

(Non) Conformal Quantum Fluids

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Why study (nearly perfect) quantum fluids?

Hard computational problem: Have to determine real time correlation functions. Achieve quantum supremacy?

Fluid dynamics is the universal effective description of non-equilibrium many body systems. Description is “most effective” in nearly perfect fluids.

Fluid-gravity correspondence: Can (strongly coupled) fluids teach us something about quantum gravity?

Effective theories for fluids (Unitary Fermi Gas, $T > T_F$)



$$\mathcal{L} = \psi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2M} \right) \psi - \frac{C_0}{2} (\psi^\dagger \psi)^2$$



$$\frac{\partial f_p}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f_p = C[f_p] \quad \omega < T$$



$$\frac{\partial}{\partial t} (\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0 \quad \omega < T \frac{s}{\eta}$$

Effective theories (Strong coupling)



$$\mathcal{L} = \bar{\lambda}(i\sigma \cdot D)\lambda - \frac{1}{4}G_{\mu\nu}^a G_{\mu\nu}^a + \dots \Leftrightarrow S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \mathcal{R} + \dots$$

$$SO(d+2, 2) \rightarrow Schr_d^2$$

$$AdS_{d+3} \rightarrow Schr_d^2$$



$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0 \quad (\omega < T)$$

Outline

- I. EFT: Gradient expansion
- II. EFT: Fluctuations
- III. Models of fluids: Kinetic theory & QFT
- IV. Models of fluids: Holography
- V. Analyzing fluids: How to measure η/s

I. Gradient expansion (simple non-relativistic fluid)

Simple fluid: Conservation laws for mass, energy, momentum

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j}^\rho = 0$$

$$\frac{\partial \epsilon}{\partial t} + \vec{\nabla} \cdot \vec{j}^\epsilon = 0$$

$$\frac{\partial \pi_i}{\partial t} + \frac{\partial}{\partial x_j} \Pi_{ij} = 0$$

Ward identity: mass current = momentum density

$$\vec{j}^\rho \equiv \rho \vec{v} = \vec{\pi}$$

Constitutive relations: Gradient expansion for currents

Energy momentum tensor

$$\Pi_{ij} = P \delta_{ij} + \rho v_i v_j + \eta \left(\partial_i v_j + \partial_j v_i - \frac{2}{3} \delta_{ij} \partial_k v_k \right) + O(\partial^2)$$

Conformal fluid dynamics: Symmetries

Symmetries of a conformal non-relativistic fluid

$$\text{Galilean boost} \quad \vec{x}' = \vec{x} + \vec{v}t \quad t' = t$$

$$\text{Scale trafo} \quad \vec{x}' = e^s \vec{x} \quad t' = e^{2s} t$$

$$\text{Conformal trafo} \quad \vec{x}' = \vec{x}/(1 + ct) \quad 1/t' = 1/t + c$$

This is known as the Schrödinger algebra (= the symmetries of the free Schrödinger equation)

Generators: Mass, momentum, angular momentum

$$M = \int dx \rho \quad P_i = \int dx j_i \quad J_{ij} = \int dx \epsilon_{ijk} x_j j_k$$

Boost, dilations, special conformal

$$K_i = \int dx x_i \rho \quad D = \int dx x \cdot j \quad C = \int dx x^2 \rho / 2$$

Spurion method: Local symmetries

Diffeomorphism invariance $\delta x_i = \xi_i(x, t)$

$$\delta g_{ij} = -\mathcal{L}_\xi g_{ij} = -\xi^k \partial_k g_{ij} + \dots$$

Gauge invariance $\delta\psi = i\alpha(x, t)\psi$

$$\delta A_0 = -\dot{\alpha} - \xi^k \partial_k A_0 - A_k \dot{\xi}^k$$

$$\delta A_i = -\partial_i \alpha - \xi^k \partial_k A_i - A_k \partial_i \xi^k + m g_{ik} \dot{\xi}^k$$

Conformal transformations $\delta t = \beta(t)$

$$\delta O = -\beta \dot{O} - \frac{1}{2} \Delta_O \beta O$$

More recent work: Newton-Cartan geometry

Example: Stress tensor

Determine transformation properties of fluid dynamic variables

$$\delta\rho = -\mathcal{L}_\xi\rho \quad \delta s = -\mathcal{L}_\xi s \quad \delta v = -\mathcal{L}_\xi v + \dot{\xi}$$

Stress tensor: Ideal fluid dynamics

$$\Pi_{ij}^0 = P g_{ij} + \rho v_i v_j,$$

$$P = \frac{2}{3}\mathcal{E}$$

First order viscous hydrodynamics

$$\delta^{(1)}\Pi_{ij} = -\eta\sigma_{ij} - \zeta g_{ij}\langle\sigma\rangle$$

$$\zeta = 0$$

$$\sigma_{ij} = \left(\nabla_i v_j + \nabla_j v_i - \frac{2}{3} g_{ij} \langle\sigma\rangle \right)$$

$$\langle\sigma\rangle = \nabla \cdot v + \frac{\dot{g}}{2g}$$

Simple application: Kubo formula

Consider background metric $g_{ij}(t, x) = \delta_{ij} + h_{ij}(t, x)$. Linear response

$$\delta\Pi^{xy} = -\frac{1}{2}G_R^{xyxy}h_{xy}$$

Harmonic perturbation $h_{xy} = h_0 e^{-i\omega t}$

$$G_R^{xyxy} = P - i\eta\omega + \dots$$

Kubo relation:
$$\eta = -\lim_{\omega \rightarrow 0} \left[\frac{1}{\omega} \text{Im} G_R^{xyxy}(\omega, 0) \right]$$

Gradient expansion:
$$\omega \leq \frac{P}{\eta} \simeq \frac{s}{\eta} T.$$

Second order conformal hydrodynamics

Second order gradient corrections to stress tensor

$$\delta^{(2)}\Pi^{ij} = \eta\tau_\pi \left[\langle D\sigma^{ij} \rangle + \frac{2}{3}\sigma^{ij}(\nabla \cdot v) \right] \\ + \lambda_1 \sigma^{\langle i}_k \sigma^{j \rangle k} + \lambda_2 \sigma^{\langle i}_k \Omega^{j \rangle k} + \lambda_3 \Omega^{\langle i}_k \Omega^{j \rangle k} + O(\nabla^2 T)$$

$$D = \partial_0 + v \cdot \nabla \quad A^{\langle ij \rangle} = \frac{1}{2} (A_{ij} + A_{ji} - \frac{2}{3} g_{ij} A^k_k) \quad \Omega^{ij} = (\nabla_i v_j - \nabla_j v_i)$$

