QCD and Dense Matter: Introduction

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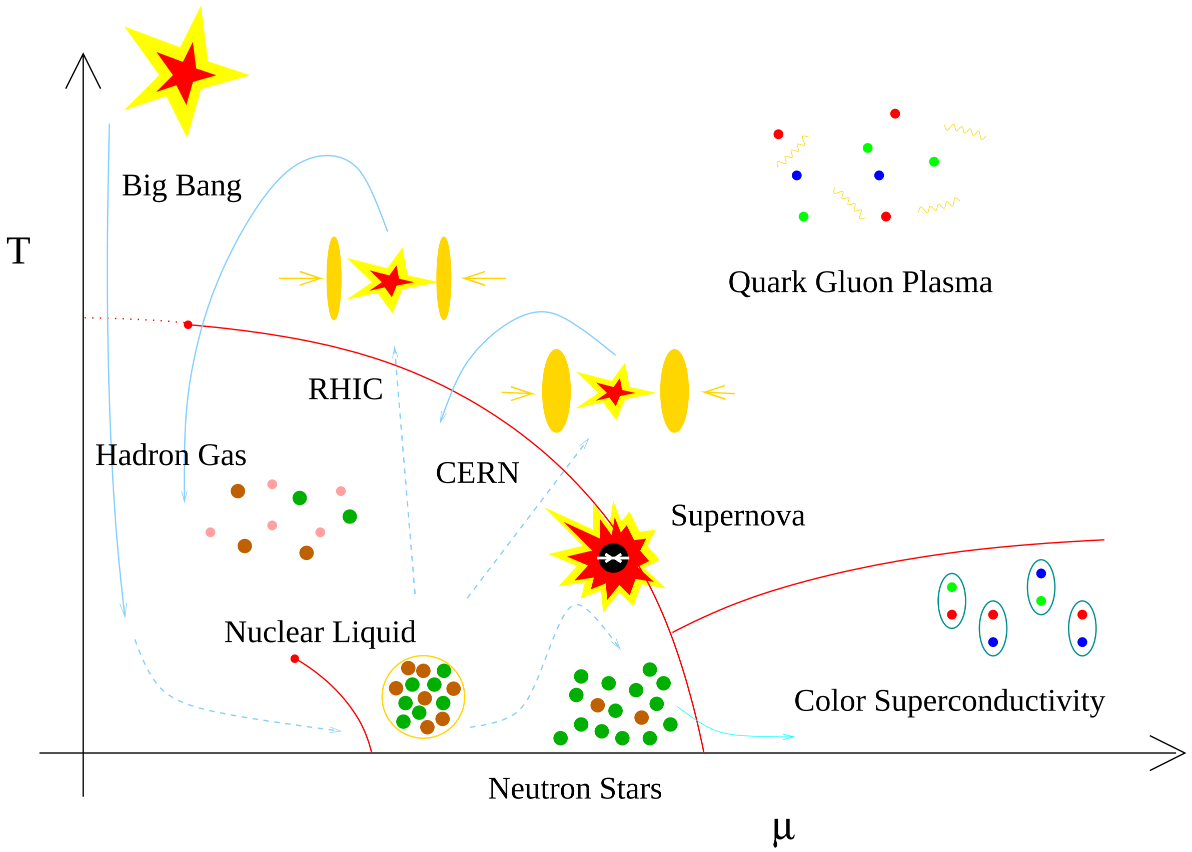
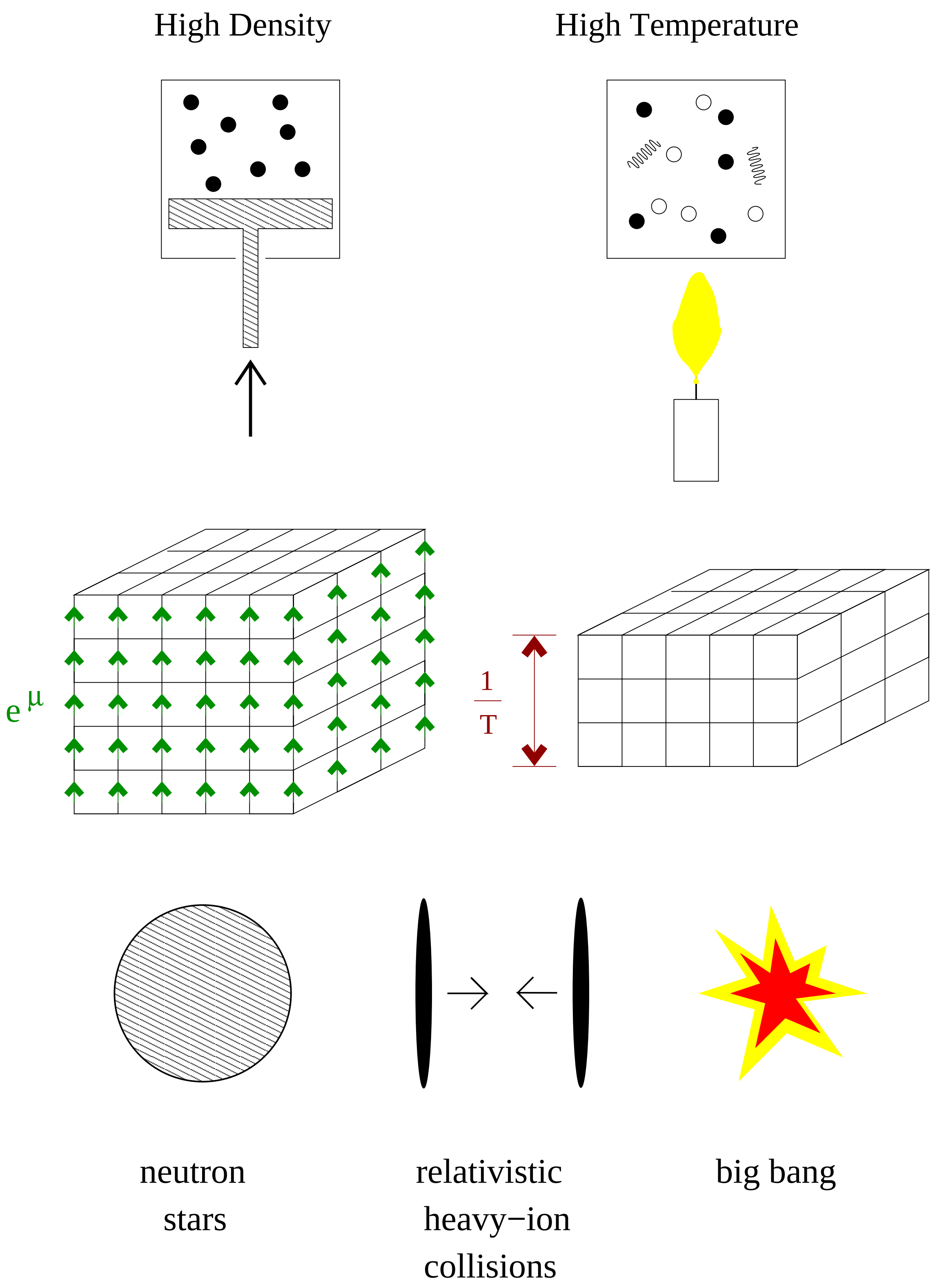


QCD in Xtreme Conditions



WHY?

- These conditions exist in the universe:

Neutron Stars, Big Bang

- Exploring the entire phase diagram is important to understanding the phase that we happen to live in

We cannot understand the **structure of hadrons** without understanding the **structure of the vacuum**.

We cannot understand the **structure of the vacuum** without understanding how it can be **modified**.

confined $\leftrightarrow$ Higgs $\leftrightarrow$ deconfined

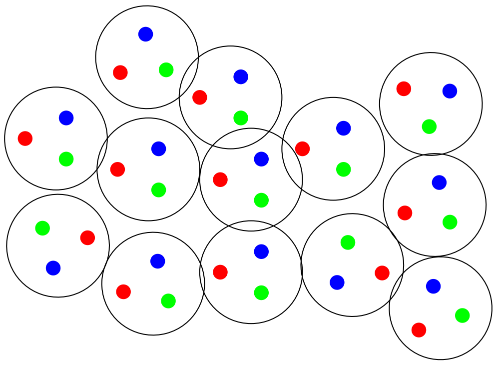
chiral symmetry breaking/restoration

- QCD simplifies in extreme environments:

Study QCD matter in a regime where **quarks and gluons** are indeed the correct **degrees of freedom**.

