Transport in Conformal Quantum Fluids

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Why study transport in (nearly perfect) quantum fluids?

Transport without quasi-particles? Model system for QGP, strange metals, etc.

Fluid dynamics is the universal effective description of non-equilibrium many body systems. Description is "most effective" in nearly perfect fluids.

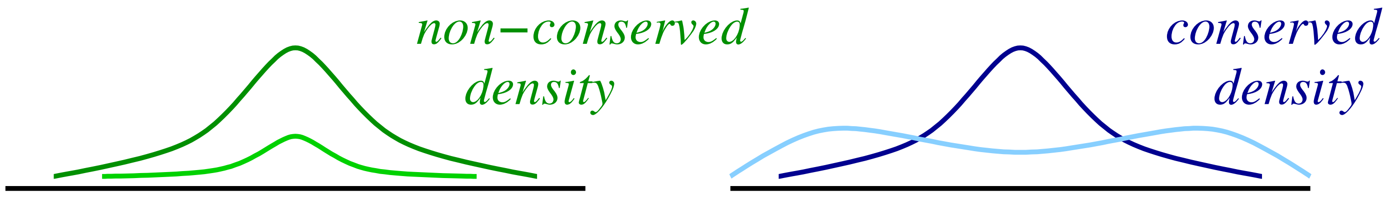
Fluid-gravity correspondence: Can (strongly coupled) fluids teach us something about quantum gravity?

Hydrodynamics

Hydrodynamics (undergraduate version): Newton's law for continuous, deformable media.

Fluids: Gases, liquids, plasmas, . . .

Hydrodynamics (postmodern): Effective theory of non-equilibrium long-wavelength, low-frequency dynamics of <u>any</u> many-body system.



$\tau \sim \tau_{micro}$ $\qquad\qquad$ $\tau \sim \lambda$

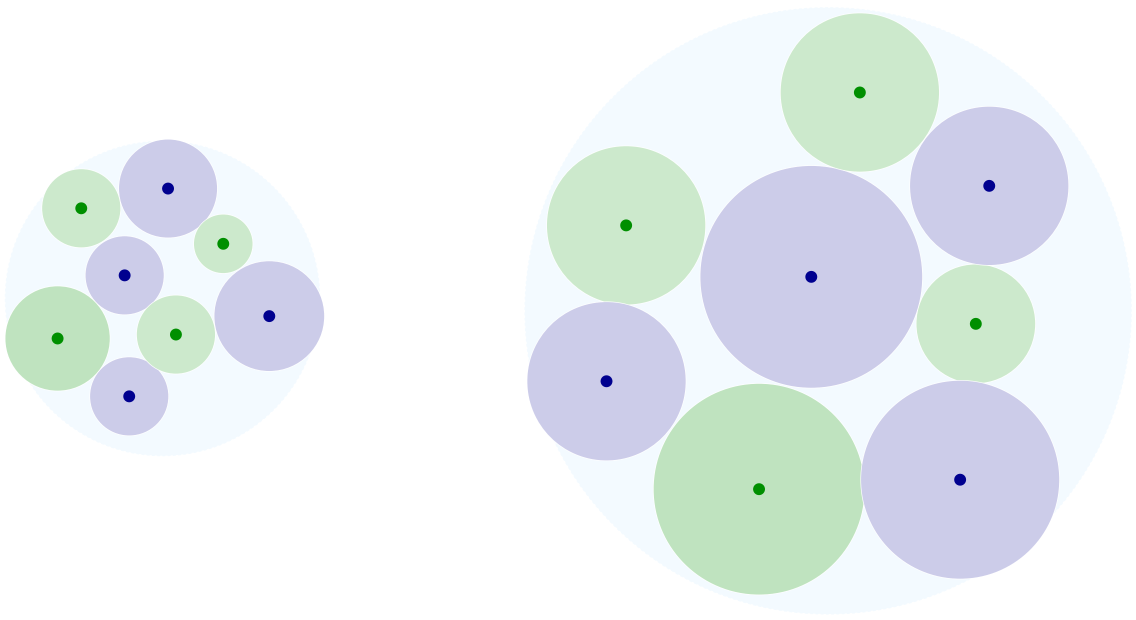
$\tau \gg \tau_{micro}$: Dynamics of conserved charges.