New transport coefficients $\tau_\pi, \lambda_i, \gamma_i$

Can be written as a relaxation equation for $\pi^{ij} \equiv \delta\Pi^{ij}$

$$\pi^{ij} = -\eta\sigma^{ij} - \tau_\pi \left[\langle D\pi^{ij} \rangle + \frac{5}{3}(\nabla \cdot v)\pi^{ij} \right] + \dots$$

Second order fluid dynamics: Causality

“Speed” of diffusive wave in Navier-Stokes theory

$$v_D = \frac{\partial|\omega|}{\partial k} = \frac{2\eta}{\rho} k$$

May encounter $v_D \gg c_s$

Not a fundamental problem (should impose $k < \Lambda$), but a nuisance in simulations.

Second order fluid dynamics, relaxation type

$$i\omega = \frac{\nu k^2}{1 - i\omega\tau_\pi} \quad (\text{“resummed hydro”})$$

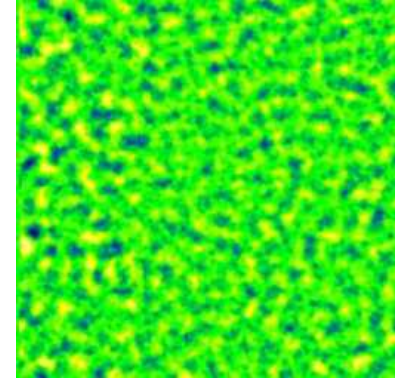
Limiting speed $v_D^\infty \sim \sqrt{\eta/(\rho\tau_\pi)}$

Find $v_D^\infty \sim c_s$ for $\tau_\pi = \eta/P$.

II. Beyond gradients: Hydrodynamic fluctuations

Hydrodynamic variables fluctuate

$$\langle \delta v_i(x, t) \delta v_j(x', t) \rangle = \frac{T}{\rho} \delta_{ij} \delta(x - x')$$



Linearized hydrodynamics propagates fluctuations as shear or sound

$$\langle \delta v_i^T \delta v_j^T \rangle_{\omega, k} = \frac{2T}{\rho} (\delta_{ij} - \hat{k}_i \hat{k}_j) \frac{\nu k^2}{\omega^2 + (\nu k^2)^2} \quad \textit{shear}$$

$$\langle \delta v_i^L \delta v_j^L \rangle_{\omega, k} = \frac{2T}{\rho} \hat{k}_i \hat{k}_j \frac{\omega k^2 \Gamma}{(\omega^2 - c_s^2 k^2)^2 + (\omega k^2 \Gamma)^2} \quad \textit{sound}$$

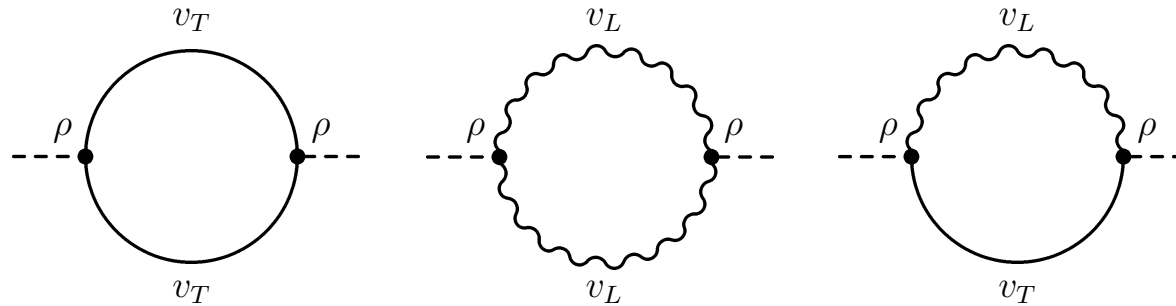
$$v = v_T + v_L: \quad \nabla \cdot v_T = 0, \quad \nabla \times v_L = 0$$

$$\nu = \eta/\rho, \quad \Gamma = \frac{4}{3}\nu + \dots$$

Hydro Loops: “Breakdown” of second order hydro

Correlation function in hydrodynamics

$$G_S^{xyxy} = \langle \{ \Pi^{xy}, \Pi^{xy} \} \rangle_{\omega, k} \simeq \rho_0^2 \langle \{ v_x v_y, v_x v_y \} \rangle_{\omega, k}$$



Match to response function in $\omega \rightarrow 0$ (Kubo) limit

$$G_R^{xyxy} = P + \delta P - i\omega[\eta + \delta\eta] + \omega^2 [\eta\tau_\pi + \delta(\eta\tau_\pi)]$$

with

$$\delta P \sim T\Lambda^3 \quad \delta\eta \sim \frac{T\rho\Lambda}{\eta} \quad \delta(\eta\tau_\pi) \sim \frac{1}{\sqrt{\omega}} \frac{T\rho^{3/2}}{\eta^{3/2}}$$

Hydro Loops: RG and “breakdown” of 2nd order hydro

Cutoff dependence can be absorbed into bare parameters. Non-analytic terms are cutoff independent.

Fluid dynamics is a “renormalizable” effective theory.

