fermions)

Instantons ensemble

- instanton ensemble



described by partition function

$$Z = \frac{1}{N_I! N_A!} \prod_I^{N_I+N_A} \int [d\Omega_I \, n(\rho_I)] \; \exp(-S_{int})$$

$$n(\rho) = C_{N_c} \left(\frac{8\pi^2}{g^2}\right)^{2N_c} \rho^{-5} \exp\left[-\frac{8\pi^2}{g(\rho)^2}\right]$$

$$C_{N_c} = \frac{0.47 \exp(-1.68 N_c)}{(N_c-1)!(N_c-2)!} \qquad \frac{8\pi^2}{g^2(\rho)} = -b \log(\rho\Lambda), \qquad b = \frac{11}{3} N_c$$

$$S_{int} = -\frac{32\pi^2}{g^2} |u|^2 \left\{ \frac{\rho_I^2 \rho_A^2}{R_{IA}^4} \left(1 - 4\cos^2\theta\right) + S_{core}\left(\frac{\rho_I^2 \rho_A^2}{R_{IA}^4}\right) \right\}$$

- complicated ensemble, size distribution

$$\rho^* \sim O(1) \qquad \begin{cases} \rho < \rho^* & n(\rho) \sim \exp(-N_c) \\ \rho \sim \rho^* & n(\rho) \sim const \end{cases}$$

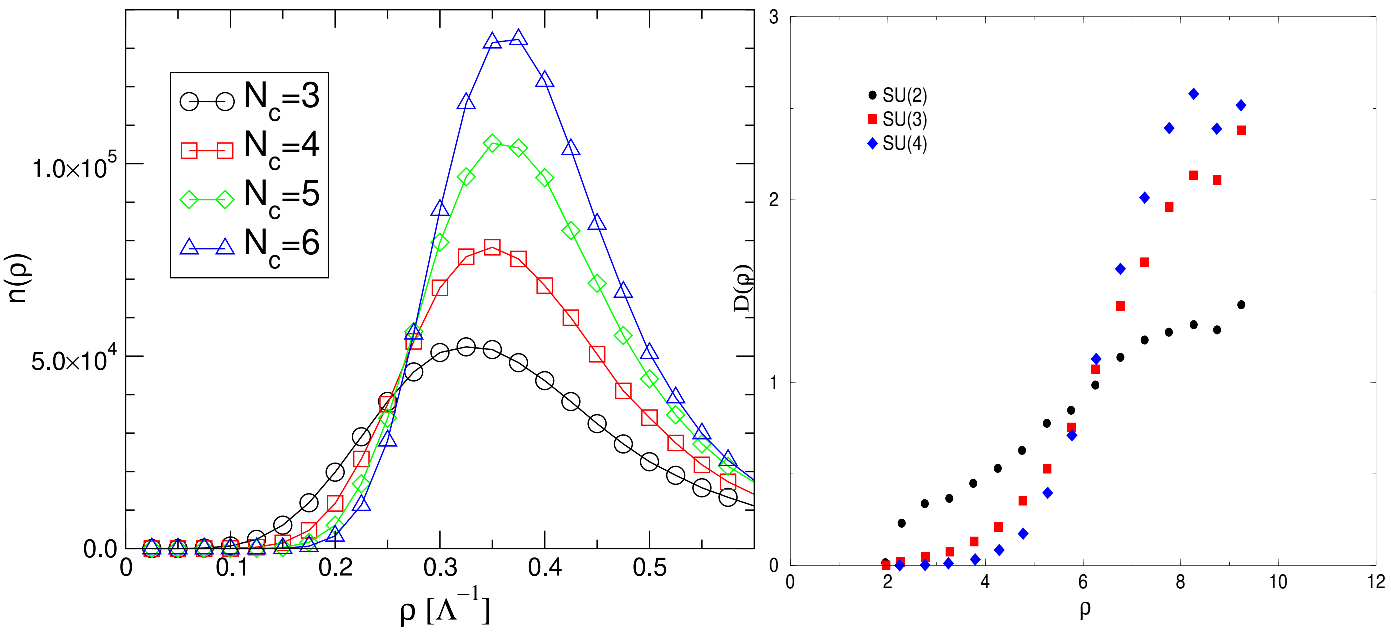
- total density determined by interactions

$$\begin{array}{ccc} S(1-body) & \sim & S(2-body) \\ N_c & \sim & N_c \;\times\; \frac{1}{N_c} \;\times\; \left(\frac{N}{V}\right) \\ classical & \sim & classical \times color\ overlap \times density \end{array}$$