Water: $(\rho, \epsilon, \vec{\pi})$

Not your grandfathers fluid

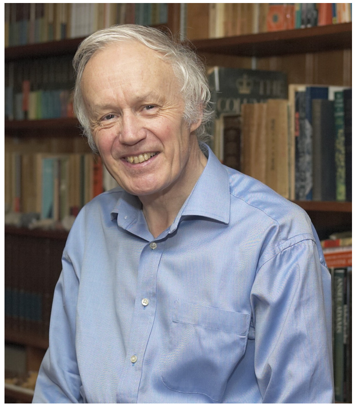
Consider a many body system with $\sigma \sim 1/k^2$

Can be made using Feshbach resonances in dilute atomic gases.



Systems remains hydrodynamic despite expansion

Effective theories for fluids (Unitary Fermi Gas, $T > T_F$)

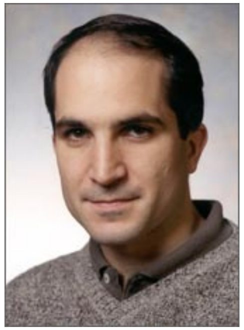


$$\mathcal{L} = \psi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2M} \right) \psi - \frac{C_0}{2} (\psi^\dagger \psi)^2$$

$$\frac{\partial f_p}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f_p = C[f_p] \qquad \omega < T$$

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0 \quad \omega < T \frac{s}{\eta}$$

Effective theories (Strong coupling)



$$\mathcal{L} = \bar{\lambda}(i\sigma \cdot D)\lambda - \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} + \ldots \quad \Leftrightarrow \quad S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g}\mathcal{R} + \ldots$$

$$SO(d+2,2) \to Schr_d^2 \qquad\qquad AdS_{d+3} \to Schr_d^2$$

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}\Pi_{ij} = 0 \quad (\omega < T)$$

Outline

I. EFT: Gradient expansion

II. EFT: Fluctuations

III. Models of fluids: Kinetic theory & QFT

IV. Models of fluids: Holography

V. Analyzing fluids: How to measure η/s

VI. Analyzing fluids: How to measure D_s

I. Gradient expansion (simple non-relativistic fluid)

Simple fluid: Conservation laws for mass, energy, momentum

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \vec{\jmath}^{\,\rho} = 0$$

$$\frac{\partial \epsilon}{\partial t} + \vec{\nabla} \vec{\jmath}^{\,\epsilon} = 0$$

$$\frac{\partial \pi_i}{\partial t} + \frac{\partial}{\partial x_j} \Pi_{ij} = 0$$

Ward identity: mass current = momentum density

$$\vec{\jmath}^{\,\rho} \equiv \rho \vec{v} = \vec{\pi}$$

Constitutive relations: Gradient expansion for currents

Energy momentum tensor

$$\Pi_{ij} = P\delta_{ij} + \rho v_i v_j + \eta \left(\partial_i v_j + \partial_j v_i - \frac{2}{3} \delta_{ij} \partial_k v_k \right) + O(\partial^2)$$

Conformal fluid dynamics: Symmetries

Symmetries of a conformal non-relativistic fluid

Galilean boost	$\vec{x}' = \vec{x} + \vec{v}t$	$t' = t$
Scale trafo	$\vec{x}' = e^s \vec{x}$	$t' = e^{2s} t$
Conformal trafo	$\vec{x}' = \vec{x}/(1+ct)$	$1/t' = 1/t + c$

This is known as the Schrödinger algebra ($=$ the symmetries of the free Schrödinger equation)

Generators: Mass, momentum, angular momentum

$$M = \int dx\, \rho \qquad P_i = \int dx\, \jmath_i \qquad J_{ij} = \int dx\, \epsilon_{ijk} x_j \jmath_k$$

Boost, dilations, special conformal

$$K_i = \int dx\, x_i \rho \qquad D = \int dx\, x \cdot \jmath \qquad C = \int dx\, x^2 \rho/2$$

Spurion method: Local symmetries

Diffeomorphism invariance $\delta x_i = \xi_i(x,t)$

$$\delta g_{ij} = -\mathcal{L}_\xi g_{ij} = -\xi^k \partial_k g_{ij} + \ldots$$

Gauge invariance $\delta\psi = i\alpha(x,t)\psi$

$$\begin{aligned}
\delta A_0 &= -\dot{\alpha} - \xi^k \partial_k A_0 - A_k \dot{\xi}^k \\
\delta A_i &= -\partial_i \alpha - \xi^k \partial_k A_i - A_k \partial_i \xi^k + m g_{ik} \dot{\xi}^k
\end{aligned}$$

Conformal transformations $\delta t = \beta(t)$

$$\delta O = -\beta \dot{O} - \frac{1}{2}\Delta_O \dot{\beta} O$$

More recent work: Newton-Cartan geometry

Son, Wingate (2006), Jensen (2014)