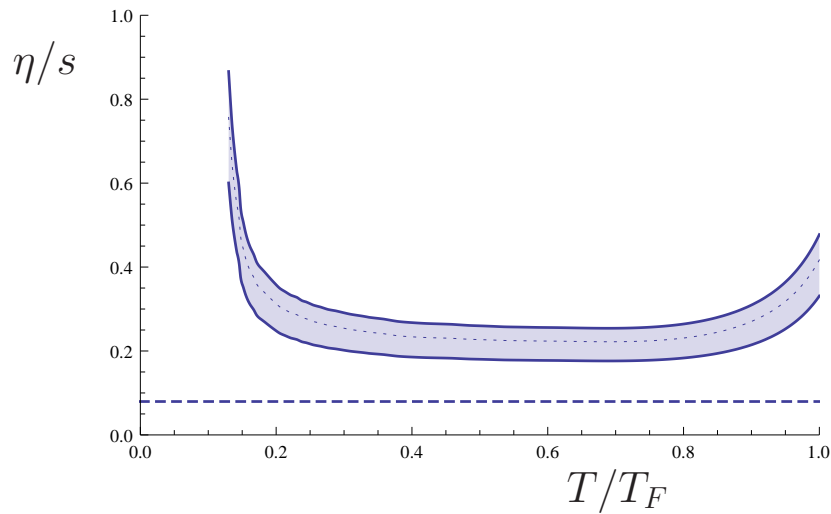
Small η enhances fluctuation corrections: $\delta\eta \sim T \left(\frac{\rho}{\eta}\right)^2 \left(\frac{P}{\rho}\right)^{1/2}$

Small η leads to large $\delta\eta$: There must be a bound on η/n .

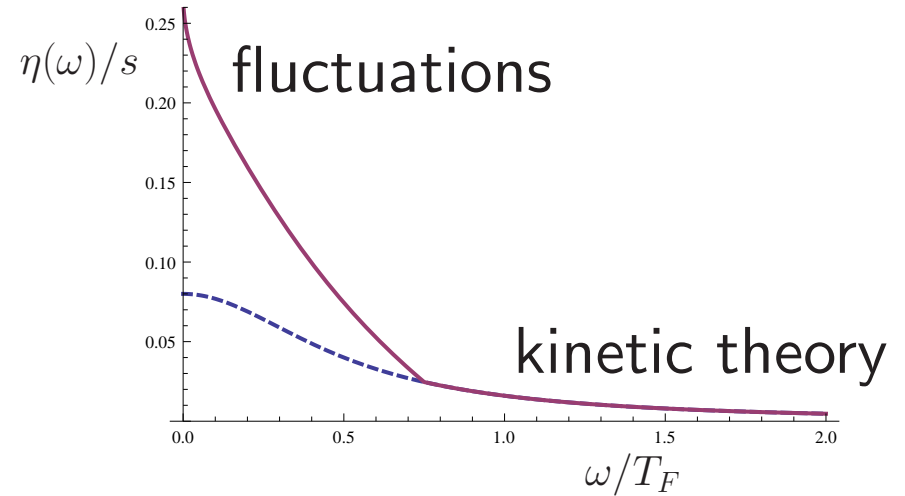
Relaxation time diverges: $\delta(\eta\tau_\pi) \sim \frac{1}{\sqrt{\omega}} \left(\frac{\rho}{\eta}\right)^{3/2}$

2nd order hydro without fluctuations inconsistent.

Fluctuation induced bound on η/s



$$(\eta/s)_{min} \simeq 0.2$$



spectral function
non-analytic $\sqrt{\omega}$ term

III. Kinetic theory

Microscopic picture: Quasi-particle distribution function $f_p(x, t)$

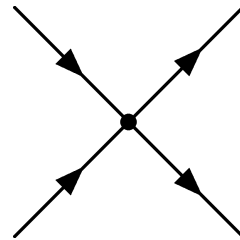
$$\rho(x, t) = \int d\Gamma_p \sqrt{g} m f_p(x, t) \quad \pi_i(x, t) = \int d\Gamma_p \sqrt{g} p_i f_p(x, t)$$

$$\Pi_{ij}(x, t) = \int d\Gamma_p \sqrt{g} p_i v_j f_p(x, t)$$

Boltzmann equation

$$\left(\frac{\partial}{\partial t} + \frac{p^i}{m} \frac{\partial}{\partial x^i} - \left(g^{il} \dot{g}_{lj} p^j + \Gamma_{jk}^i \frac{p^j p^k}{m} \right) \frac{\partial}{\partial p^i} \right) f_p(t, x,) = C[f]$$

$$C[f] =$$



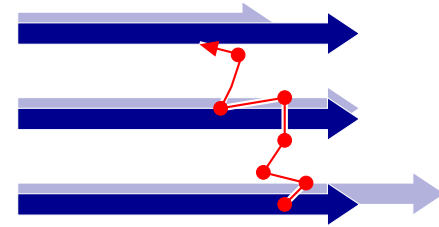
Solve order-by-order in Knudsen number $Kn = l_{mfp}/L$

Kinetic theory: Knudsen expansion

Chapman-Enskog expansion $f = f_0 + \delta f_1 + \delta f_2 + \dots$

Gradient exp. $\delta f_n = O(\nabla^n)$

\equiv Knudsen exp. $\delta f_n = O(Kn^n)$



First order result

Bruun, Smith (2005)

$$\delta^{(1)}\Pi^{ij} = -\eta\sigma^{ij} \quad \eta = \frac{15}{32\sqrt{\pi}}(mT)^{3/2}$$

Second order result

Chao, Schaefer (2012), Schaefer (2014)

$$\delta^{(2)}\Pi^{ij} = \frac{\eta^2}{P} \left[\langle D\sigma^{ij} \rangle + \frac{2}{3}\sigma^{ij}(\nabla \cdot v) \right] + \frac{\eta^2}{P} \left[\frac{15}{14}\sigma^{i \langle k} \sigma^{j \rangle k} - \sigma^{i \langle k} \Omega^{j \rangle k} \right] + O(\kappa\eta\nabla^i\nabla^j T)$$

relaxation time $\tau_\pi = \eta/P$

Frequency dependence, breakdown of kinetic theory

Consider harmonic perturbation $h_{xy}e^{-i\omega t+ikx}$. Use schematic collision term $C[f_p^0 + \delta f_p] = -\delta f_p/\tau$.

$$\delta f_p(\omega, k) = \frac{1}{2T} \frac{-i\omega p_x v_y}{-i\omega + i\vec{v} \cdot \vec{k} + \tau_0^{-1}} f_p^0 h_{xy}.$$

Leads to Lorentzian line shape of transport peak

$$\eta(\omega) = \frac{\eta(0)}{1 + \omega^2 \tau_0^2}$$

Pole at $\omega = i\tau_0^{-1}$ ($\tau_0 = \eta/(sT)$) controls range of convergence of gradient expansion.