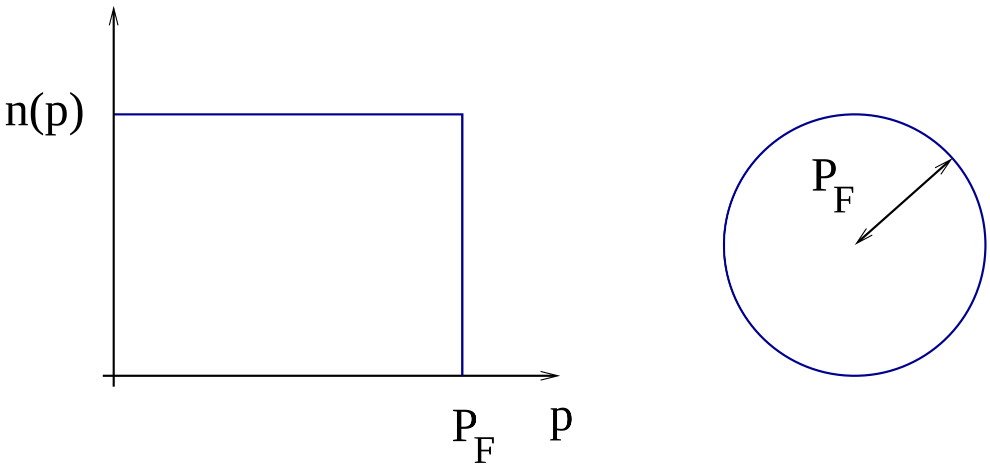
Dense Hadronic Matter

- consider baryon density $n_B \gg 1\,\mathrm{fm}^{-3}$



quarks expected to move freely

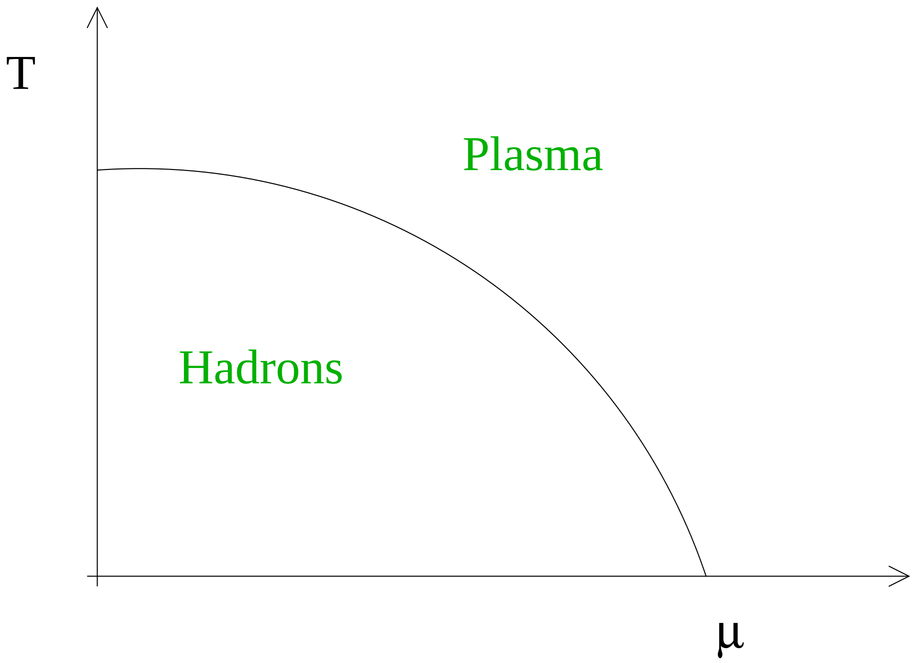
- ground state: cold quark matter (quark fermi liquid)



- only quarks with $p \sim p_F \gg \Lambda_{QCD}$ scatter
- $p_F \gg \Lambda_{QCD} \rightarrow$ effective coupling is weak

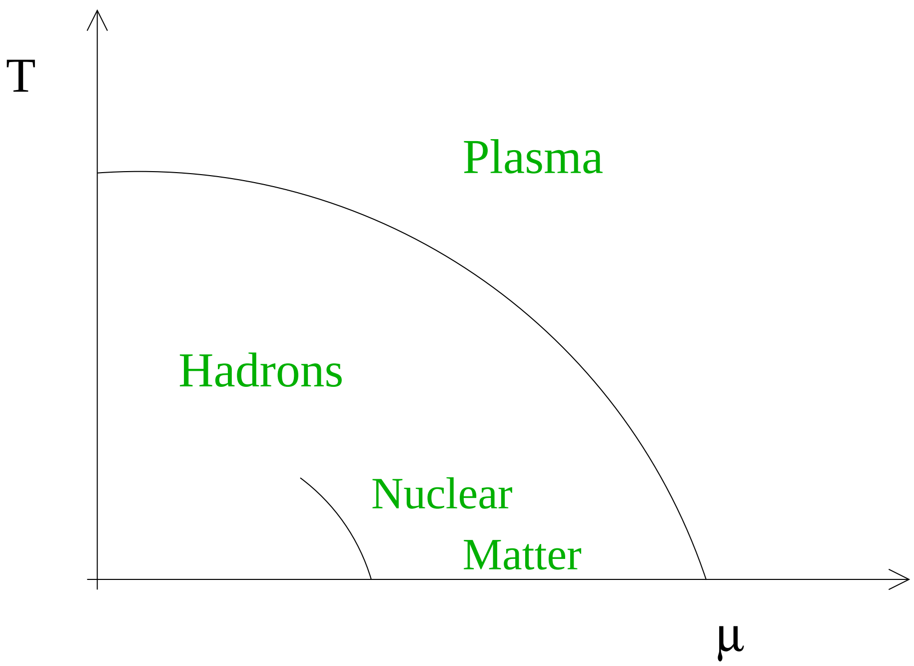
  $\rightarrow$ **no chiral symmetry breaking**
  $\rightarrow$ **no confinement**
  $\rightarrow$ **no dynamically generated masses**

Phase Diagram: First Try

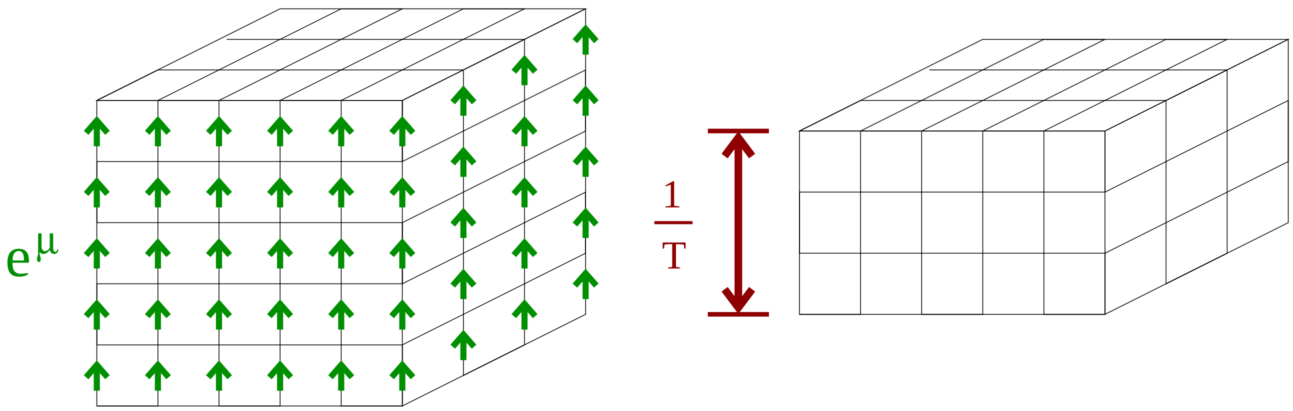


Where are we?

Phase Diagram: Second Try



Xtreme QCD on the Lattice



- Partition function

$$Z = \int dA \underbrace{\det(\not{D} + i\mu\gamma_4)}_{\text{complex}} e^{-S}$$

"sign" problem: no importance sampling

- progress in simulating small μ, $T \sim T_c$
  - improved re-weighting (Fodor, Katz)
  - imaginary chemical potential (Forcrand, Philipsen, ...)
  - Taylor expansion (Karsch et al.)
- or: evading the sign problem
  - $N_c = 2$, adjoint fermions, four-fermion models
  - $\mu_I \neq 0, \mu_B = 0$

Questions: Short Term

- Compare different methods (home work problem?)
- Trace E in $\mu - m_s$ plane
- Simulate Fermi surface (?)

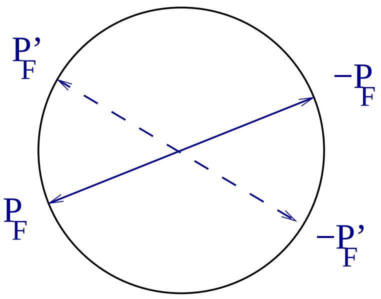
Questions: Long Term

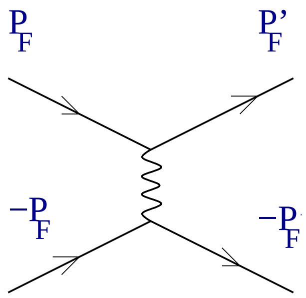
- What does the sign problem "mean"? Are QCD-like theories with a sign problem "intrinsically more complicated" than QCD-like theories without a sign problem?
- Solving rather than evading the sign problem

Superfluidity

- is the quark liquid stable?

dominant interaction



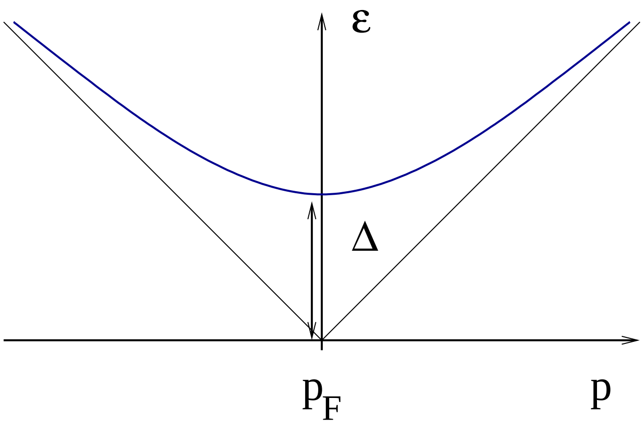


Uses the whole
Fermi surface
coherently

- very degenerate perturbation theory

$\Rightarrow$ instability (BCS) in weak coupling
$\Rightarrow$ $q(p_F)q(-p_F)$ condensate

- arbitrarily weak (attractive!) interaction leads to superfluidity/superconductivity (pairs neutral/charged)



$$\epsilon(p) = \sqrt{(p - p_F)^2 + \Delta^2}$$

$\rightarrow$ gap in spectrum, transport without dissipation, ...

