Scale invariant fluid dynamics for the dilute Fermi gas at unitarity

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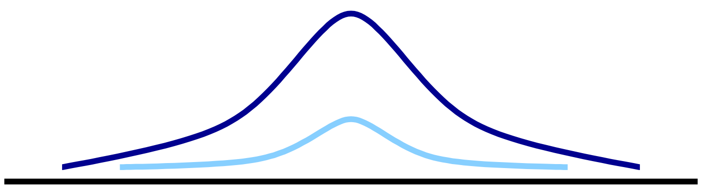
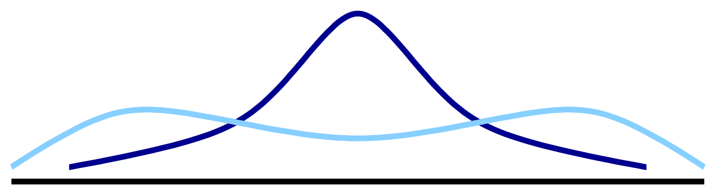
Fluids: Gases, Liquids, Plasmas, . . .

Hydrodynamics: Long-wavelength, low-frequency dynamics of conserved or spontaneously broken symmetry variables.



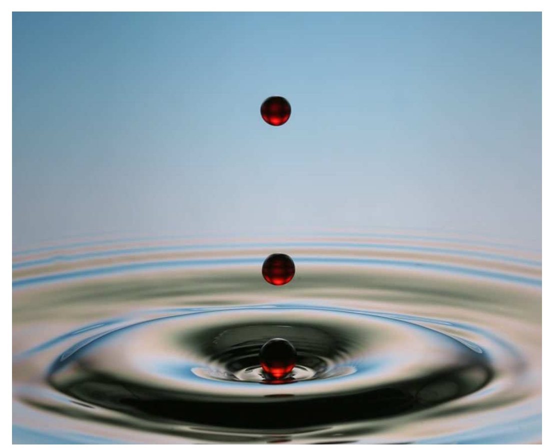
$\tau \sim \tau_{micro}$



$\tau \sim \lambda$

Historically: Water

$(\rho, \epsilon, \vec{\pi})$



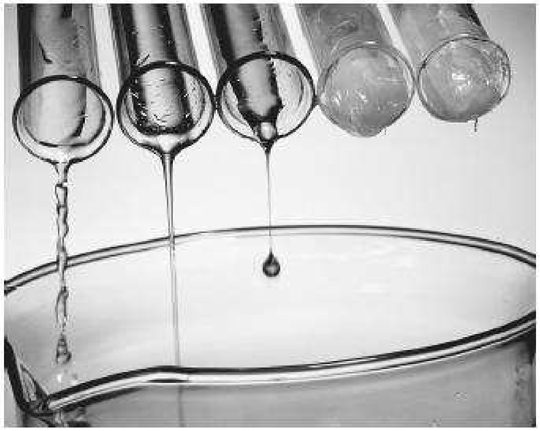
Simple non-relativistic fluid

Simple fluid: Conservation laws for mass, energy, momentum

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0$$

$$\frac{\partial \epsilon}{\partial t} + \vec{\nabla} \vec{j}^{\epsilon} = 0$$

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0$$



Constitutive relations: Energy momentum tensor

$$\Pi_{ij} = P\delta_{ij} + \rho v_i v_j + \eta \left(\partial_i v_j + \partial_j v_i - \frac{2}{3} \delta_{ij} \partial_k v_k \right) + O(\partial^2)$$

reactive dissipative 2nd order

Expansion $\Pi^0_{ij} \gg \delta\Pi^1_{ij} \gg \delta\Pi^2_{ij}$

Regime of applicability

Expansion parameter $Re^{-1} = \frac{\eta(\partial v)}{\rho v^2} = \frac{\eta}{\rho L v} \ll 1$

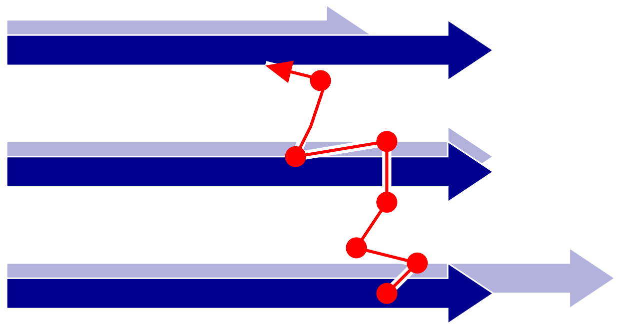
$$Re = \underbrace{\frac{\hbar n}{\eta}}_{\text{fluid property}} \times \underbrace{\frac{mvL}{\hbar}}_{\text{flow property}}$$

Consider $mvL \sim \hbar$: Hydrodynamics requires $\eta/(\hbar n) < 1$

Shear viscosity in kinetic theory

Kinetic theory: conserved quantities carried by quasi-particles

$$\frac{\partial f_p}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f_p + \vec{F} \cdot \vec{\nabla}_p f_p = C[f_p]$$



$$\eta \sim \frac{1}{3} n \, \bar{p} \, l_{mfp}$$

Weakly interacting gas: $l_{mfp} \sim 1/(n\sigma)$ $\Rightarrow$ $\eta \sim \frac{1}{3} \frac{\bar{p}}{\sigma}$

$$\eta(\sigma \to 0) \to \infty$$

Strongly interacting gas:

$$\frac{\eta}{n} \sim \bar{p} l_{mfp} \geq \hbar$$

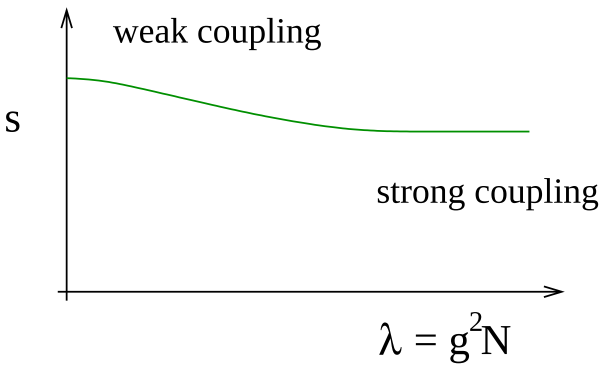
but: kinetic theory not reliable!