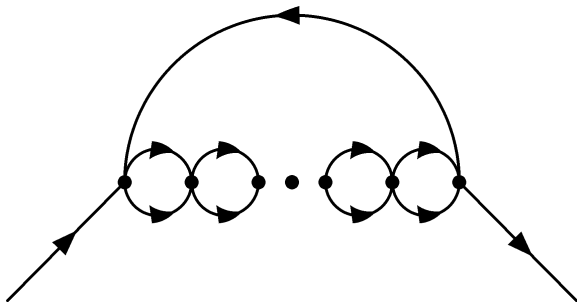
High frequency behavior misses short range correlations for $\omega > T$.

Bulk viscosity and conformal symmetry breaking

Conformal symmetry breaking (thermodynamics)

$$1 - \frac{2\mathcal{E}}{3P} = \frac{\langle \mathcal{O}_c \rangle}{12\pi m a P} \sim \frac{1}{6\pi} n \lambda^3 \frac{\lambda}{a}$$

How does this translate into $\zeta \neq 0$? Momentum dependent $m^*(p)$.



$$Im \Sigma(k) \sim zT \sqrt{\frac{T}{\epsilon_k}} Erf \left(\sqrt{\frac{\epsilon_k}{T}} \right) \ll T$$

$$Re \Sigma(k) \sim zT \frac{\lambda}{a} \sqrt{\frac{T}{\epsilon_k}} F_D \left(\sqrt{\frac{\epsilon_k}{T}} \right)$$

Bulk viscosity

$$\zeta = \frac{1}{24\sqrt{2}\pi} \lambda^{-3} \left(\frac{z\lambda}{a} \right)^2$$

$$\zeta \sim \left(1 - \frac{2\mathcal{E}}{3P} \right)^2 \eta$$

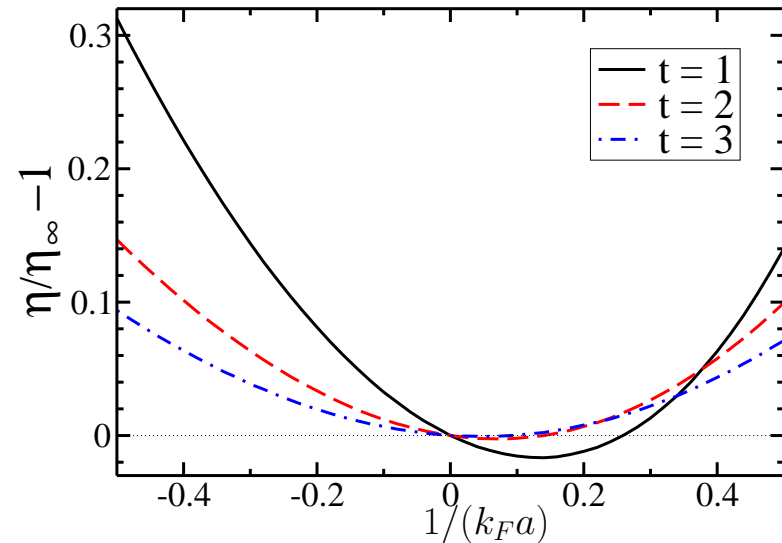
Shear viscosity and conformal symmetry breaking

Consider shear viscosity at $a \neq \infty$

$$\eta = \eta_0 \left\{ 1 + O\left(\frac{\lambda^2}{a^2}\right) + O\left(\frac{z\lambda}{a}\right) + \dots \right\}$$