- conclude

$$\boxed{\left(\frac{N}{V}\right) = O(N_c)}$$

- instanton size distribution



B. Lucini, M. Teper, hep-lat/0103027.

- fluctuations in N are $1/N_c$ suppressed

$$\langle N^2\rangle - \langle N\rangle^2 = \frac{4}{b}\langle N\rangle \qquad (\text{not } \langle N\rangle!)$$

low energy theorem

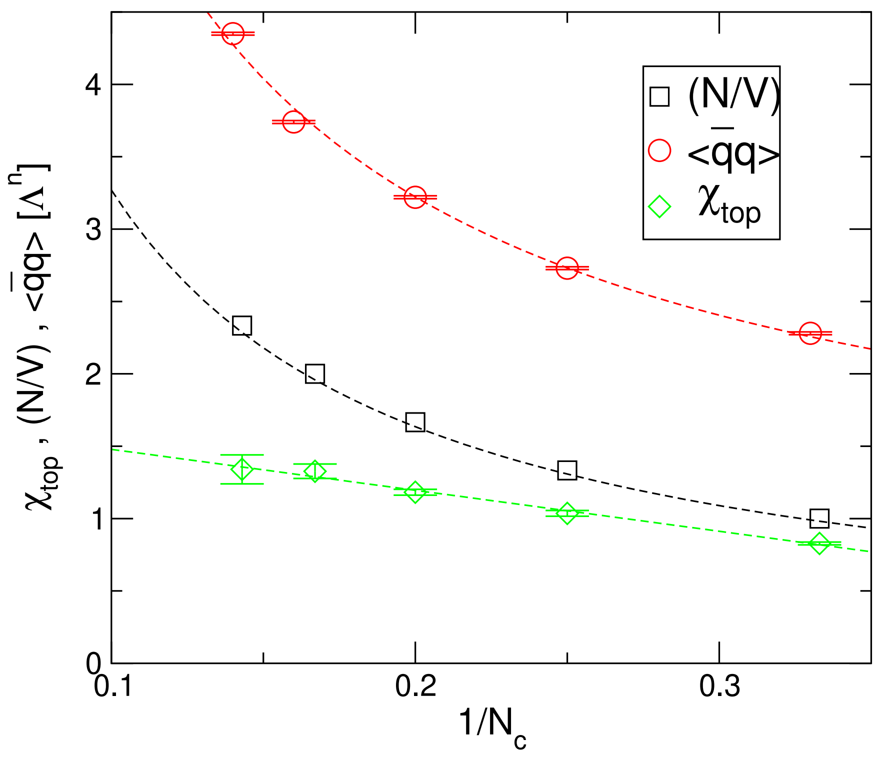
$$\frac{1}{(32\pi^2)^2}\int d^4x \, \left\{\langle g^2G^2(0)g^2G^2(x)\rangle - \langle g^2G^2(0)\rangle^2\right\} = \frac{4}{b}\frac{1}{32\pi^2}\langle g^2G^2\rangle$$

V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B191, 301 (1981).