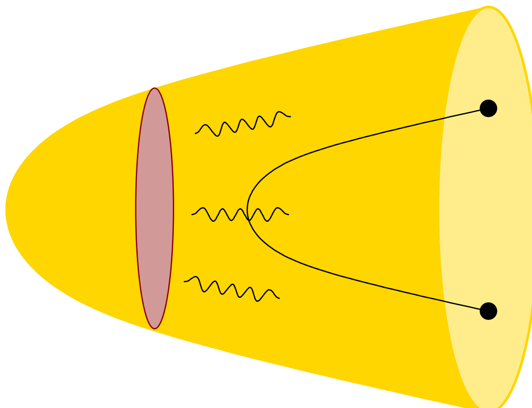
Holographic duals at finite temperature

Thermal (conformal) field theory $\equiv AdS_5$ black hole

CFT temperature $\Leftrightarrow$ Hawking temperature of black hole

CFT entropy $\Leftrightarrow$ Hawking-Bekenstein entropy $\sim$ area of event horizon





$$s(\lambda \to \infty) = \frac{\pi^2}{2} N_c^2 T^3 = \frac{3}{4} s(\lambda = 0)$$

Gubser and Klebanov

Holographic duals: Transport properties

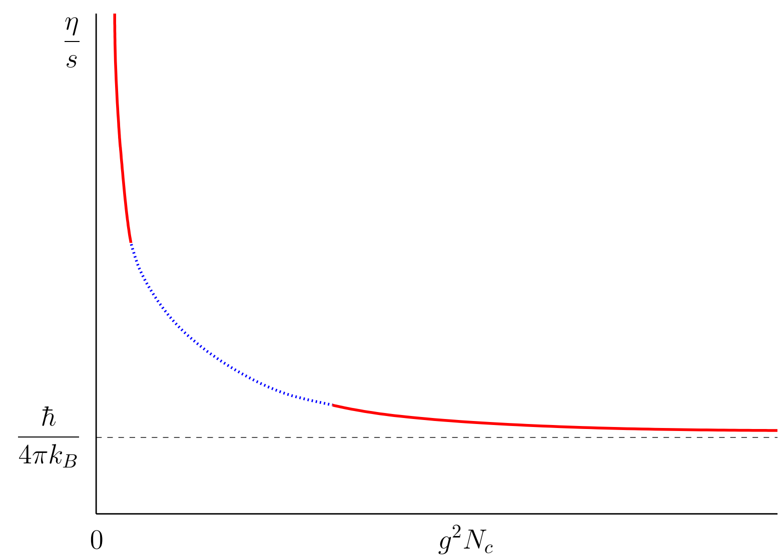
Thermal (conformal) field theory $\equiv AdS_5$ black hole

CFT entropy	$\Leftrightarrow$	Hawking-Bekenstein entropy $\sim$ area of event horizon
shear viscosity	$\Leftrightarrow$	Graviton absorption cross section $\sim$ area of event horizon

Strong coupling limit

$$\frac{\eta}{s} = \frac{\hbar}{4\pi k_B}$$

Son and Starinets (2001)



Strong coupling limit universal? Provides lower bound for all theories?

Answer appears to be no; e.g. theories with higher derivative gravity duals.

Effective theories for fluids (Unitary Fermi Gas, $T > T_F$)

$$\mathcal{L} = \psi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2M} \right) \psi - \frac{C_0}{2} (\psi^\dagger \psi)^2$$

$$\frac{\partial f_p}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f_p = C[f_p] \qquad \omega < T$$

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0 \quad \omega < T \left(\frac{T_F}{T} \right)^{3/2}$$

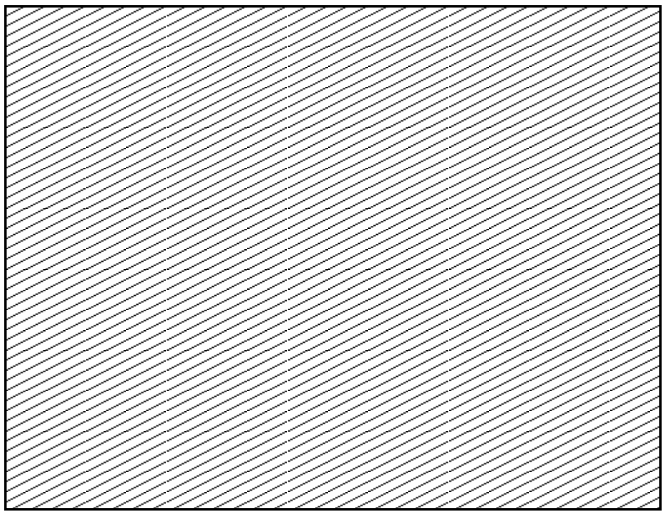
Effective theories (Strong coupling)

$$\mathcal{L} = \bar{\lambda}(i\sigma \cdot D)\lambda - \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} + \ldots \;\Leftrightarrow\; S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g}\mathcal{R} + \ldots$$

$$SO(d+2,2) \rightarrow Schr(d) \qquad\qquad AdS_{d+3} \rightarrow \mathcal{X}_{d+3}$$