Medium effects at $O(z\lambda/a)$: Self energy, in-medium scattering

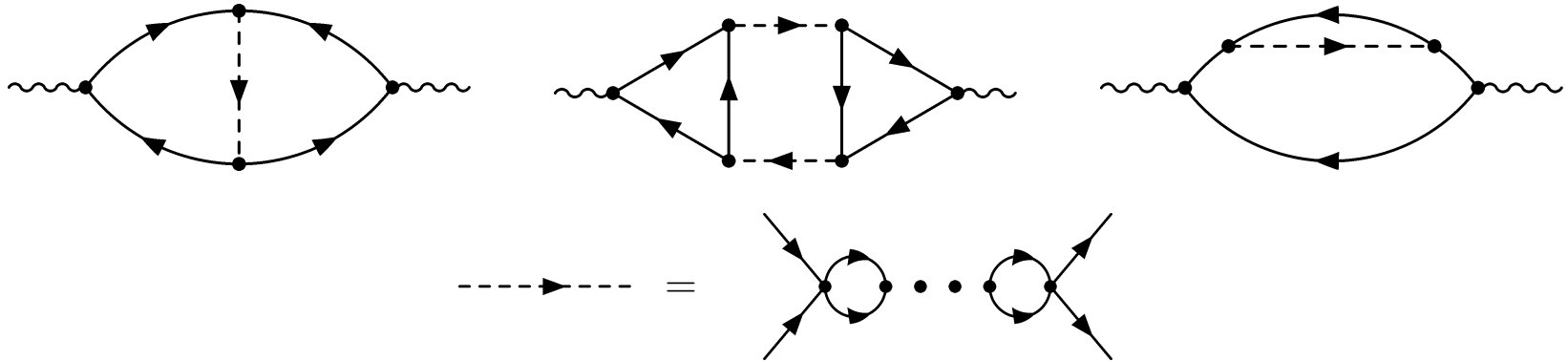
$$\Pi(P, q) = \text{Diagram}$$



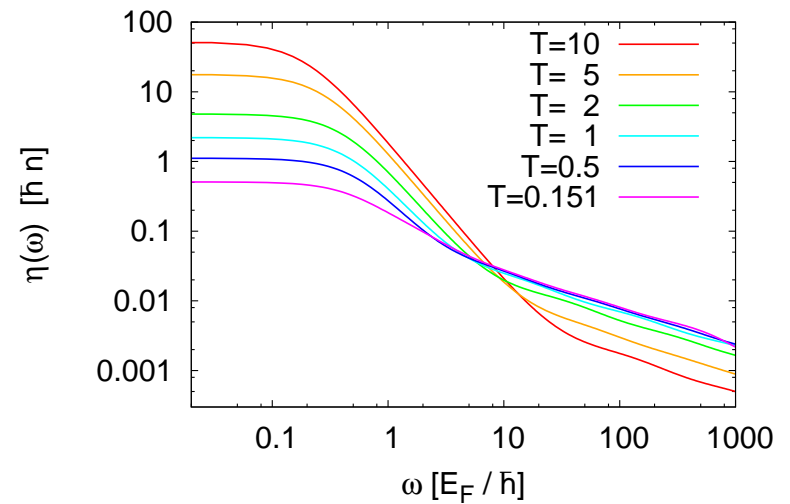
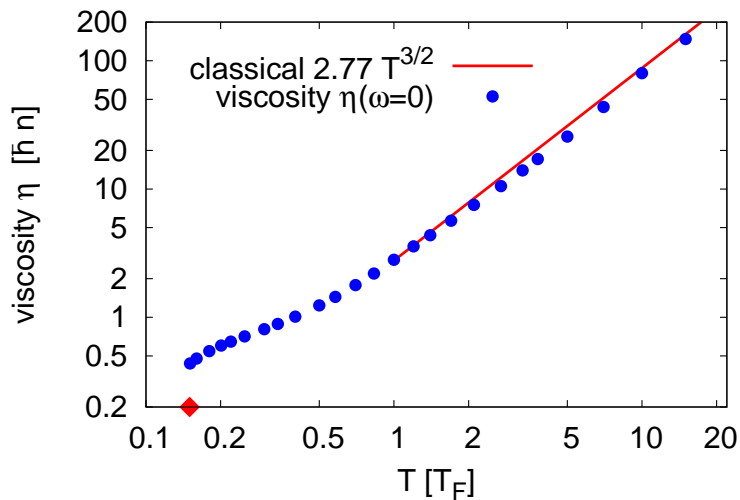
Minimum shear viscosity achieved on BEC side

IIIb. Quantum Field Theory

The diagrammatic content of the Boltzmann equation is known: Kubo formula with “Maki-Thompson” + “Azlamov-Larkin” + “Self-energy”



Can be used to extrapolate Boltzmann result to $T \sim T_F$



Short time behavior: OPE

Operator product expansion (OPE)

$$\eta(\omega) = \sum_n \frac{\langle \mathcal{O}_n \rangle}{\omega^{(\Delta_n - d)/2}} \quad \mathcal{O}_n(\lambda^2 t, \lambda x) = \lambda^{-\Delta_n} \mathcal{O}_n(t, x)$$

Leading operator: Contact density (Tan)

$$\mathcal{O}_c = C_0^2 \psi \psi \psi^\dagger \psi^\dagger = \Phi \Phi^\dagger \quad \Delta_c = 4$$

$\eta(\omega) \sim \langle \mathcal{O}_c \rangle / \sqrt{\omega}$. Asymptotic behavior + analyticity gives sum rule

$$\frac{1}{\pi} \int d\omega \left[\eta(\omega) - \frac{\langle \mathcal{O}_c \rangle}{15\pi \sqrt{m\omega}} \right] = \frac{\mathcal{E}}{3}$$

Randeria, Taylor (2010), Enss, Zwerger (2011), Hoffman (2013)

IV. Holography

DLCQ idea: Light cone compactification of relativistic theory in $d+2$

$$p_\mu p^\mu = 2p_+ p_- - p_\perp^2 = 0 \quad p_- = \frac{p_\perp^2}{2p_+} \quad p_+ = \frac{2n+1}{L}$$

Galilean invariant theory in $d+1$ dimensions.

String theory embedding: Null Melvin Twist

$$AdS_{d+3} \xrightarrow{\text{NMT}} Schr_d^2$$

$$Iso(AdS_{d+3}) = SO(d+2, 2) \supset Schr(d)$$

Son (2008), Balasubramanian et al. (2008)

Other ideas: Horava-Lifshitz (Karch, 2013)

Schrödinger Metric

Coordinates (u, v, \vec{x}, r) , periodic in v , $\vec{x} = (x, y)$

$$ds^2 = \frac{r^2}{k(r)^{2/3}} \left\{ \left[\frac{1 - f(r)}{4\beta^2} - r^2 f(r) \right] du^2 + \frac{\beta^2 r_+^4}{r^4} dv^2 - [1 + f(r)] du dv \right\} \\ + k(r)^{1/3} \left\{ r^2 d\vec{x}^2 + \frac{dr^2}{r^2 f(r)} \right\}$$

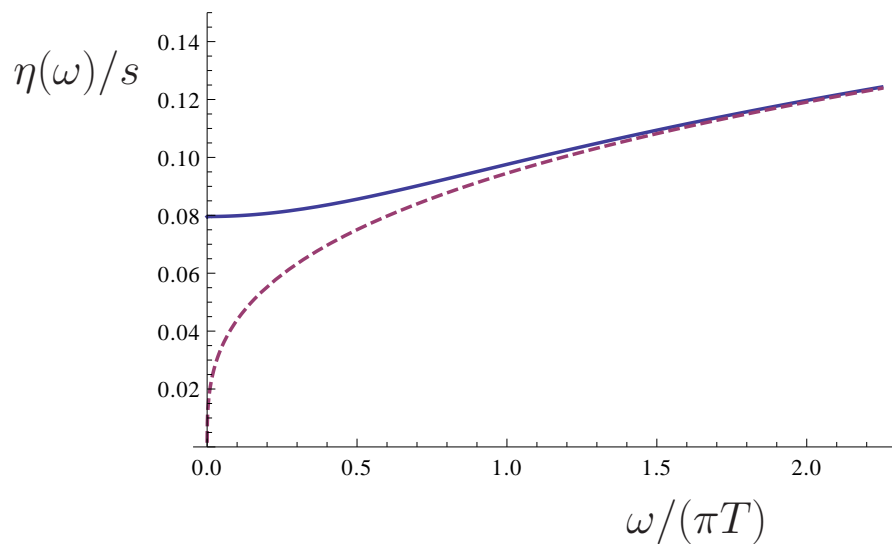
Fluctuations $\delta g_x^y = e^{-i\omega u} \chi(\omega, r)$ satisfy ($u = (r_+/r)^2$)

$$\chi''(\omega, u) - \frac{1 + u^2}{f(u)u} \chi'(\omega, u) + \frac{u}{f(u)^2} \omega^2 \chi(\omega, u) = 0$$

Retarded correlation function

$$G_R(\omega) = \frac{\beta r_+^3 \Delta v}{4\pi G_5} \left. \frac{f(u) \chi'(\omega, u)}{u \chi(\omega, u)} \right|_{u \rightarrow 0}.$$