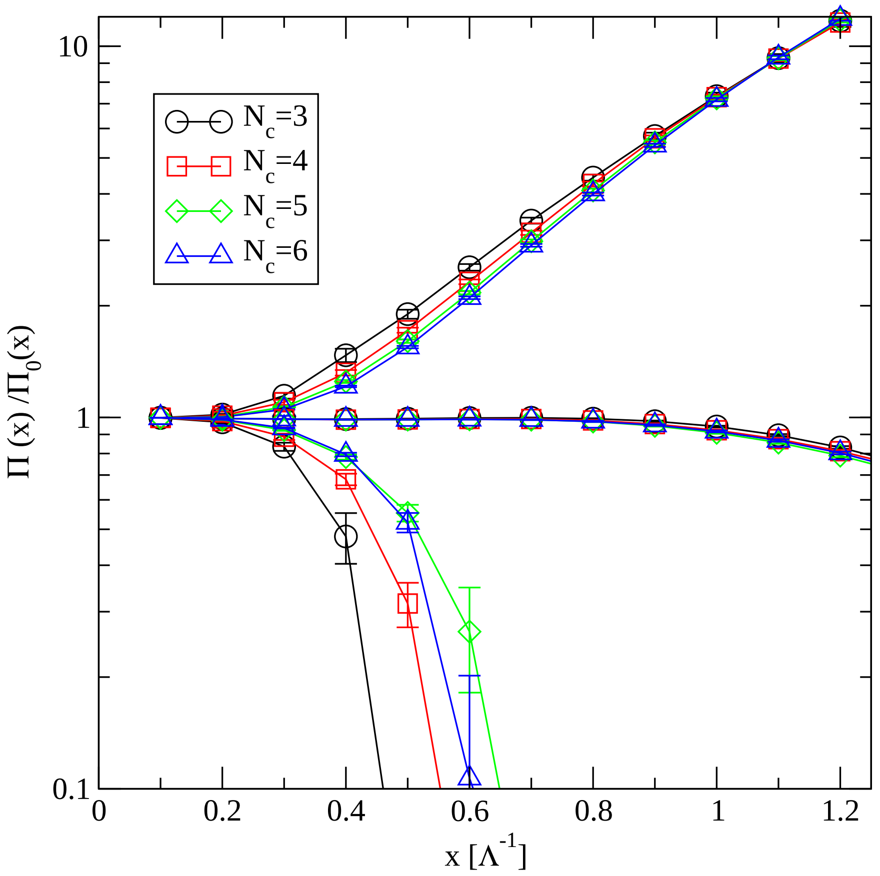
- also true for topological charge

$$\langle Q^2\rangle = \frac{4}{b-r(b-4)}\langle N\rangle$$

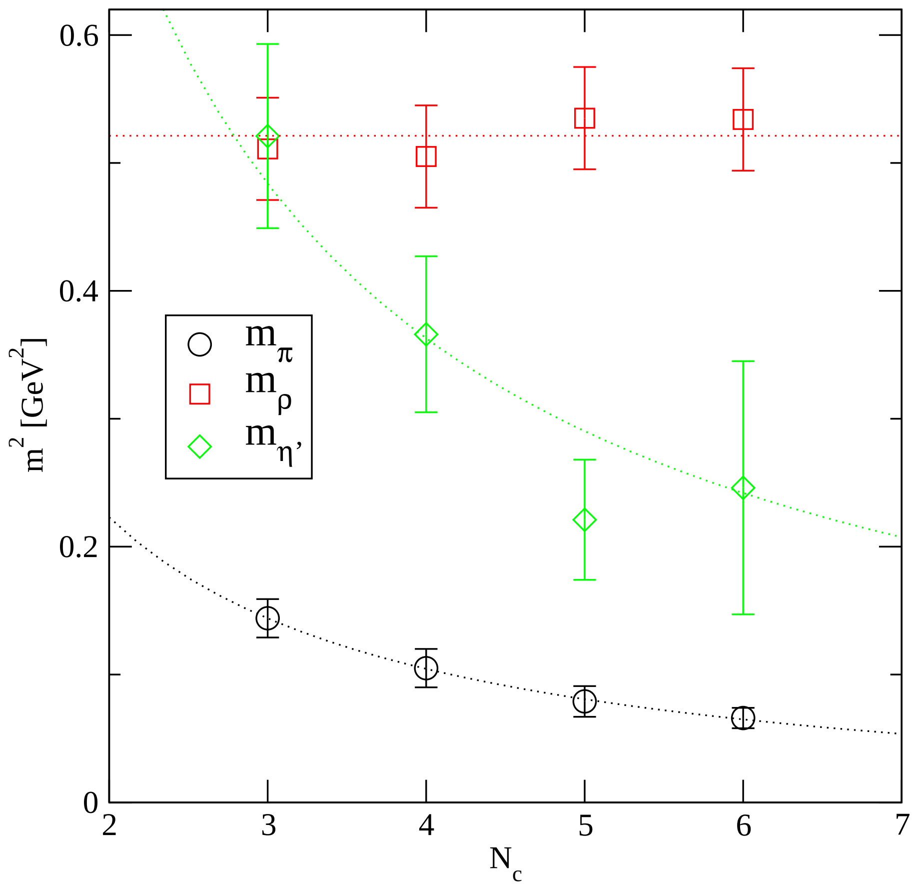
- instanton density, quark condensate, topological susceptibility