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0 \;\; (\omega < T)$$

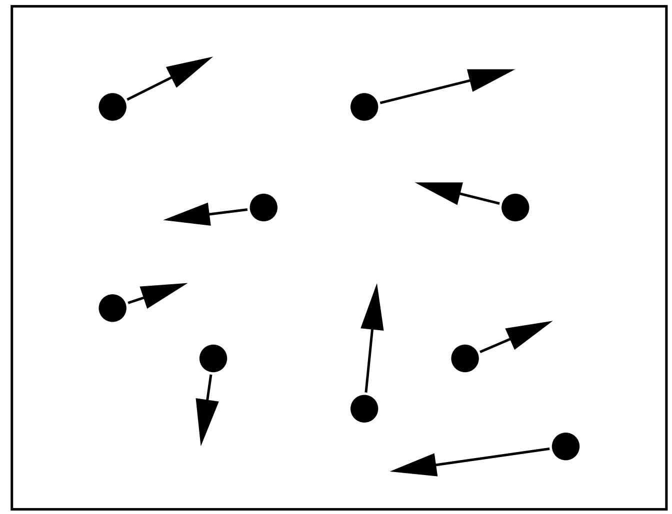
Kinetics vs No-Kinetics



AdS/CFT low viscosity goo

gravitational dual

$\eta/s \simeq 1/(4\pi)$



pQCD kinetic plasma

quasi-particles

$\eta/s \gtrsim 1$

Outline

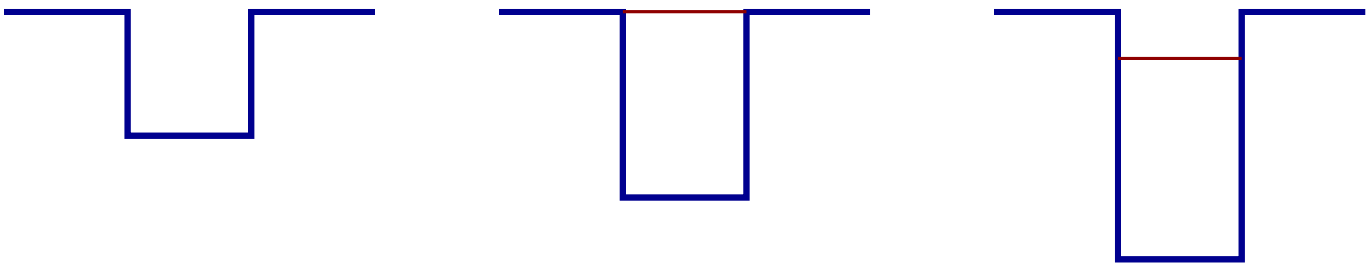
I. Conformal second order hydrodynamics

II. Kinetic theory

III. Transport coefficients from elliptic flow

I. Non-relativistic fermions in unitarity limit

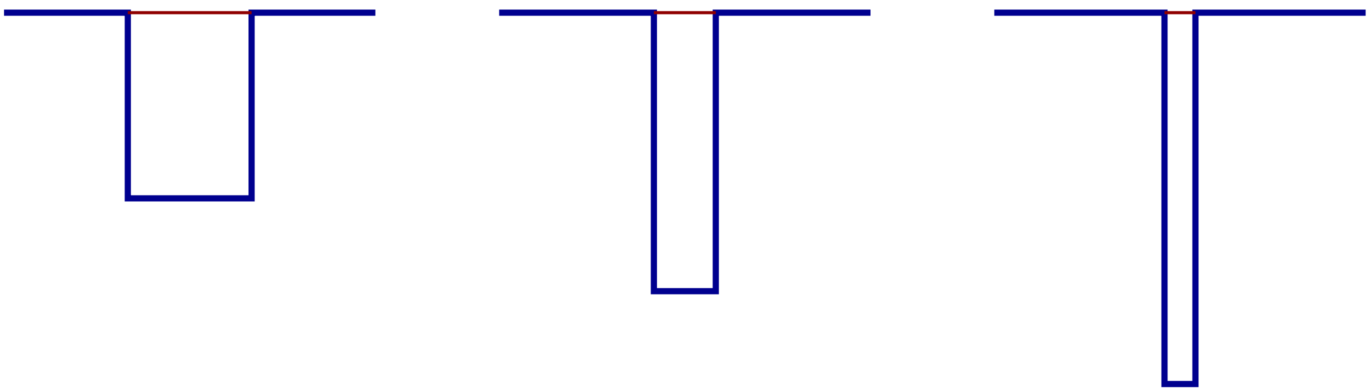
Consider simple square well potential



$a < 0$ $\quad$ $a = \infty,\ \epsilon_B = 0$ $\quad$ $a > 0,\ \epsilon_B > 0$

Non-relativistic fermions in unitarity limit

Now take the range to zero, keeping $\epsilon_B \simeq 0$

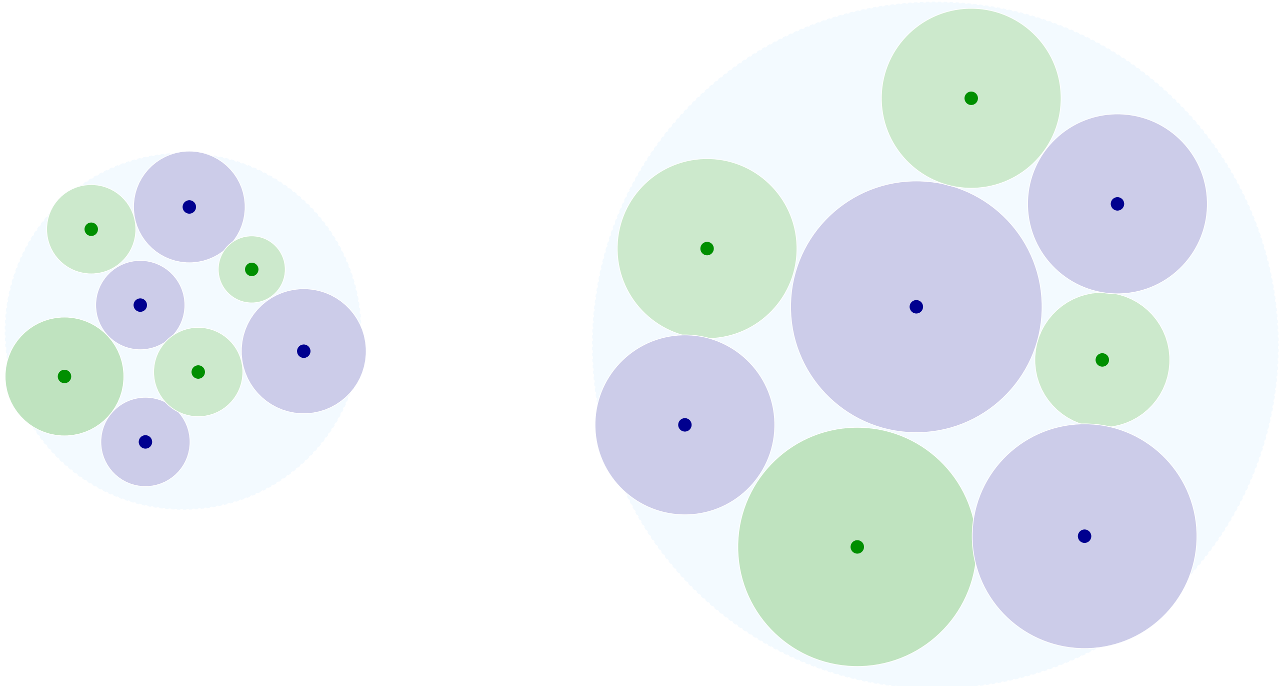


Universal relations

$$\mathcal{T} = \frac{1}{ik + 1/a} \qquad \epsilon_B = \frac{1}{2ma^2} \qquad \psi_B \sim \frac{1}{\sqrt{a}r} \exp(-r/a)$$

Universal fluid dynamics

Many body system: Effective cross section $\sigma_{tr} \sim n^{-2/3}$ (or $\sigma_{tr} \sim \lambda^2$)



Systems remains hydrodynamic despite expansion

Scale and conformal symmetry

Gallilean boosts	$\vec{x}' = \vec{x} + t$	$t' = t$
scale trafo	$\vec{x}' = e^{s}\vec{x}$	$t' = e^{2s}t$
conformal trafo	$\vec{x}' = \vec{x}/(1+ct)$	$1/t' = 1/t + c$

Ideal fluid dynamics

$$\Pi^0_{ij} = P\delta_{ij} + \rho v_i v_j , \qquad P = \frac{2}{3}\mathcal{E}$$

First order viscous hydrodynamics

$$\delta^{(1)}\Pi_{ij} = -\eta\sigma_{ij}, \;\; \sigma_{ij} = \left(\nabla_i v_j + \nabla_j v_i - \frac{2}{3}\delta_{ij}(\nabla\cdot v)\right), \qquad \zeta = 0$$