Spectral function



$$\eta(0)/s = 1/(4\pi)$$

$$\eta(\omega \rightarrow \infty) \sim \omega^{1/3}$$

Kubo relation (incl. τ_π): $G_R(\omega) = P - i\eta\omega + \tau_\pi\eta\omega^2 + \kappa_R k^2$

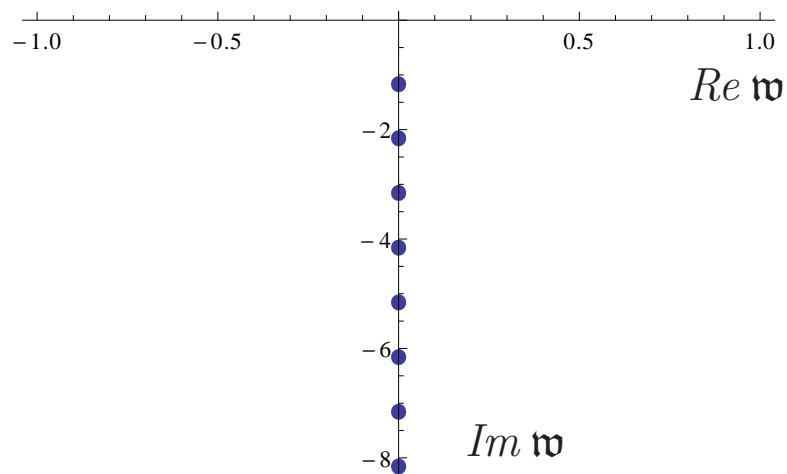
$$\tau_\pi T = -\frac{\log(2)}{2\pi}$$

$$AdS_5 : \tau_\pi T = \frac{2 - \log(2)}{2\pi}$$

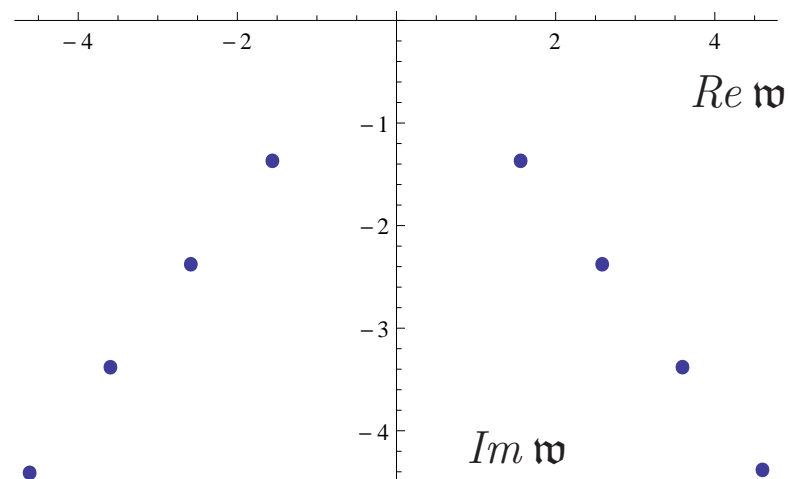
Range of validity of fluid dynamics: $\omega < T$

*Sch*₂: Cannot be matched to relaxation type hydro?

Quasi-normal modes



Sch_2^2



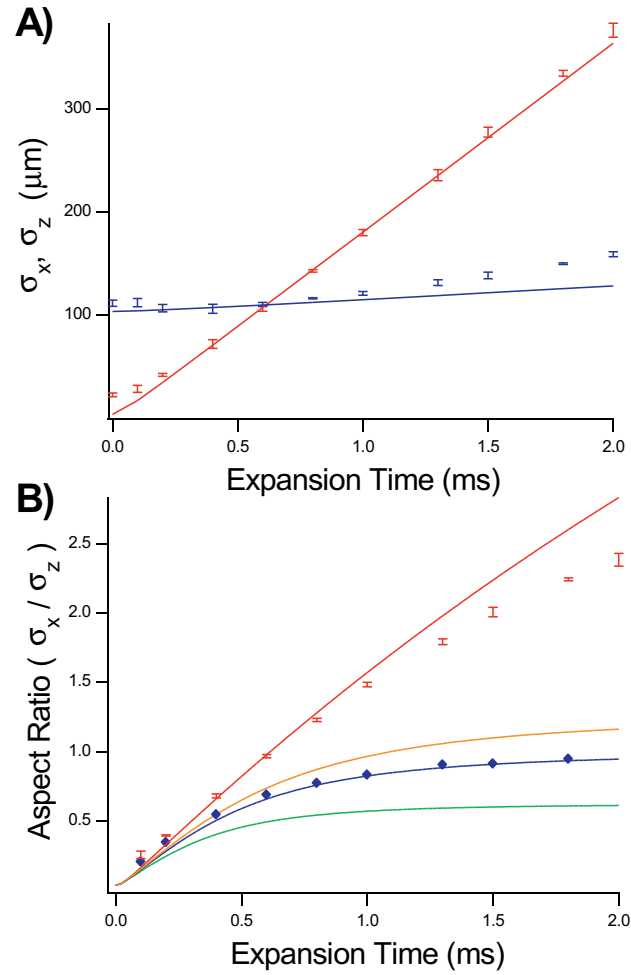
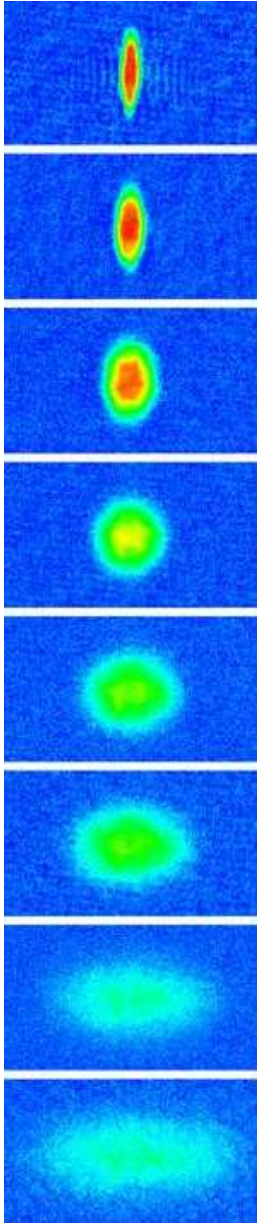
AdS_5

QNM's are stable, $Im \lambda < 0$.