- meson correlation functions (π, ρ, η')



- meson masses: $m_\pi^2, m_\rho^2 \sim 1,\ m_{\eta'}^2 \sim 1/N_c$



Note that $m_{\eta'}^2 \sim 1/N_c$ even though $(N/V) \sim N_c$

Summary

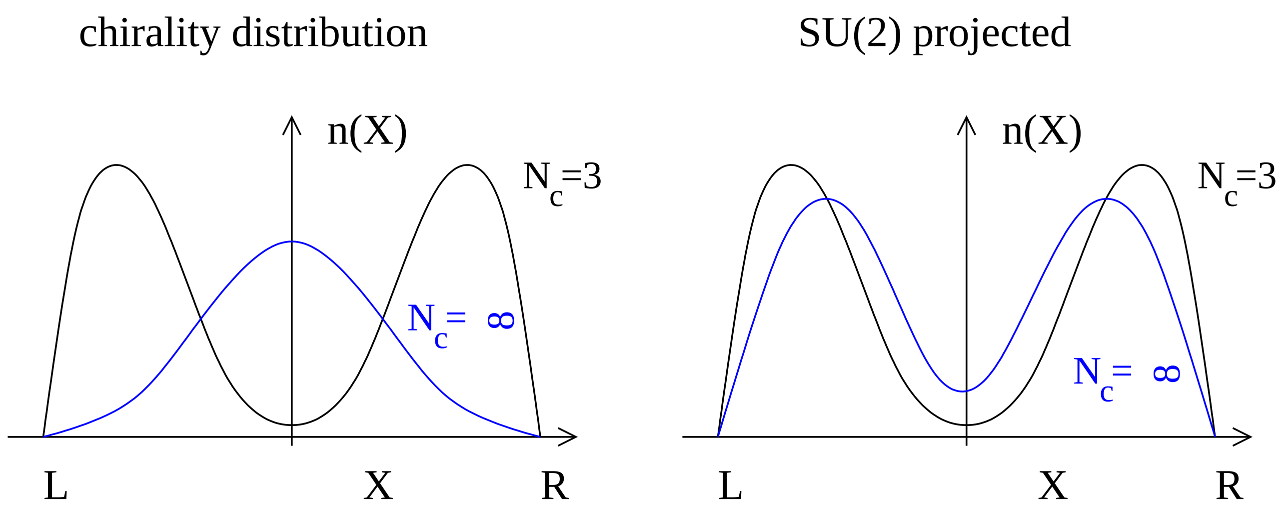
- nice example for semi-classical instanton liquid: QCD at large baryon density

  can be studied on the lattice for $N_c = 2$ or $\mu_I \neq 0$

- instanton liquid can have a smooth large N_c limit

$$\left(\frac{N}{V}\right) = O(N_c), \qquad \chi_{top} = O(1), \qquad m^2_{\eta'} = O(1/N_c)$$

- how can we check this?



- smooth large N_c limit does not imply that $N_c = 3$ is similar to $N_c = \infty$ in all regards

η', scalar mesons (and fluctuations of $\langle \bar{q}q \rangle$), pentaquarks (?)