Scale invariant superfluid hydrodynamics

Momentum density: $\pi_i = \rho_n v_{n,i} + \rho_s v_{s,i}$

Stress tensor $\Pi_{ij} + \delta\Pi_{ij}$ with

$$\delta\Pi_{ij} = -\eta\left(\nabla_i v_{n,j} + \nabla_j v_{n,i} - \frac{2}{3}\delta_{ij}\nabla_i v_{n,i}\right)$$

$$- \delta_{ij}\Big(\zeta_1 \nabla_i \left(\rho_s \left(v_{s,i} - v_{n,i}\right)\right) + \zeta_2 \left(\nabla_i v_{n,i}\right)\Big)$$

Equation of motions for v_s: $\dot{v}_s + \frac{1}{2}\nabla(v_s^2) = -\nabla(\mu + H)$ with

$$H = -\zeta_3 \nabla_i \left(\rho_s \left(v_{s,i} - v_{n,i}\right)\right) - \zeta_4 \nabla_i v_{n,i}$$

Conformal symmetry: $\zeta_1 = \zeta_2 = \zeta_4 = 0$

Son (2007)

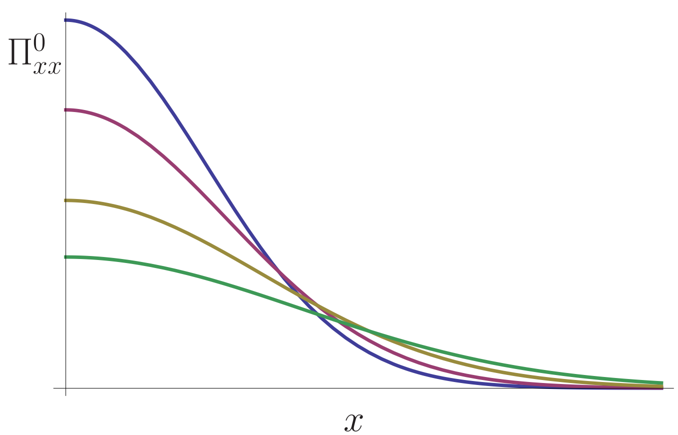
Why second order fluid dynamics?

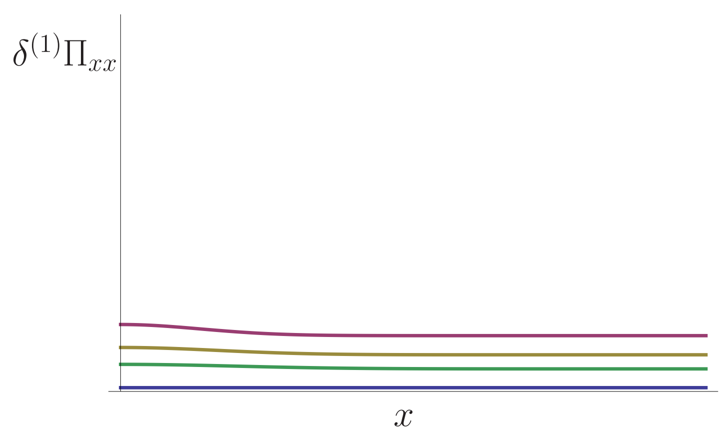
Consider ideal expansion after release from a harmonic trap

$$\rho(x_i,t) = \rho_0(b_i(t)x_i), \qquad v_i(x_j,t) = \alpha_i(t)x_i, \qquad \alpha_i(t) = \dot{b}_i(t)/b_i(t)$$

Compare ideal and dissipative stresses

$$\Pi^0_{ii} = P + \rho\alpha_i x_i^2, \qquad\qquad \delta^{(1)}\Pi_{ii} = \eta\left(\alpha_i - \tfrac{4}{3}\textstyle\sum_j \alpha_j\right)$$





Ideal stresses propagate with speed $\sim c_s$, dissipative stresses propagate with infinite speed. Hydro always breaks down in the dilute corona.

Second order conformal hydrodynamics

Relaxation of shear stress is a second order hydro term. Complete list

$$\begin{aligned}\delta^{(2)}\Pi^{ij} &= \eta\tau_\pi\left[{}^{\langle}D\sigma^{ij\rangle}+\frac{2}{3}\sigma^{ij}(\nabla\cdot v)\right]\\ &\quad+\lambda_1\sigma^{\langle i}{}_k\sigma^{j\rangle k}+\lambda_2\sigma^{\langle i}{}_k\Omega^{j\rangle k}+\lambda_3\Omega^{\langle i}{}_k\Omega^{j\rangle k}\\ &\quad+\gamma_1\nabla^{\langle i}T\nabla^{j\rangle}T+\gamma_2\nabla^{\langle i}P\nabla^{j\rangle}P+\gamma_3\nabla^{\langle i}\nabla^{j\rangle}T+\ldots\end{aligned}$$

$$A^{\langle ij\rangle}=\frac{1}{2}\left(A_{ij}+A_{ji}-\frac{2}{3}g_{ij}A^k{}_k\right)\quad\Omega^{ij}=\left(\nabla_i v_j-\nabla_j v_i\right)$$

New transport coefficients $\tau_\pi,\lambda_i,\gamma_i$

Can be written as a relaxation equation for $\pi^{ij}\equiv\delta\Pi^{ij}$

$$\pi^{ij}=-\eta\sigma^{ij}-\tau_\pi\left[{}^{\langle}D\pi^{ij\rangle}+\frac{5}{3}(\nabla\cdot v)\pi^{ij}\right]+\ldots$$

Chao, Schaefer (2011)

II. Linear response and kinetic theory

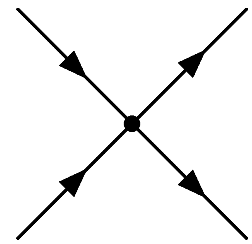
Consider background metric $g_{ij}(t,\mathbf{x}) = \delta_{ij} + h_{ij}(t,\mathbf{x})$. Linear response

$$\delta\Pi^{ij} = \frac{\delta\Pi^{eq}_{ij}}{\delta h_{ij}} h^{ij} - \frac{1}{2} G_R^{ijkl} h_{kl}$$

Kubo relation: $\eta(\omega) = \frac{1}{\omega}\mathrm{Im}G_R^{xyxy}(\omega,0)$

Kinetic theory: Boltzmann equation

$$\left(\frac{\partial}{\partial t} + \frac{p^i}{m}\frac{\partial}{\partial x^i} - \left(g^{il}\dot{g}_{lj}p^j + \Gamma^i_{jk}\frac{p^j p^k}{m}\right)\frac{\partial}{\partial p^i}\right) f(t,\mathbf{x},\mathbf{p}) = C[f]$$