- QCD: gluon exchange attractive in $\bar{3}$ channel

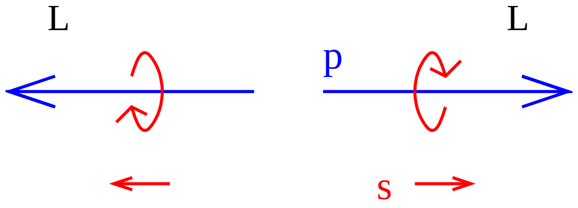
$$3 \times 3 = 6_S + 3_A \quad \text{flux reduced} \Rightarrow \text{attractive}$$

- spin-flavor-color wave function

$$(\uparrow\downarrow - \downarrow\uparrow) \times (ud - du) \times (rb - br) \qquad s = 0,\ I = 0,\ c = \bar{3}$$

order parameter: $\quad \Phi^a = \epsilon^{abc}\langle q^b C\gamma_5\tau_2 q^c\rangle$

- chiral symmetry is not broken



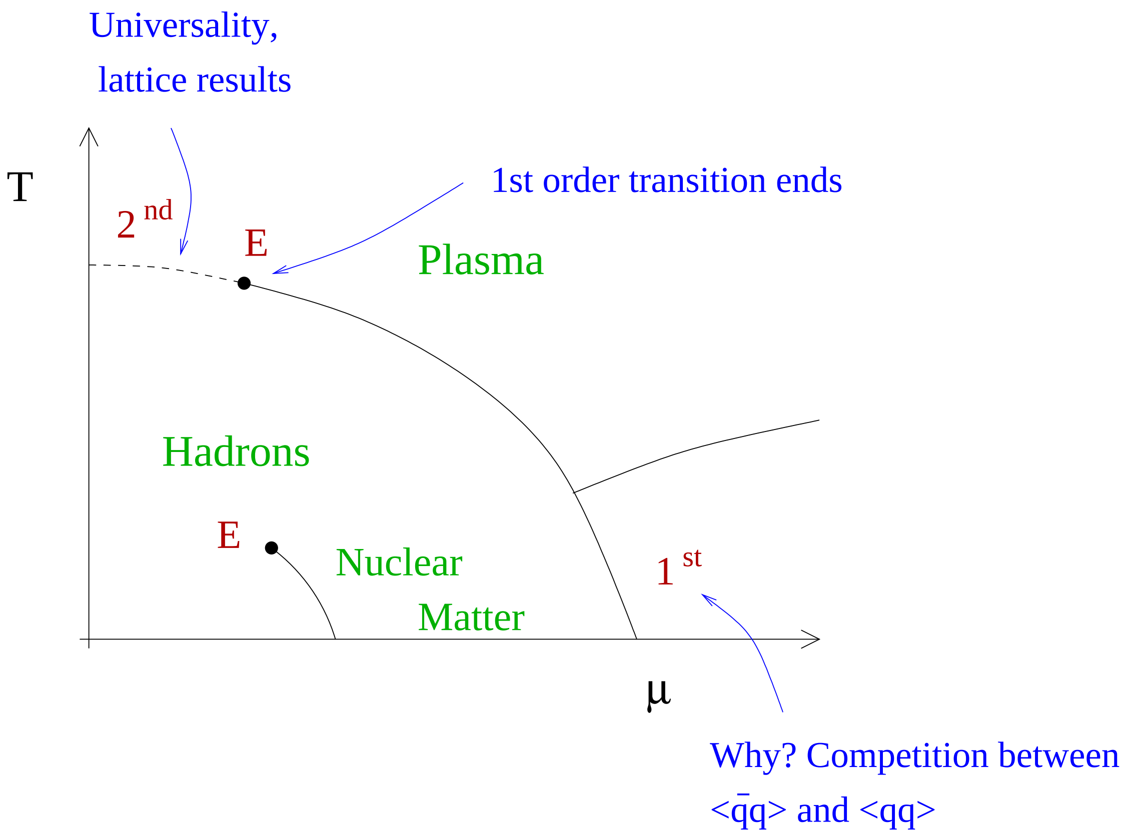
$$\Phi \sim \langle q_L q_L\rangle - \langle q_R q_R\rangle$$

- color symmetry broken by Higgs mechanism (Meissner effect)

$$\Phi^a \in [\bar{3}];\ \Rightarrow\ SU(3) \to SU(2)$$

– 5/8 gluons acquire mass via Higgs mechanism
– $SU(2)$ is confined

Phase Diagram: Second Revision

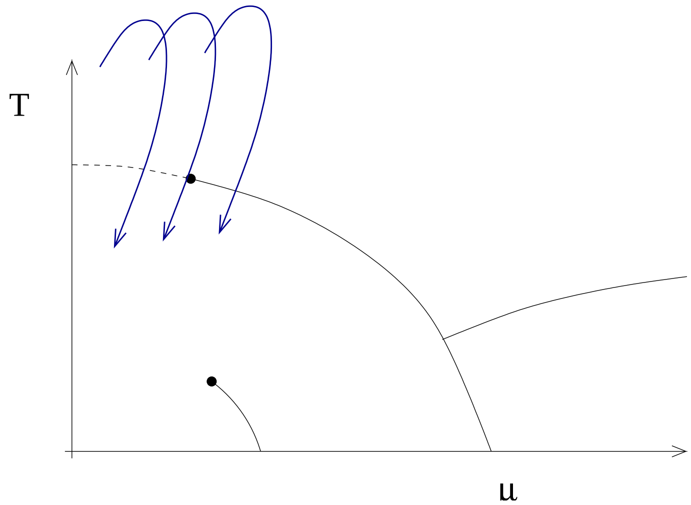


critical endpoint (E) persists even if $m \neq 0$

Questions

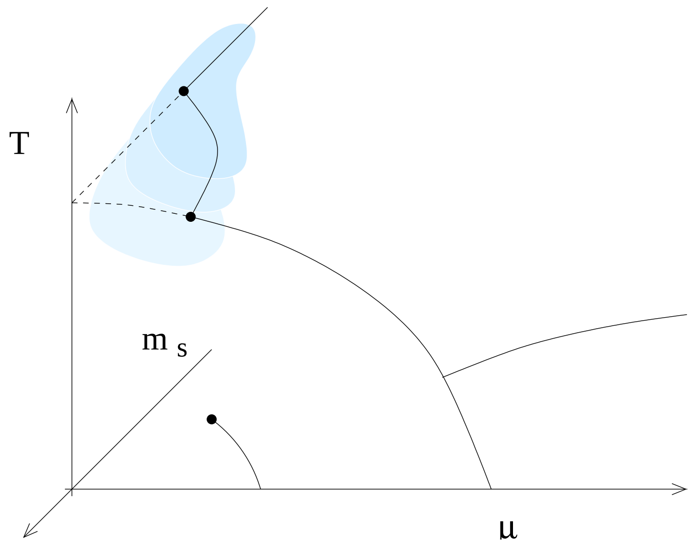
- Critical fluctuations in heavy ion collisions?

  energy scan



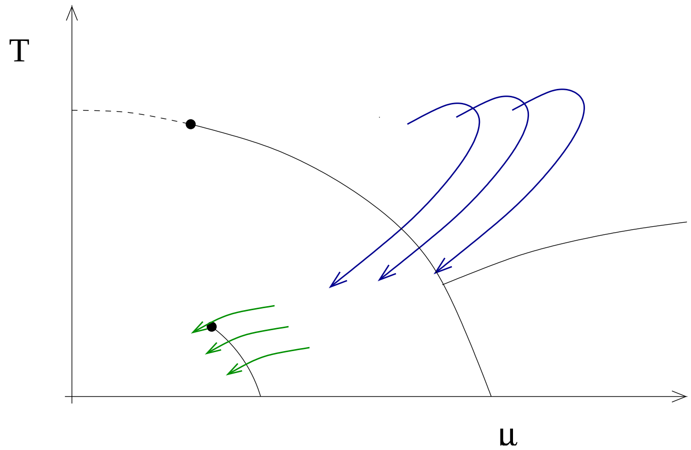
- Can we trace E in the $\mu - m_s$ plane?

  m_s scan

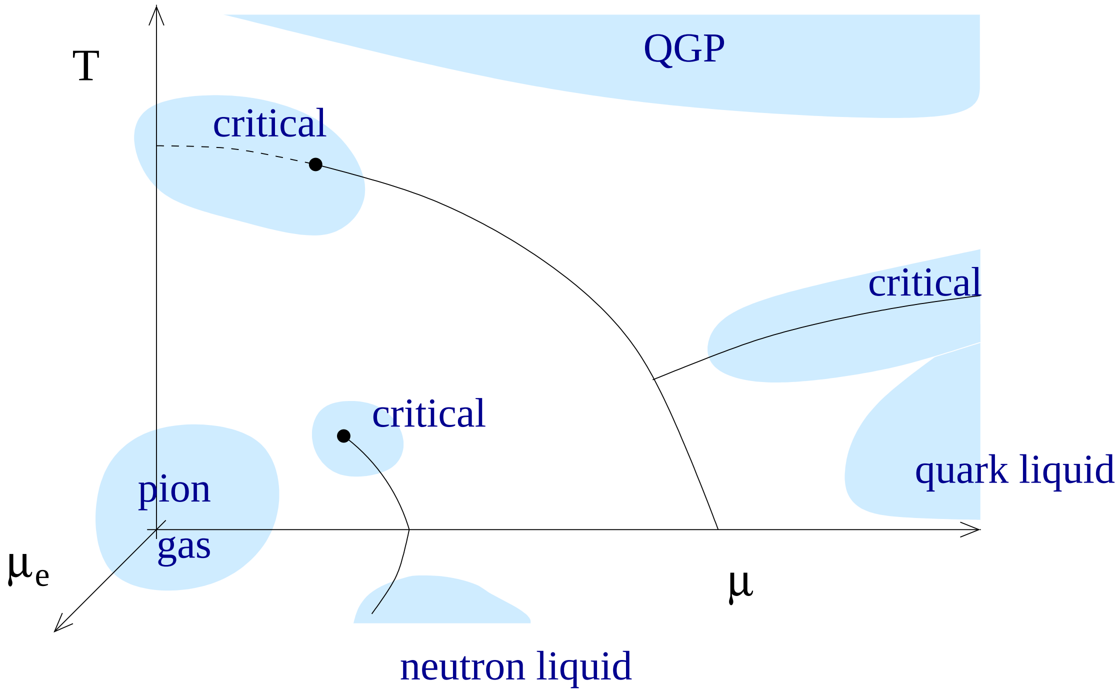


- More critical phenomena?

  energy/system size



Phase Diagram: Small Parameters?

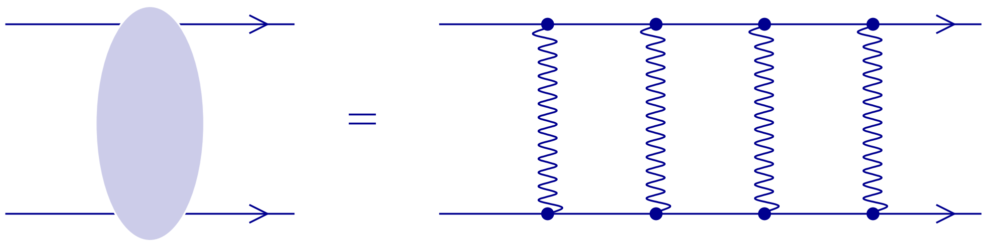


Other directions?