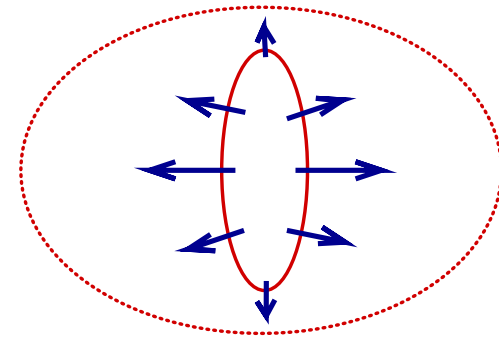
Pole at $\omega \sim iT$ limits convergence of fluid dynamics.

Modes overdamped in Sch_2^2 .

V. Experiments: Elliptic flow

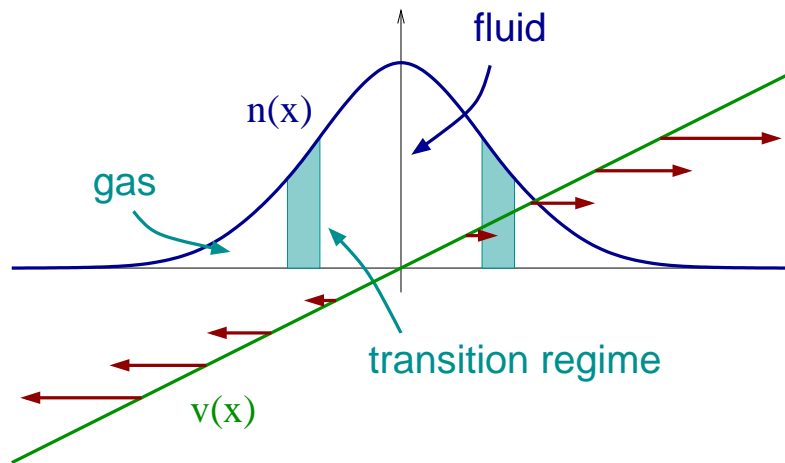


Hydrodynamic expansion converts
coordinate space
anisotropy
to momentum space
anisotropy

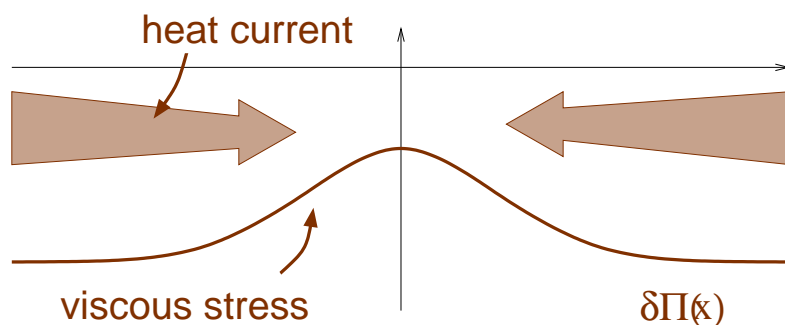


Determination of $\eta(n, T)$

Measurement of $A_R(t, E_0)$ determines $\eta(n, T)$. But:



The whole cloud is not a fluid.
Can we ignore this issue?



No. Hubble flow & low density
viscosity $\eta \sim T^{3/2}$ lead to
paradoxical fluid dynamics.

Possible Solutions

Combine hydrodynamics & Boltzmann equation. Not straightforward.

Hydrodynamics + non-hydro degrees of freedom (\mathcal{E}_a ; $a = x, y, z$)

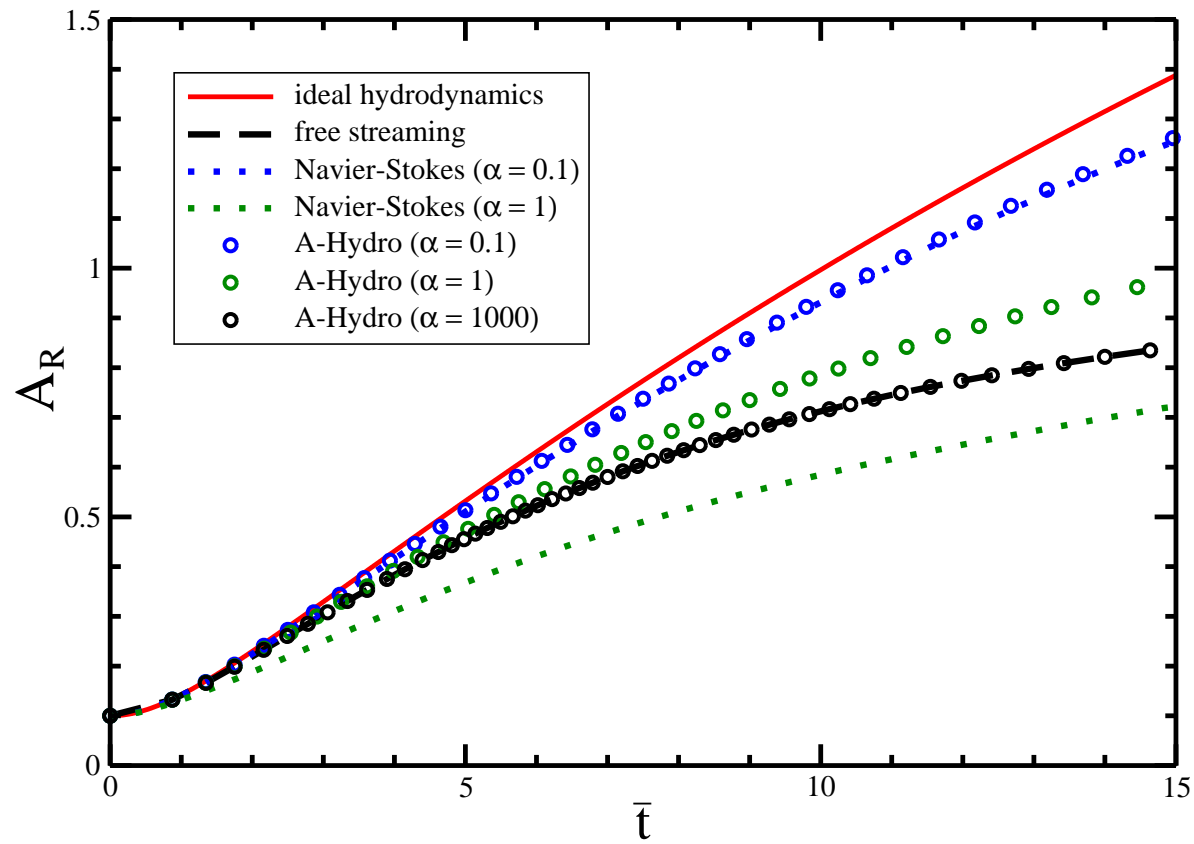
$$\frac{\partial \mathcal{E}_a}{\partial t} + \vec{\nabla} \cdot \vec{j}_a^\epsilon = -\frac{\Delta P_a}{2\tau} \quad \Delta P_a = P_a - P$$