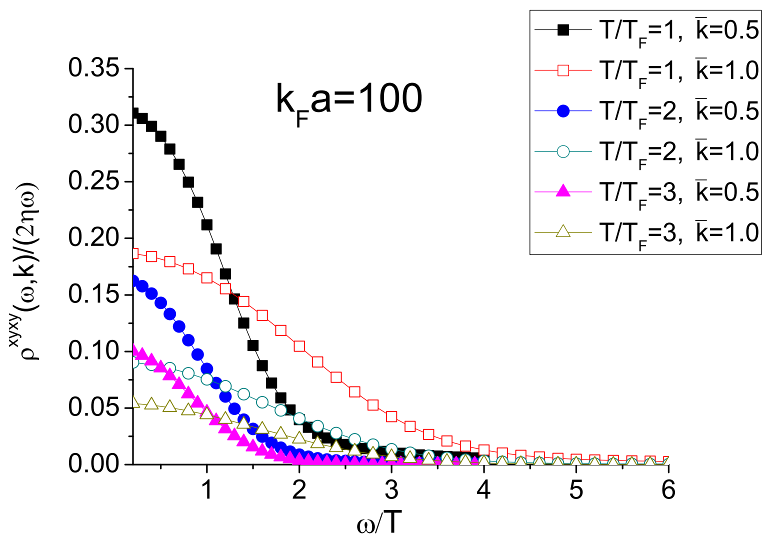
$C[f] =$ 

Kinetic theory

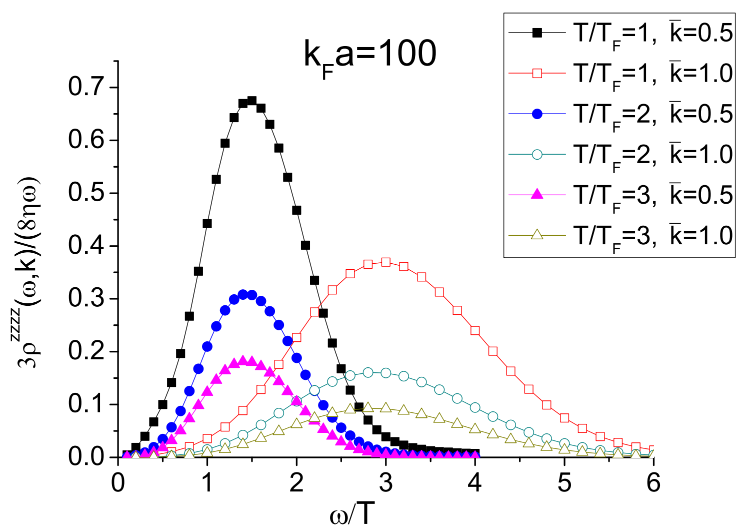
linearize $f = f_0 + \delta f$, solve for δf, $\hookrightarrow \delta\Pi_{ij}$, $\hookrightarrow G_R$, $\hookrightarrow \eta(\omega)$

$$\eta(\omega) = \frac{\eta}{1+\omega^2\tau_R^2} \qquad \eta = \frac{15}{32\sqrt{\pi}}(mT)^{3/2} \qquad \tau_R = \frac{\eta}{nT}$$

shear channel



sound channel



Braby, Chao, T.S. (2010)

Shear viscosity: Sum rules

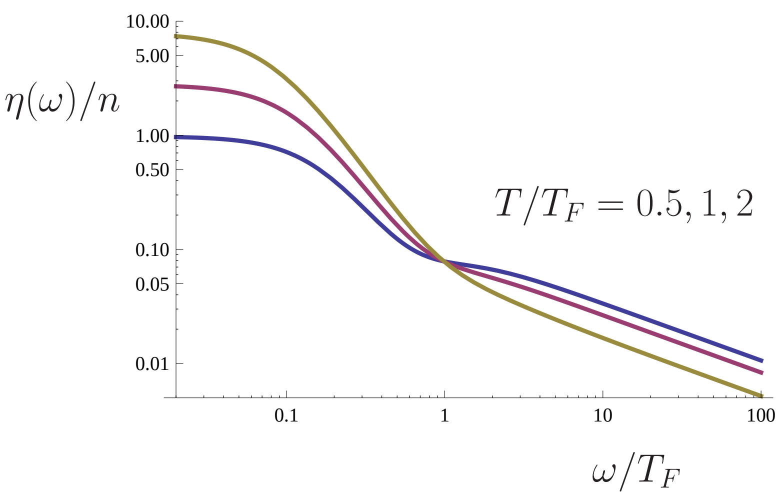
Randeria & Taylor proved the sum rules (corrected by Enss & Zwerger)

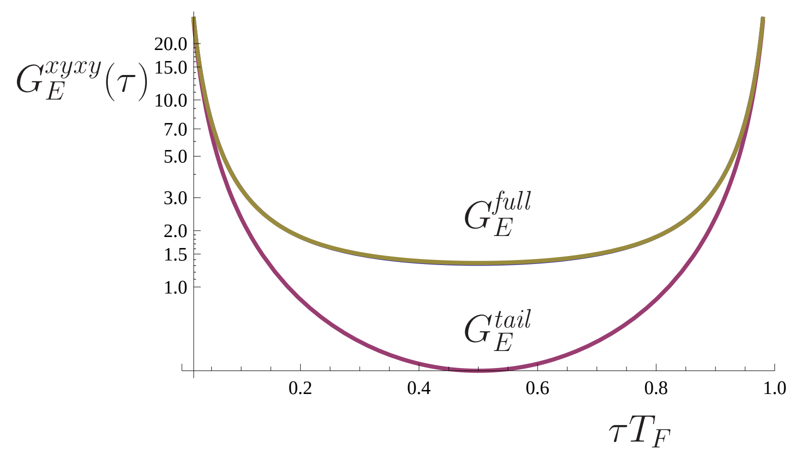
$$\frac{1}{\pi}\int dw\,\left[\eta(\omega)-\frac{C}{15\pi\sqrt{m\omega}}\right]=\frac{\mathcal{E}}{3}-\frac{C}{10\pi ma}$$

$$\frac{1}{\pi}\int dw\,\zeta(\omega)=\frac{1}{72\pi ma^2}\left(\frac{\partial C}{\partial a^{-1}}\right)$$

where C is Tan's contact, $\rho(k)\sim C/k^4$.

Sum rules constrain spectral function and euclidean correlator





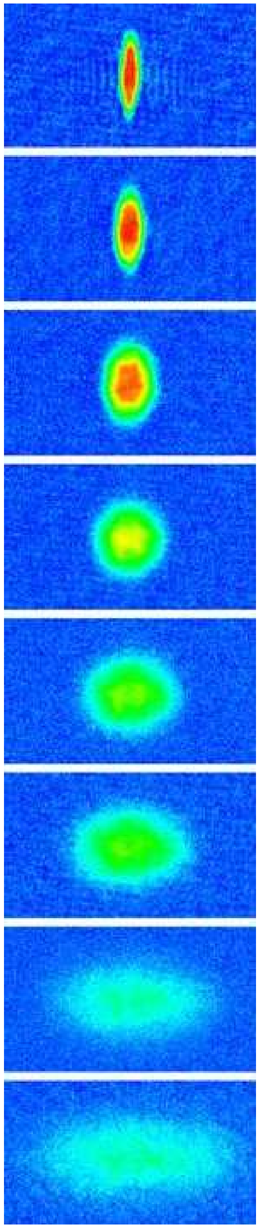
Second order hydrodynamics from kinetic theory