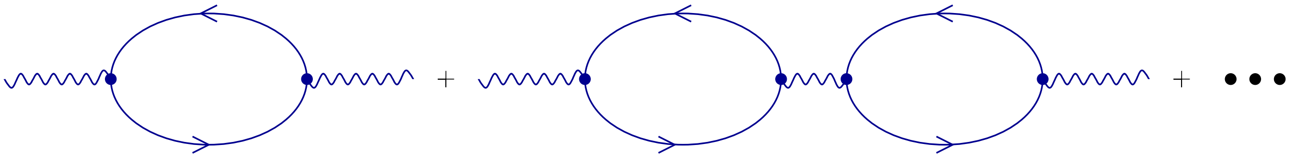
$$m_{u,d}, m_s, N_f, N_c, \ldots$$

Very large density: gap equation

- $\mu \gg \Lambda_{QCD}$: perturbative forces dominate



- gluon exchange: small angle scattering dominates $\rightarrow$ medium effects important



$$D_E = \frac{1}{\vec{q}^2 + 2m^2} \qquad m^2 = \frac{N_f}{4\pi^2} g^2 \mu^2 \qquad \text{Debye screening}$$

$$D_M = \frac{1}{\vec{q}^2 + i\frac{\pi}{2} m \frac{\omega}{q}} \qquad \omega < q \qquad \text{Landau damping}$$

- consider $\omega \simeq \Delta$. Typical momenta

$$q_E \simeq g\mu, \qquad q_M \simeq (g^2 \mu^2 \Delta)^{1/3}$$

Superconductivity driven by magnetic forces!

- Note: Perturbation theory is self-consistent

$$\mu \to \infty \quad \Rightarrow \quad q_M \gg \Lambda_{QCD}$$

Eliashberg Equation

- retardation important: Eliashberg theory

$$\Delta(p_0) = \frac{g^2}{18\pi^2}\int dq_0 \left\{ \log\left(\frac{b_M}{|p_0-q_0|}\right) + \ldots \right\} \frac{\Delta(q_0)}{\sqrt{q_0^2+\Delta(q_0)^2}}$$

collinear log BCS log

- double logarithmic behavior

$$\Delta_0 \sim \alpha_s \Delta_0 \left[\log\left(\frac{\mu}{\Delta_0}\right)\right]^2 \quad \rightarrow \quad \Delta_0 \sim \exp\left(-\frac{c}{\sqrt{\alpha_s}}\right)$$

- more careful analysis (2SC phase)

$$\Delta_0 = 512\pi^4 \mu g^{-5} \exp\left(-\frac{\pi^2+4}{8}\right) \exp\left(-\frac{3\pi^2}{\sqrt{2}g}\right)$$

Son (1998)
Schafer, Wilczek (1999), Pisarski, Rischke (1999), Hong et al (1999)
Brown, Liu, Ren (1999)

- condensation energy

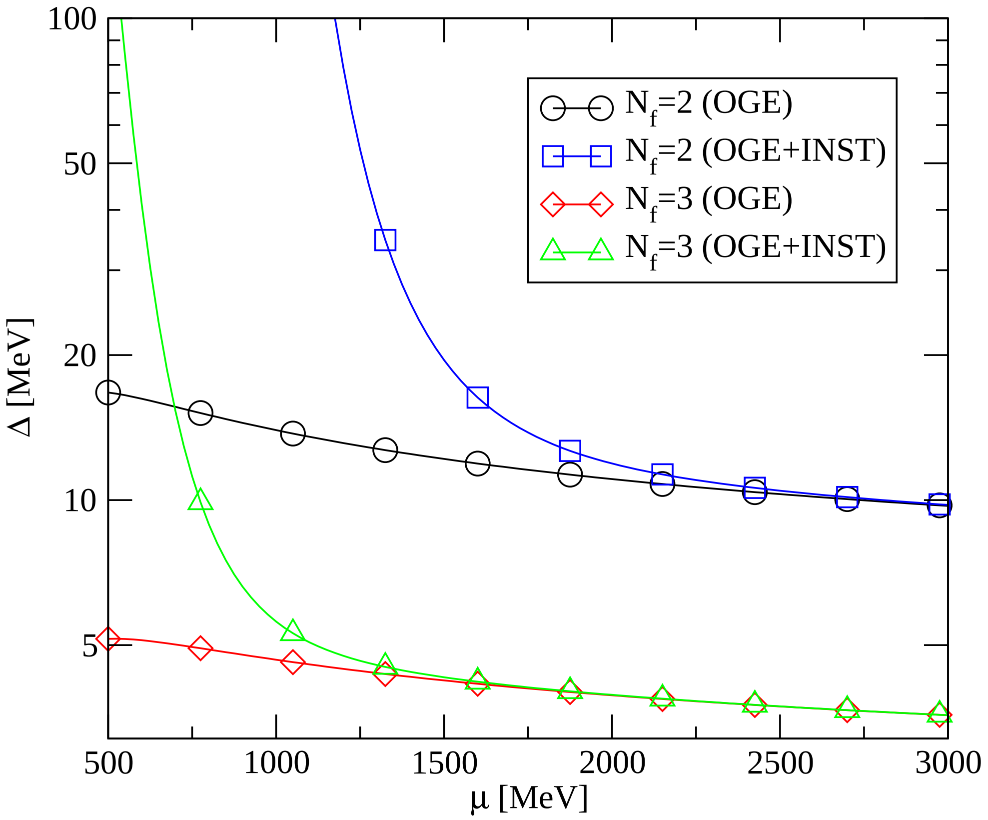
$$\epsilon \simeq N\Delta_0^2 \left(\frac{\mu^2}{4\pi^2}\right)$$

- typical numbers

$$\Delta_0 \simeq (5-20)\,\text{MeV} \qquad (\rho \sim 5\rho_0)$$

Corrections?

Example: Instantons $\Delta = \Delta_0 \exp(G_I)$



How to get control over the magnitude of the gap:

- numbers: check $N_c = 2$, $\mu_I \neq 0$ on the lattice
- OGE vs Instantons: check $m_{GB}[U(1)_A] > \Delta$?

Beautiful Case: $N_f = 3$

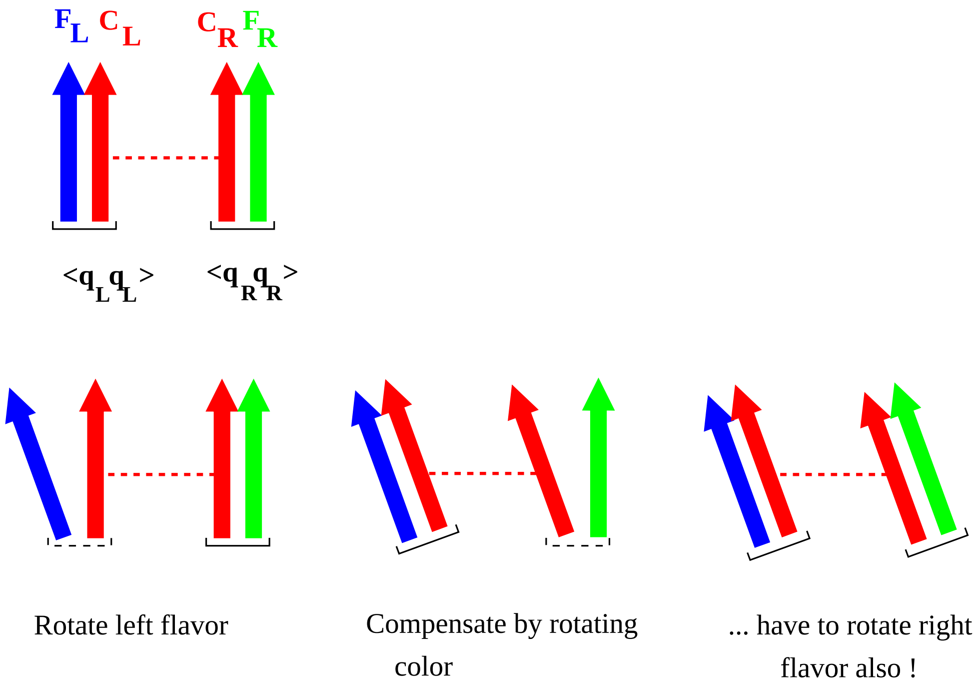
- quarks q_i^a $[i = (u, d, s);\ a = (r, b, g)]$. Order parameter?

$$\langle q_i^a q_j^b \rangle = \phi\, \epsilon^{abI} \epsilon_{ijI} \quad \text{Color} - \text{Flavor} - \text{Locking}$$

$$\begin{array}{ll} \langle ud \rangle = \langle us \rangle = \langle ds \rangle & \text{Color and Flavor} \\ \langle rb \rangle = \langle rg \rangle = \langle bg \rangle & \text{direction aligned} \end{array}$$

- symmetry breaking pattern:

$$SU(3)_L \times SU(3)_R \times [SU(3)]_C \times U(1) \to SU(3)_{C+F}$$



- novel mechanism for chiral symmetry breaking:

$$\langle \psi_L \psi_L \rangle = -\langle \psi_R \psi_R \rangle$$

breaks chiral $SU(3)_L \times SU(3)_R$ symmetry because

$$SU(3)_L \overset{Lock}{\longleftrightarrow} SU(3)_C \overset{Lock}{\longleftrightarrow} SU(3)_R$$