$$\frac{\partial \mathcal{E}}{\partial t} + \vec{\nabla} \cdot \vec{j}^\epsilon = 0 \quad \mathcal{E} = \sum_a \mathcal{E}_a$$

τ small: Fast relaxation to Navier-Stokes with $\tau = \eta/P$

τ large: Additional conservation laws. Ballistic expansion.

Anisotropic Hydrodynamics: Aspect ratio



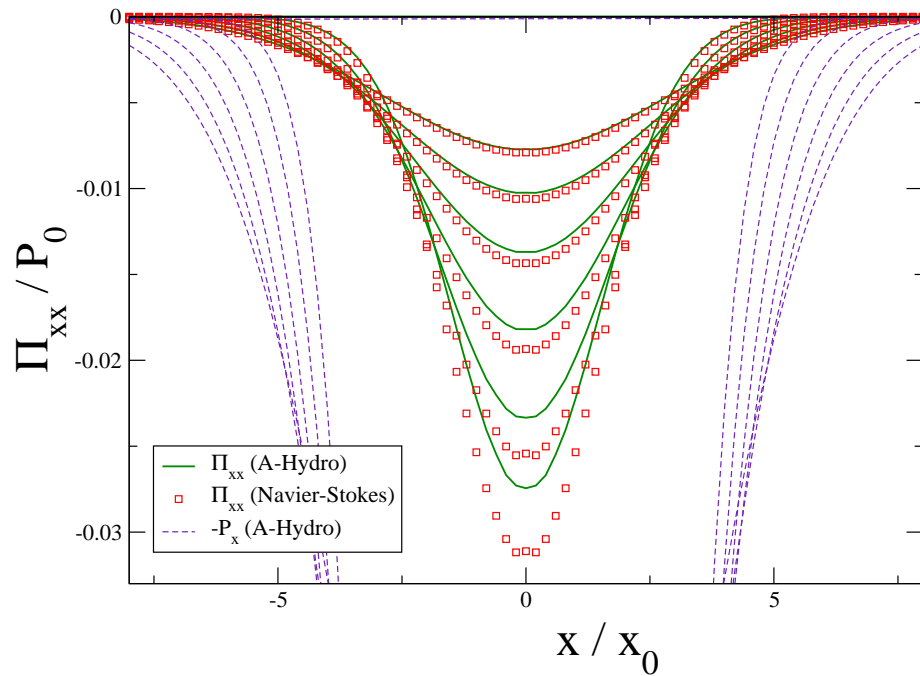
Consider $\eta = \alpha n$ and $\alpha \in [0, \infty)$

Navier-Stokes: Ideal hydro \rightarrow very viscous hydro.

A-hydro: Ideal hydro \rightarrow ballistic expansion.

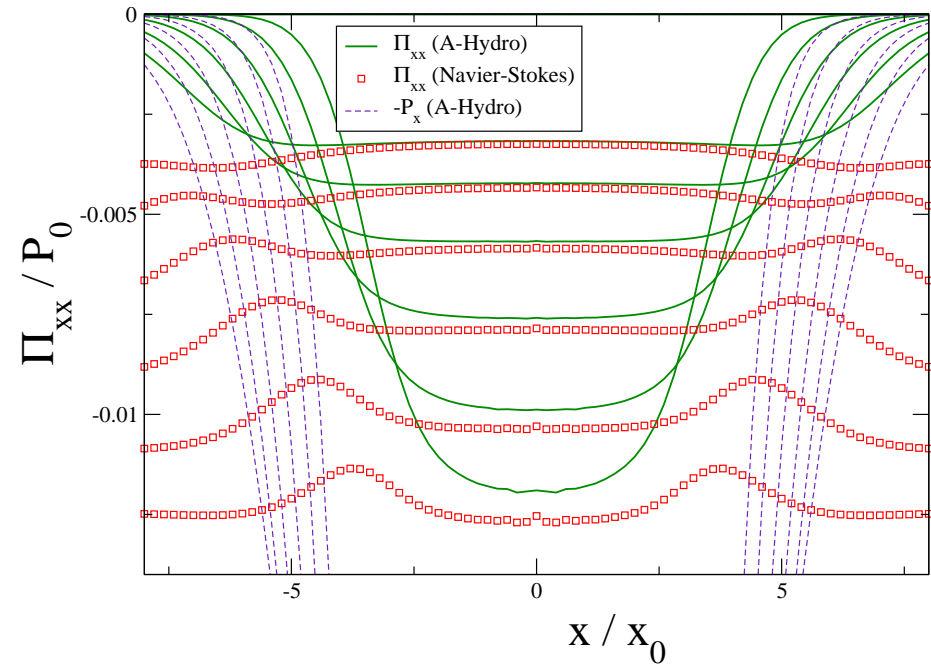
Anisotropic Hydrodynamics

$$\eta = \alpha_n n$$



Π_{xx} (Navier-Stokes)

$$\eta = \alpha_T (mT)^{3/2}$$



Π_{xx} (A-Hydro)

Outlook

Fluid dynamics as an E(F)T: Many interesting questions remain.

Experiment: Main issue is temperature, density dependence of η/s . How to unfold?

Need hydro codes that exit “gracefully” (anisotropic hydro, hydro+cascade, or LBE)

Quasi-particles vs quasi-normal modes (kinetics vs holography) unresolved. Need better holographic models, improved lattice calculations.