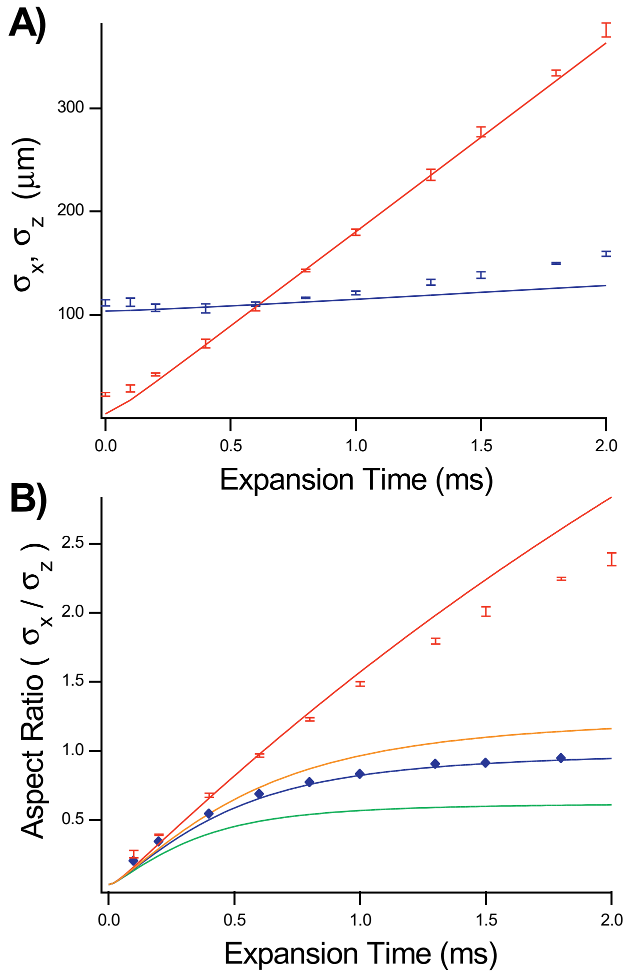
Boltzmann equation (BGK approximation)

$$\delta^{(2)}\Pi^{ij} = \frac{\eta^2}{P}\left[\langle D\sigma^{ij}\rangle + \frac{2}{3}\sigma^{ij}(\nabla\cdot v)\right] + \frac{\eta^2}{P}\left[\sigma^{\langle i}{}_k\sigma^{j\rangle k} + \sigma^{\langle i}{}_k\Omega^{j\rangle k}\right] + O(\kappa\eta\nabla^i\nabla^j T)$$

relaxation time $\tau_R = \frac{\eta}{P} \simeq \frac{\eta}{nT}$

III. Almost ideal fluid dynamics (cold Fermi gas)



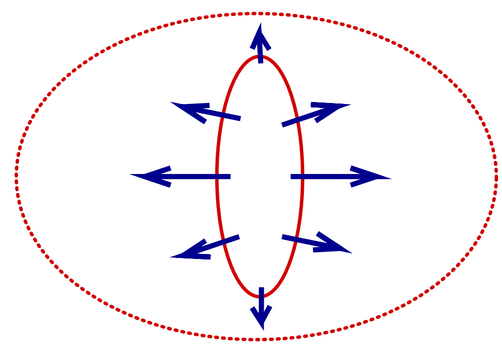


O'Hara et al. (2002)