- all quarks and gluons acquire a mass gap

CFL Phase: Excitations

- quark pairs are charged, but $U(1)_{EM}$ remains unbroken

$$\gamma^* = \alpha_{11}\gamma + \alpha_{12}g_3 + \alpha_{13}g_8$$

Material is a transparent insulator

- excitations

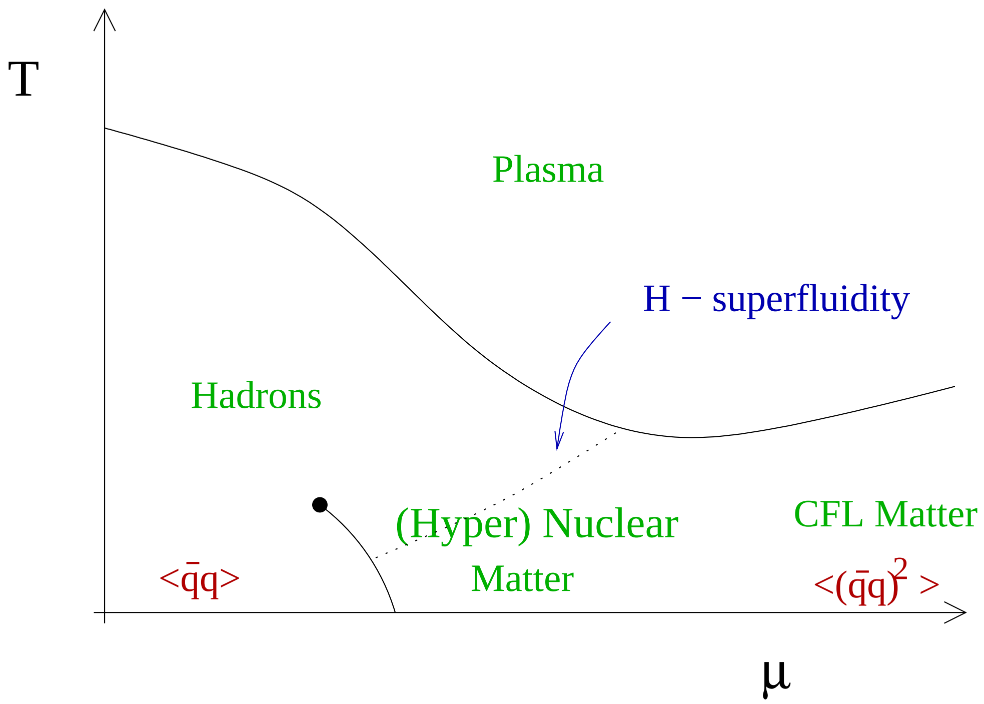
$[8]$	$Q = 0, \pm 1$	Goldstones (π, K, η)	$(QQ)(QQ)^{-1}, \ldots$
$[8] + [1]$	$Q = 0, \pm 1$	baryons (p, n, Λ, Σ)	$(Q)(QQ), \ldots$
$[8]$	$Q = 0, \pm 1$	vectors $(\rho, K^*, \ldots)$	$g, QQ^{-1}, \ldots$
$[1]$	$Q = 0$	$U(1)_A$ Goldstone (η')	$QQ^{-1}, (QQ)(QQ)^{-1}, \ldots$
$[1]$	$Q = 0$	$U(1)_B$ Goldstone	$QQ^{-1}, (QQ)(QQ)^{-1}, \ldots$

- order parameters

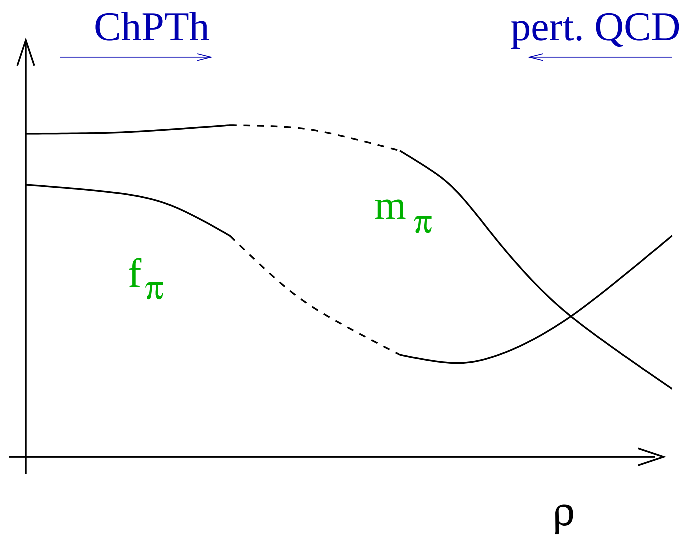
$\langle \bar{q}q \rangle$	chiral symmetry breaking
$\langle (uds)(uds) \rangle$	H superfluidity

Continuity between quark and hadron matter?

Phase Diagram: Third Revision



Density dependence of hadronic parameters calculable at both low density (effective theory) and high density (perturbative QCD).



Questions

- less symmetric states

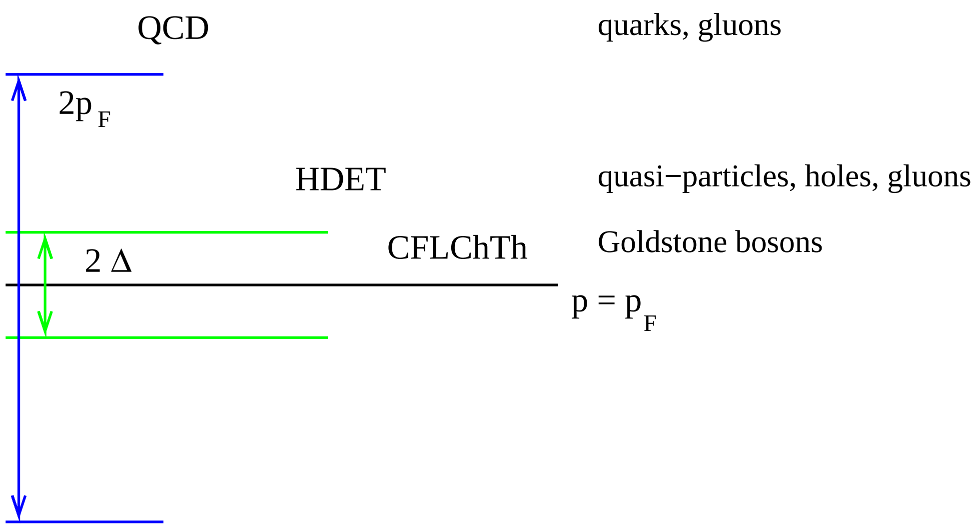
$$m_s \neq 0, \qquad \mu_e \neq 0$$

- spectrum of excitations

$$m_\pi,\, m_K,\; f_\pi, \ldots$$

- transport properties
  - mean free path $(\nu, \gamma, \ldots)$
  - specific heat, thermal conductivity
  - neutrino emissivity

Effective Field Theories



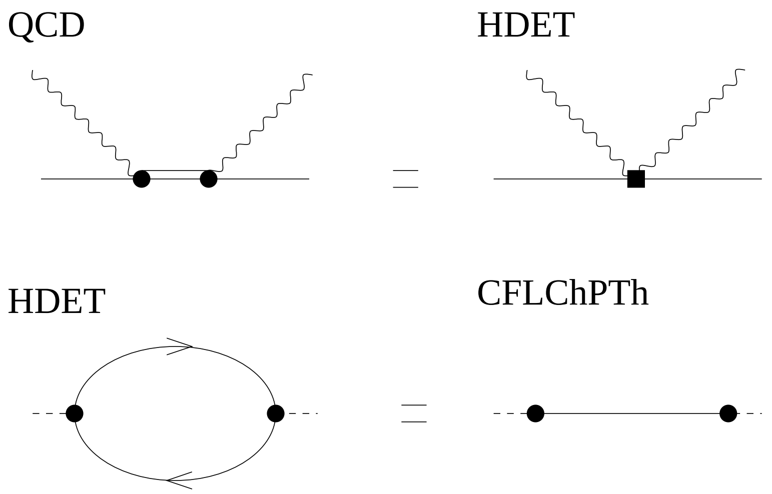
hierarchy of effective theories

QCD $\quad \mathcal{L} = \bar{\psi}(i\not{D} + \mu\gamma_0)\psi - \frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu}$

HDET $\quad \mathcal{L} = \psi_v^\dagger(iv \cdot D)\psi_v - \frac{\Delta}{2}\psi_{-v}^T C\psi_v - \frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} + \ldots$

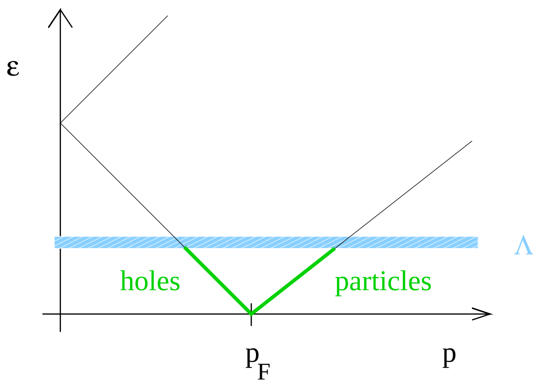
CFLChPTh $\quad \mathcal{L} = \frac{f_\pi^2}{4}\mathrm{Tr}\left[\nabla_0\Sigma\nabla_0\Sigma^\dagger - v^2\vec{\nabla}\Sigma\vec{\nabla}\Sigma^\dagger\right] + \ldots$

coefficients determined by matching Green functions



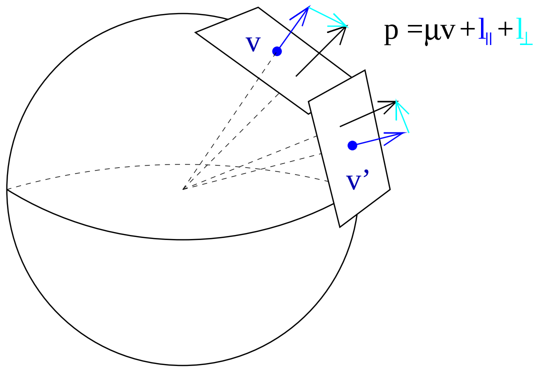
High density effective theory

- quasi-particles (holes)



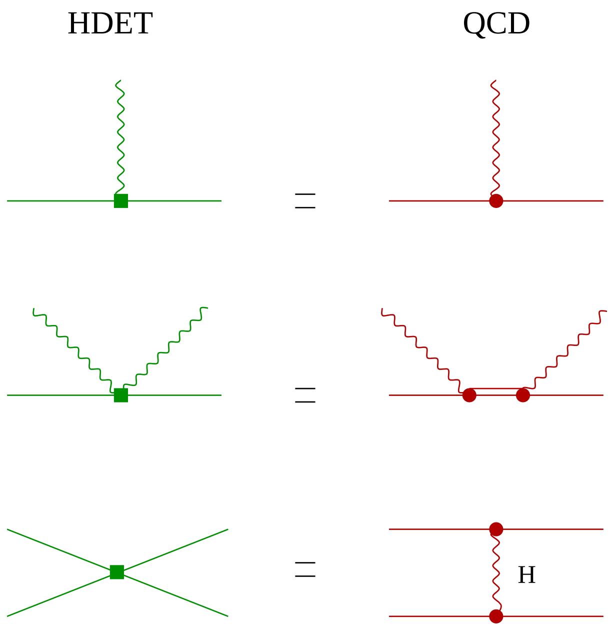
$$E_\pm = -\mu + \sqrt{\vec{p}^2 + m^2}$$
$$\simeq -\mu \pm |\vec{p}|$$

- effective field theory on v-patches



$$\psi_{v\pm} = e^{-i\mu v\cdot x}\left(\frac{1 \pm \vec{\alpha}\cdot\vec{v}}{2}\right)\psi$$

- effective lagrangian for ψ_{v+}

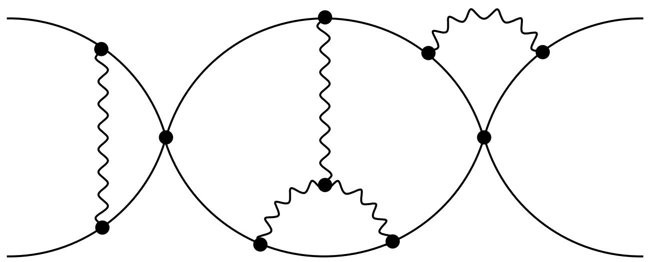


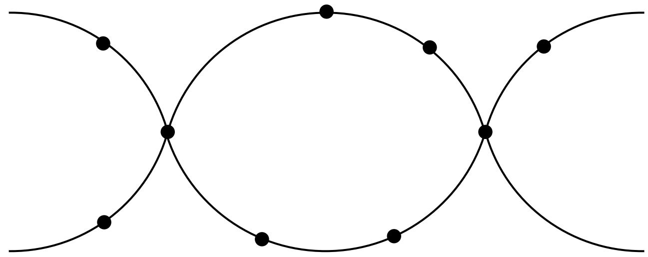
$$\mathcal{L} = \psi_v^\dagger (iv\cdot D)\psi_v$$

$$+ \frac{1}{2p_F}\psi_v^\dagger\left[(\vec{\alpha}_\perp\cdot\vec{D})^2 + MM^\dagger\right]\psi_v$$

$$+ \frac{\Gamma_{\vec{v}\cdot\vec{v}'}}{p_F^2}(\psi_{v'}^\dagger\psi_v)(\psi_{v'}^\dagger\psi_v) + \ldots$$

Power Counting





$$\boxed{\delta = \sum_i \left[(k-4)V_k^S + (k-2-f_k)V_k^H \right] + E_Q + 4 - 2N_c}$$

V_k^S	soft vertices of $O(l^k)$
V_k^H	hard vertices of $O(l^k)$
f_k	number of fermion fields in V_k^H
E_Q	external quark lines
N_c	connected graphs in hard graph

- resum quark loops in gluon n-pt fcts at $l \sim g\mu$ (HDL)
  ... but not quark self energies or vertex fcts
- four quark operators are leading order
- six quark operators (etc) are suppressed

Effective Chiral Theory

- oscillations of the order parameter

$$X_i^a = \epsilon_{ijk}\epsilon^{abc}(\psi_L)_j^b(\psi_L)_k^c, \qquad \langle X_i^a\rangle \sim \delta_i^a$$

$$Y_i^a = \epsilon_{ijk}\epsilon^{abc}(\psi_R)_j^b(\psi_R)_k^c, \qquad \langle Y_i^a\rangle \sim \delta_i^a$$

- low energy degrees of freedom

$$\Sigma = XY^\dagger = \exp\left(\frac{i\phi^a\lambda^a}{f_\pi}\right) \qquad \phi^a = (\pi, K, \eta, \eta')$$

$$\text{e.g. } K^0 \sim \epsilon^{abc}\epsilon_{ade}(\bar{u}_R^b C \bar{s}_R^c)(d_L^d C u_L^e)$$

- effective theory

$$\begin{aligned} \mathcal{L} &= \tfrac{f_\pi^2}{4}\mathrm{Tr}\left(\nabla_0\Sigma\nabla_0\Sigma^\dagger - v^2\vec{\nabla}\Sigma\vec{\nabla}\Sigma^\dagger\right) \\ &\quad + A\mathrm{Tr}(M\Sigma^\dagger)Ve^{i\Theta} \qquad \leftarrow \quad U_A(1) \text{ anomaly} \\ &\quad + B_1\left[\mathrm{Tr}(M\Sigma^\dagger)\right]^2 + \ldots \end{aligned}$$

- expansion

$$\mathcal{L} \sim f_\pi^2\Delta^2\left(\frac{\vec{\partial}}{\Delta}\right)^{N_1}\left(\frac{\partial_0 + MM^\dagger/p_F}{\Delta}\right)^{N_2}\left(\frac{MM}{p_F^2}\right)^{N_3}(\Sigma)^{N_4}(\Sigma^\dagger)^{N_4}$$

Note:

$$\Lambda_{\chi SB} \sim \Delta \ll 4\pi f_\pi$$

$$\frac{M^2}{p_F^2} \ll \frac{M^2}{p_F\Delta}$$

Quark Mass Expansion

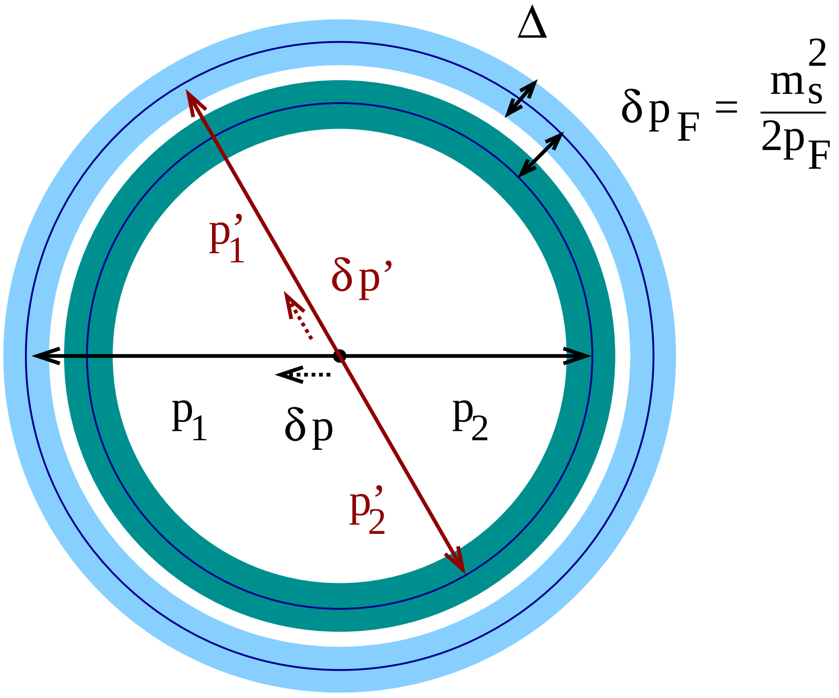
- expansion parameter

$$(m/p_F), (m/\Delta), (m/\Lambda), \ldots ?$$

- two parameters appear in matching procedure

$$\left(\frac{m^2}{p_F\Delta}\right), \quad \left(\frac{m^2}{p_F^2}\right) = \left(\frac{\Delta}{p_F}\right)\left(\frac{m^2}{p_F\Delta}\right)$$

- scale $m^2/(p_F\Delta)$ appears in BCS: consider pairing between two species with different Fermi momenta

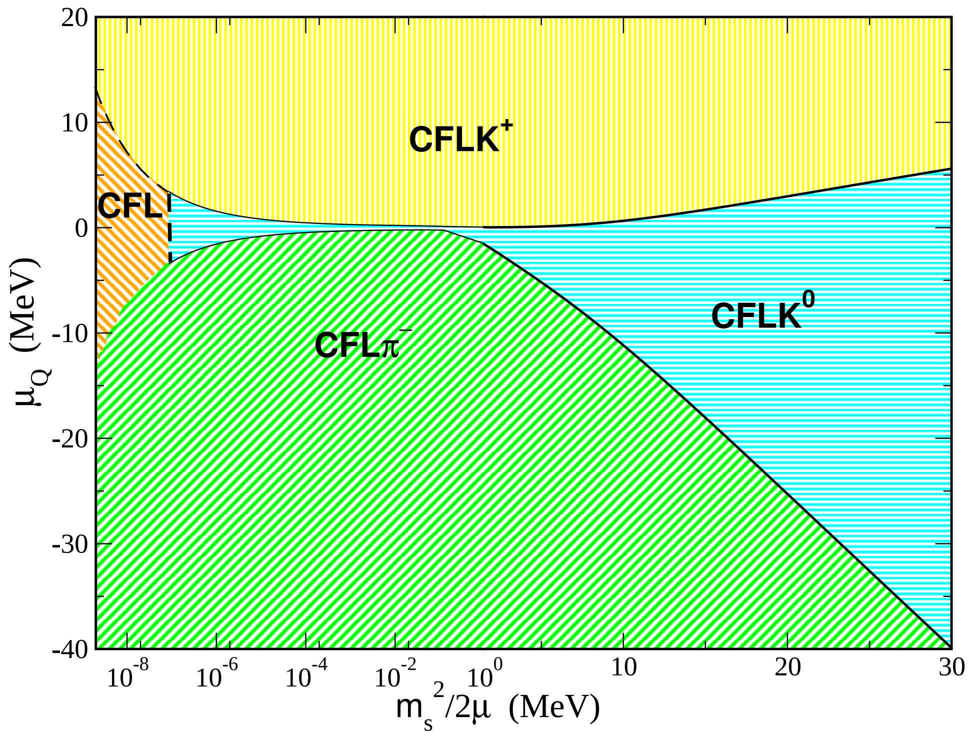


$$\delta p_f \sim \Delta \longrightarrow \text{unlocking transition}$$

$$\Rightarrow \text{mass expansion breaks down if } \frac{\delta m^2}{p_F\Delta} \sim 1$$

Phase Diagram of CFL phase

$$V(\Sigma) = \frac{f_\pi^2}{2}\mathrm{Tr}\left(W_L \Sigma W_R \Sigma^\dagger\right) - A\mathrm{Tr}(M\Sigma^\dagger) - B_1\left[\mathrm{Tr}(M\Sigma^\dagger)\right]^2 + \ldots$$