Hydrodynamic expansion converts

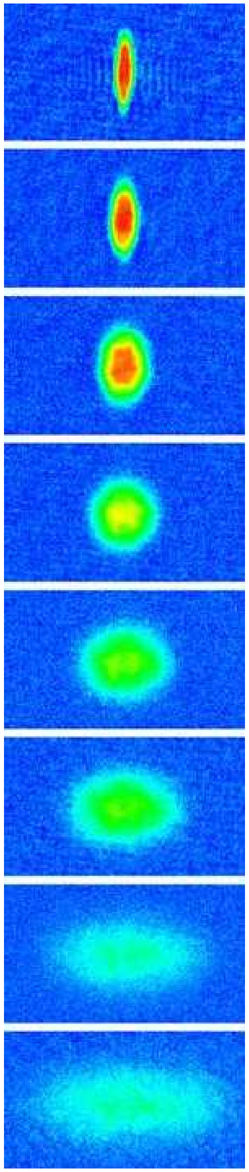
coordinate space anisotropy

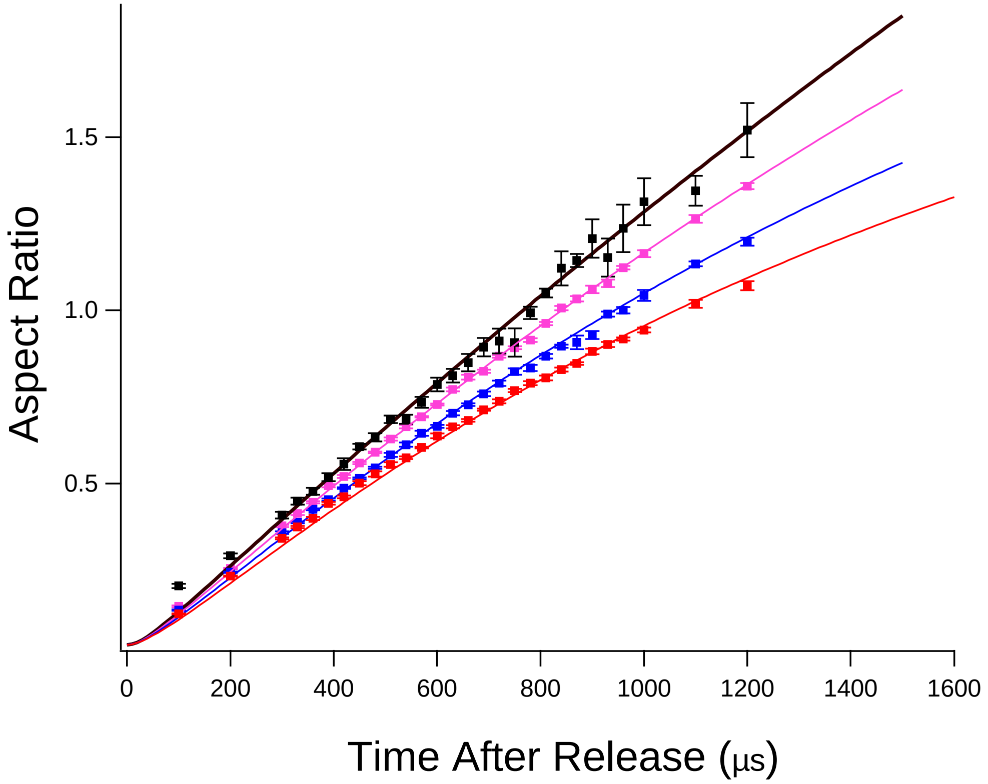
to momentum space anisotropy



Elliptic flow: High T limit

Quantum viscosity $\eta = \eta_0 \frac{(mT)^{3/2}}{\hbar^2}$





$\eta = \eta_0 (mT)^{3/2}$

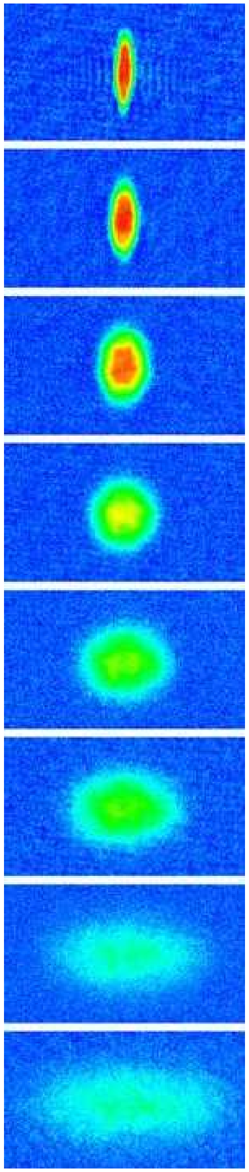
$\tau_R = \eta / P$

Cao et al., Science (2010)

fit: $\eta_0 = 0.33 \pm 0.04$

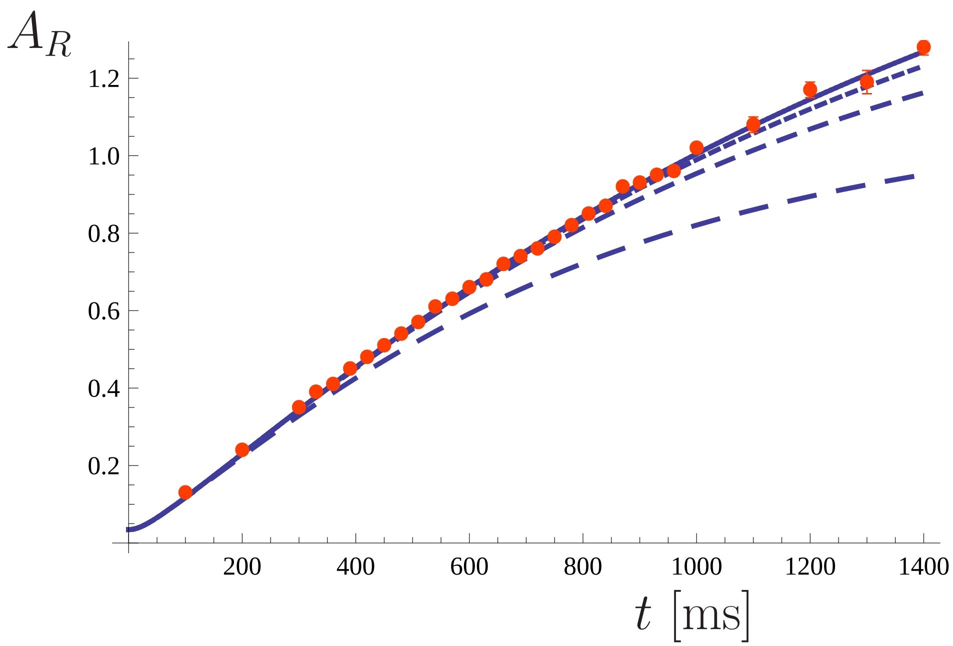
theory: $\eta_0 = \frac{15}{32\sqrt{\pi}} = 0.26$

Elliptic flow: Freezeout?



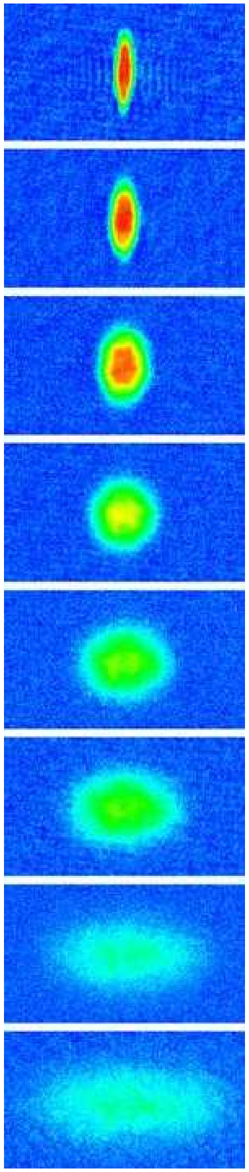
switch from hydro to (weakly collisional) kinetics