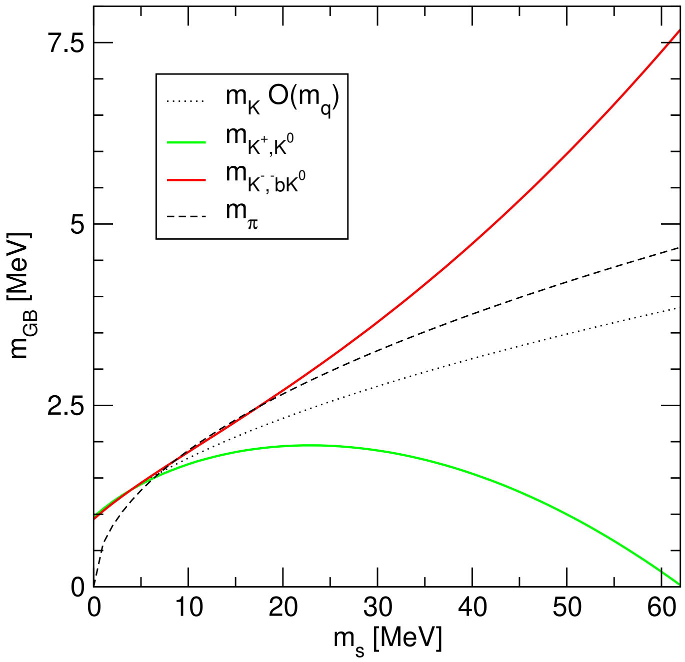


T. Schäfer, P. Bedaque (2002)

D. Kaplan, S. Reddy (2002)

The Role of the Strange Quark Mass

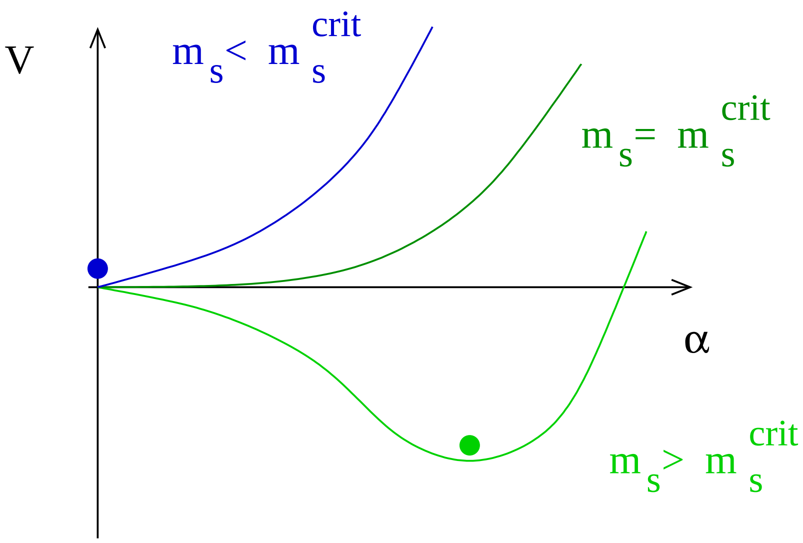
- competition between mass term and effective chemical potential $\Rightarrow K^0$ mass vanishes if $m_s > m_s|_{crit}$



$$m_s|_{crit} \simeq 3 m_u^{1/3} \Delta^{2/3}$$

$\Rightarrow$ kaon condensation

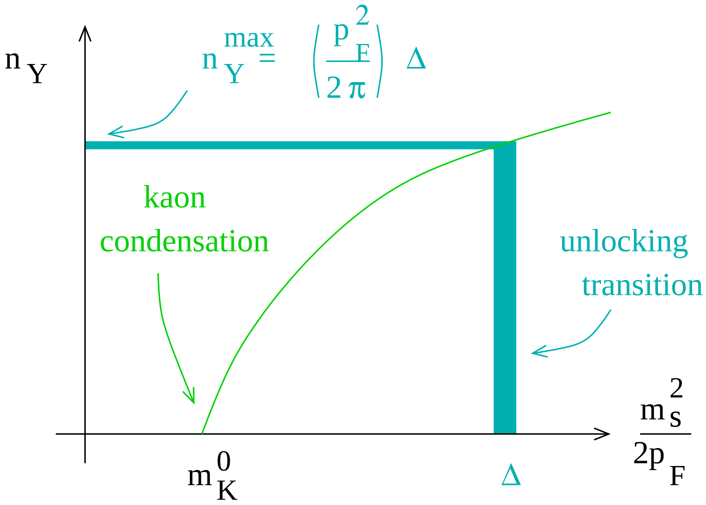
- new vacuum state

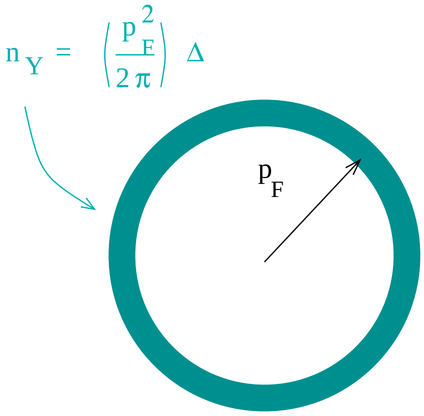


$$\Sigma = \exp(i\alpha\lambda_4)$$

$$V(\alpha) = -f_\pi^2 \left\{ \frac{m_s^2}{4p_F} \sin^2(\alpha) + (m_K^{(0)})^2 (\cos(\alpha) - 1) \right\}$$

- hypercharge density

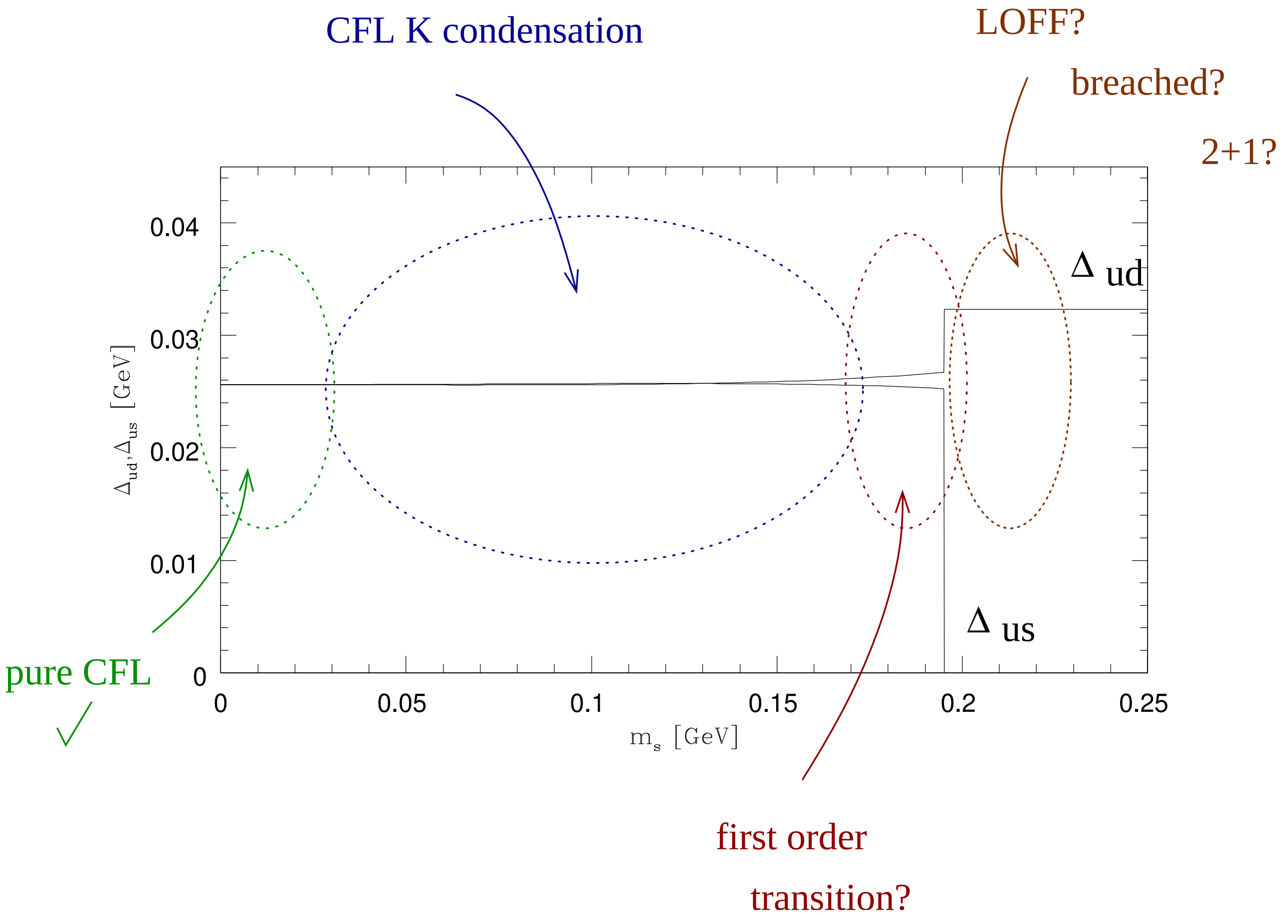




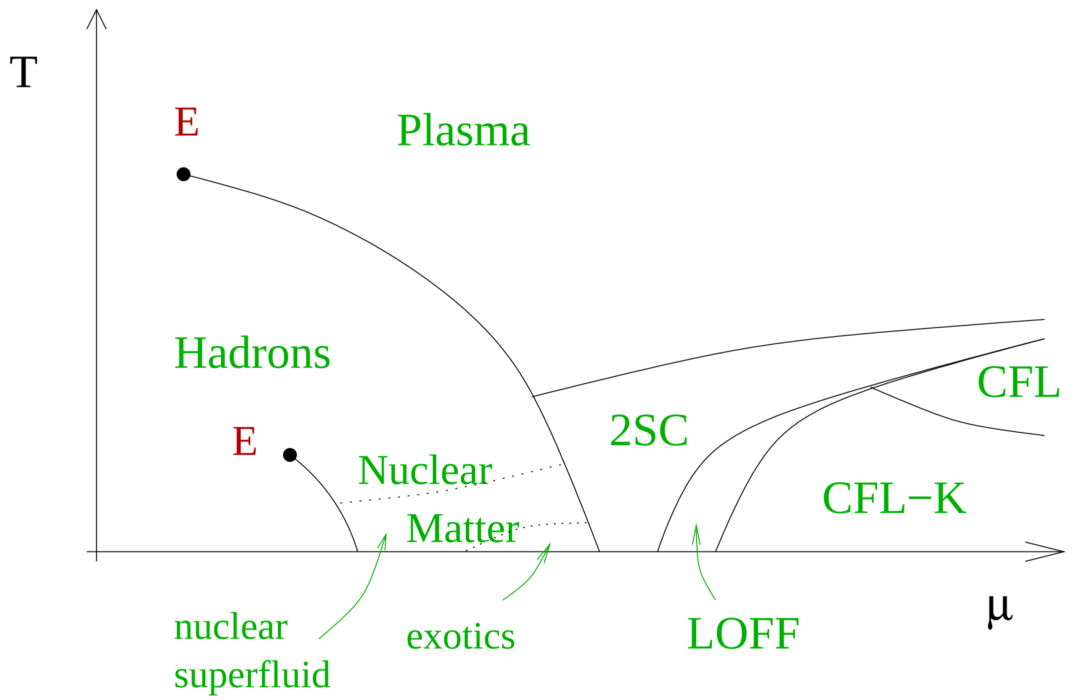
$$n_Y = f_\pi^2 \mu_{eff} \left\{ 1 - \frac{(m_K^{(0)})^4}{\mu_{eff}^4} \right\}, \qquad \mu_{eff} = \frac{m_s^2}{2p_F}$$

$n_s < n_u, n_d$ agrees with "naive" expectation

- compare to BCS calculation



Phase Diagram: $m_s \neq 0$



phase structure at moderate μ (and $m_s, \mu_e \neq 0$)
complicated and poorly understood

- systematic calculations:

$$m_s^2 \ll \mu^2, m_s\Delta \ll \mu^2, g \ll 1$$

- use neutron stars to rule out certain phases

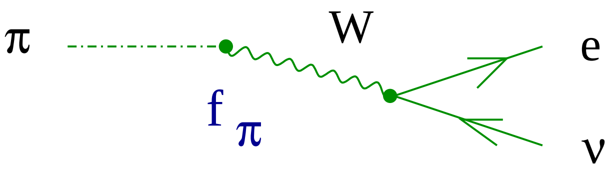
What are the most useful observables?

Extra

Matching, part I

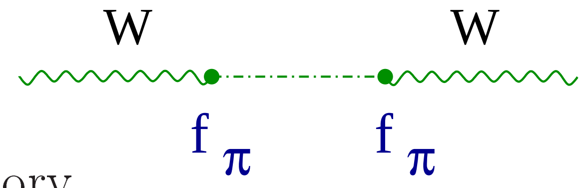
- Compute f_π: Gauge $SU(3)_L \times SU(3)_R$ flavor symmetry

$$\partial_\mu \Sigma \to \nabla_\mu \Sigma = \partial_\mu \Sigma - i W_\mu^L \Sigma + i \Sigma W_\mu^R$$

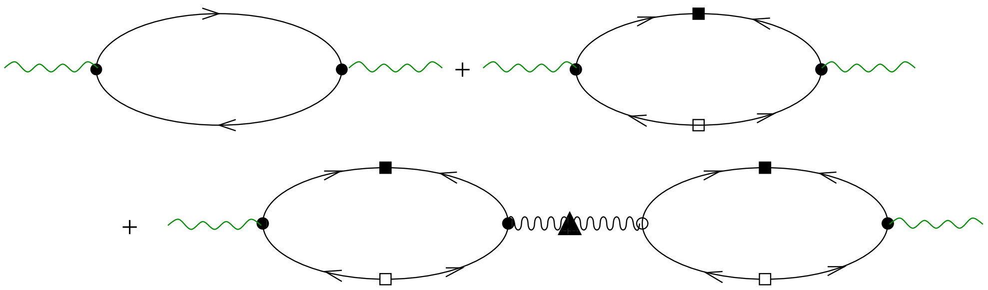


- Higgs phenomenon

$$\mathcal{L} = \frac{f_\pi^2}{4} \mathrm{Tr}\left[(W_0^L - W_0^R)^2 \right] + \ldots \qquad m_W^2 = f_\pi^2$$



- microscopic theory



$$\boxed{f_\pi^2 = \frac{21 - 8\log(2)}{18} \left(\frac{\mu^2}{2\pi^2} \right)}$$

- Note: $\quad 4\pi f_\pi \sim p_F \gg \Delta$

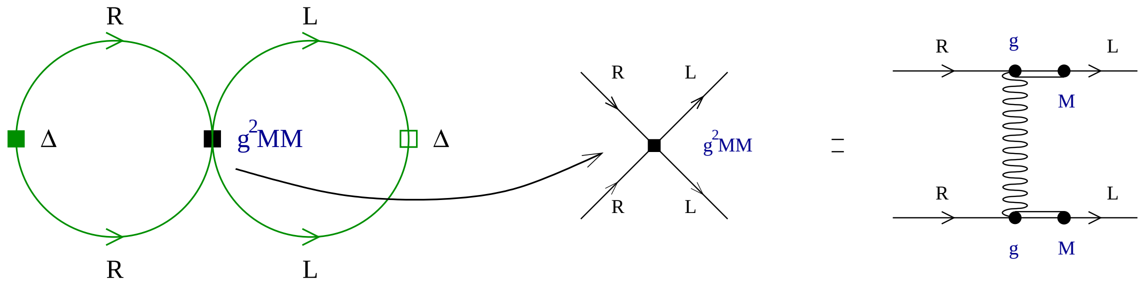
$$\rho \sim 5\rho_0 \ \Rightarrow \ p_F \sim 500\,\mathrm{MeV} \ \Rightarrow \ f_\pi \sim 100\,\mathrm{MeV}$$

Matching, part II

- Compute m dependence of vacuum energy

$$\Delta\mathcal{E} = -B_1\,[\mathrm{Tr}(M)]^2 - B_2\mathrm{Tr}(M^2) \qquad \Sigma = 1$$

- microscopic theory



$$\Delta\mathcal{E} \;\sim\; \left(\frac{g^2}{p_F^4}\right)\left[p_F^2\Delta\log(\Delta)\right]^2\left\{[\mathrm{Tr}(M)]^2 - \mathrm{Tr}(M^2)\right\}$$

$$\sim \Delta^2\left\{[\mathrm{Tr}(M)]^2 - \mathrm{Tr}(M^2)\right\}$$

$$B_1 = -B_2 = \frac{3\Delta^2}{4\pi^2} \qquad \left[\;\sim f_\pi^2\Delta^2\left(\frac{m^2}{p_F^2}\right)\;\right]$$

- meson masses

$$\boxed{\begin{aligned} m_\pi^2 &= \frac{3\Delta^2}{4f_\pi^2}(m_u+m_d)m_s \\ m_{K^\pm}^2 &= \frac{3\Delta^2}{4f_\pi^2}(m_u+m_s)m_d \end{aligned}}$$

- Note:

$$m_{GB} \sim 10\ \mathrm{MeV} \qquad m_K < m_\pi$$

Matching, part III

- consider $1/p_F$ expansion

$$\mathcal{L} = \psi_L^\dagger \left(p_0 - \epsilon_p - \frac{MM^\dagger}{2p_F} \right) \psi_L + \frac{\Delta}{2} \psi_L C \psi_L$$

$$+ (L \leftrightarrow R, M \leftrightarrow M^\dagger) + O(1/p_F^2)$$

- $MM^\dagger$ and $M^\dagger M$ enter as gauge fields

$$W_L = \frac{MM^\dagger}{2p_F} \qquad \psi_L \to L\psi_L, \ \ \psi_R \to R\psi_R$$

$$W_R = \frac{M^\dagger M}{2p_F} \qquad W_L \to L W_L L^\dagger + iL\partial_0 L^\dagger, \ldots$$

- implement gauge symmetry in effective lagrangian

$$\mathcal{L} = \frac{f_\pi^2}{4} \mathrm{Tr} \left(\nabla_0 \Sigma \nabla_0 \Sigma^\dagger \right) + \ldots$$

$$\nabla_0 \Sigma = \partial_0 \Sigma + i \frac{MM^\dagger}{2p_F} \Sigma - i\Sigma \frac{M^\dagger M}{2p_F}$$

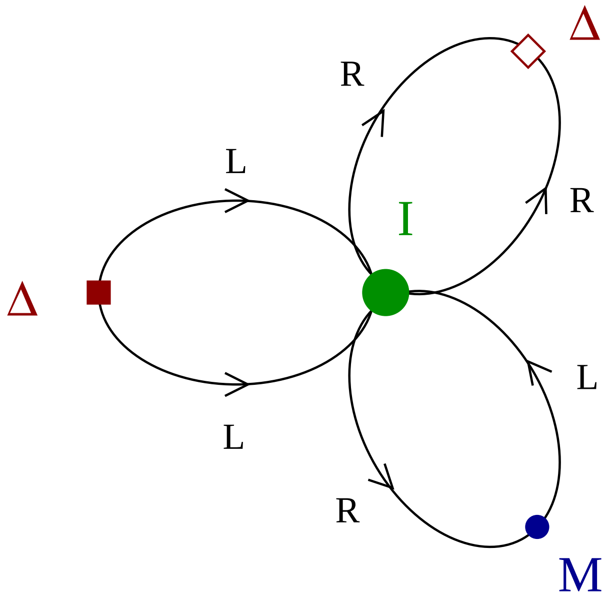
acts like an effective chemical potential $\hat{\mu} = MM^\dagger/(2p_F)$

- Note:

$$\frac{\nabla_0}{\Delta} \sim \left(\frac{\partial_0}{\Delta}, \frac{MM^\dagger}{p_F \Delta} \right)$$

Matching, anomalous part

- linear term $\mathrm{Tr}(M\Sigma)$ in vacuum energy related to instantons (violates $(Z_2)_A$ symmetry)



Instanton size

$$\rho \sim \mu^{-1} \ll \Lambda_{QCD}^{-1}$$

- instanton contribution to vacuum energy: $\mathcal{E} = A\mathrm{Tr}(M)$

$$A = C_{N_c}^{N_f} \langle \psi C \psi \rangle^2 \left(b \log \left(\frac{\mu}{\Lambda_{QCD}} \right) \right)^6 \left(\frac{\Lambda_{QCD}}{\mu} \right)^1 2\Lambda_{QCD}^{-3} e^{i\Theta}$$

$$\langle \psi C \psi \rangle = \frac{3\sqrt{2}\pi}{g} \Delta \left(\frac{\mu^2}{2\pi^2} \right)$$

- Note: $\langle \bar{\psi}\psi \rangle = -A$

$$\langle \bar{\psi}\psi \rangle \sim \left(\frac{\Lambda_{QCD}}{\mu} \right)^8 \Lambda_{QCD}^{-3} \ll \Lambda^{-3}$$