at scale factor $b_\perp^{fr} = 1, 5, 10, 20$

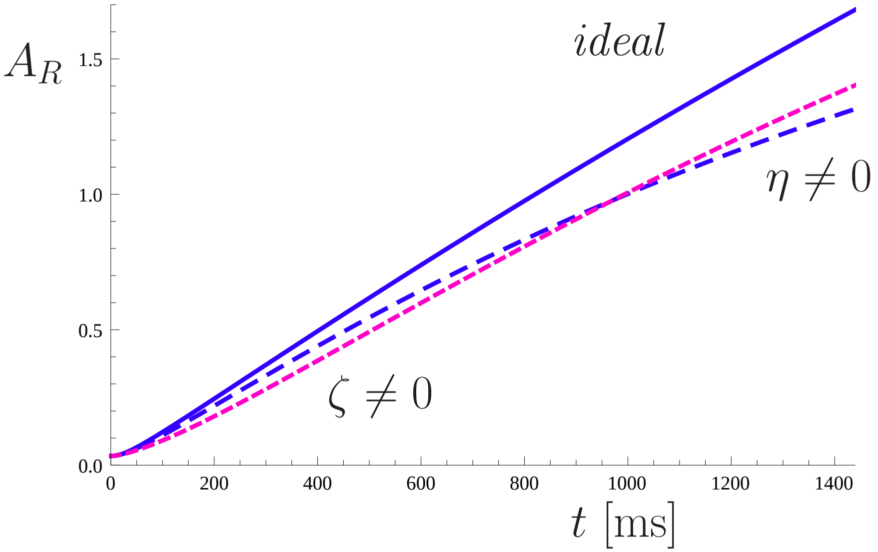


no freezeout seen in the data

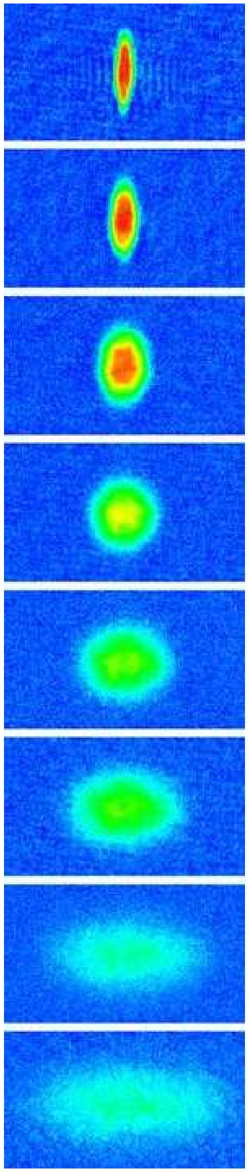
Elliptic flow: Shear vs bulk viscosity



Dissipative hydro with both η, ζ

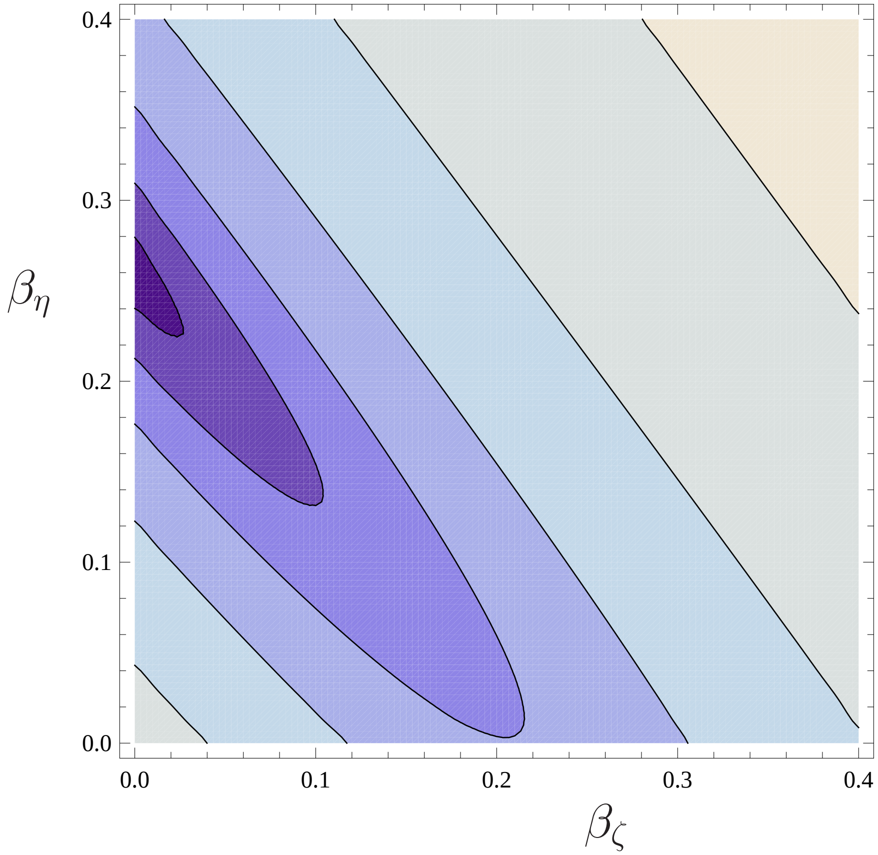


Elliptic flow: Shear vs bulk viscosity



Dissipative hydro with both η, ζ

$$\beta_{\eta,\zeta} = (\eta, \zeta)\frac{E_F}{E}\frac{1}{(3\lambda N)^{1/3}}$$



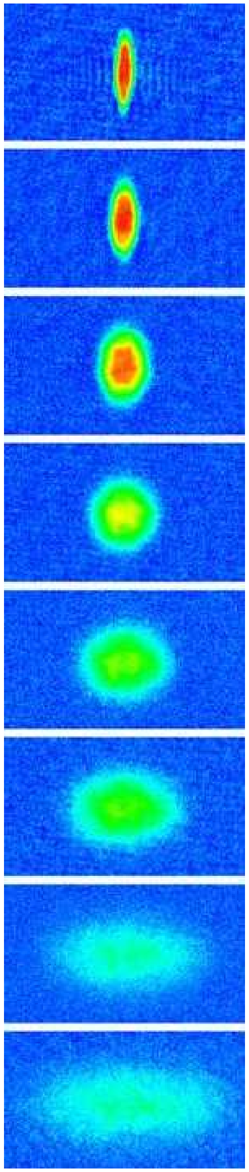
$\eta \gg \zeta$

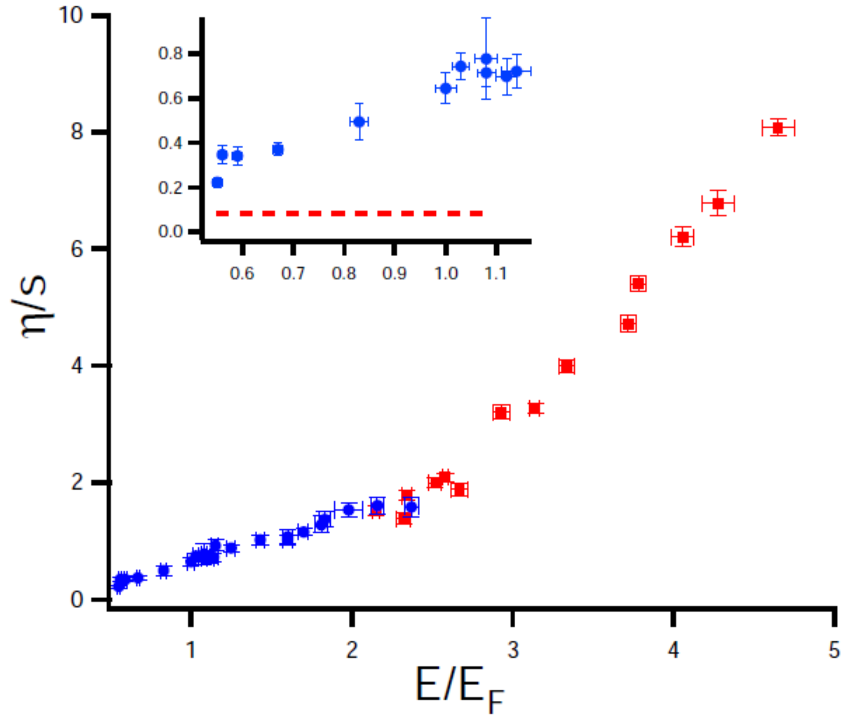
Dusling, Schaefer (2010)

Viscosity to entropy density ratio

consider both collective modes (low T)

and elliptic flow (high T)





Cao et al., Science (2010)

$$\eta/s \leq 0.4$$

Outlook

Experimental determination of transport properties: Collective modes and elliptic flow give $\langle \eta/s \rangle \lesssim 0.4$.

Local analysis requires second order hydro or hydro+kinetic. (I am working on this.)

Shear viscous relaxation time can be measured by comparing collective modes and elliptic flow.

Can we observe breaking of scale invariance and the return of bulk viscosity away from unitarity? Can we measure η and ζ_3 in the superfluid phase?

Note: Experiment (Helium vs Fermi gas at unitarity)