Example: Stress tensor

Determine transformation properties of fluid dynamic variables

$$\delta\rho = -\mathcal{L}_\xi \rho \qquad \delta s = -\mathcal{L}_\xi s \qquad \delta v = -\mathcal{L}_\xi v + \dot{\xi}$$

Stress tensor: Ideal fluid dynamics

$$\Pi^0_{ij} = P g_{ij} + \rho v_i v_j , \qquad \boxed{P = \frac{2}{3}\mathcal{E}}$$

First order viscous hydrodynamics

$$\delta^{(1)}\Pi_{ij} = -\eta\sigma_{ij} - \zeta g_{ij}\langle\sigma\rangle \qquad \boxed{\zeta = 0}$$

$$\sigma_{ij} = \left(\nabla_i v_j + \nabla_j v_i + \dot{g}_{ij} - \frac{2}{3} g_{ij}\langle\sigma\rangle\right) \qquad \langle\sigma\rangle = \nabla\cdot v + \frac{\dot{g}}{2g}$$

Son (2007)

Simple application: Kubo formula

Consider background metric $g_{ij}(t,x) = \delta_{ij} + h_{ij}(t,x)$. Linear response

$$\delta\Pi^{xy} = -\frac{1}{2} G_R^{xyxy} h_{xy}$$

Harmonic perturbation $h_{xy} = h_0 e^{-i\omega t}$

$$G_R^{xyxy} = P - i\eta\omega + \ldots$$

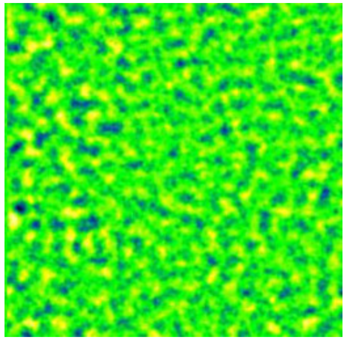
Kubo relation: $$\eta = -\lim_{\omega\to 0}\left[\frac{1}{\omega}\mathrm{Im}G_R^{xyxy}(\omega,0)\right]$$

Gradient expansion: $\omega \leq \dfrac{P}{\eta} \simeq \dfrac{s}{\eta} T$.

II. Beyond gradients: Hydrodynamic fluctuations

Hydrodynamic variables fluctuate

$$\langle \delta v_i(x,t)\delta v_j(x',t)\rangle = \frac{T}{\rho}\delta_{ij}\delta(x-x')$$



Linearized hydrodynamics propagates fluctuations as shear or sound

$$\langle \delta v_i^T \delta v_j^T\rangle_{\omega,k} = \frac{2T}{\rho}(\delta_{ij}-\hat{k}_i\hat{k}_j)\frac{\nu k^2}{\omega^2+(\nu k^2)^2} \qquad \textit{shear}$$

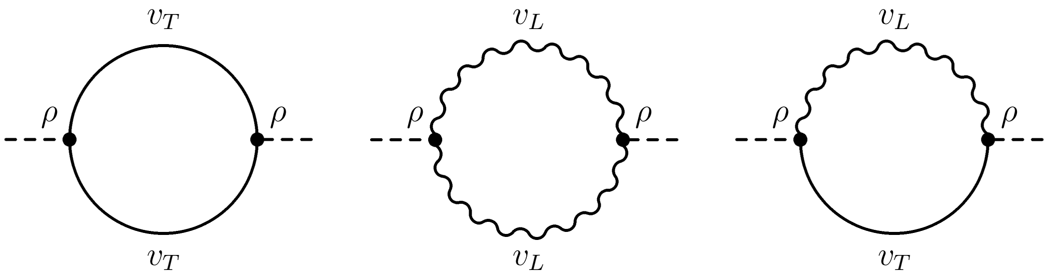
$$\langle \delta v_i^L \delta v_j^L\rangle_{\omega,k} = \frac{2T}{\rho}\hat{k}_i\hat{k}_j\frac{\omega k^2\Gamma}{(\omega^2-c_s^2k^2)^2+(\omega k^2\Gamma)^2} \qquad \textit{sound}$$

$v = v_T + v_L$: $\nabla\cdot v_T = 0$, $\nabla\times v_L = 0$ $\qquad \nu = \eta/\rho$, $\Gamma = \frac{4}{3}\nu + \ldots$

Hydro Loops: "Breakdown" of second order hydro

Correlation function in hydrodynamics

$$G_S^{xyxy} = \langle \{ \Pi^{xy}, \Pi^{xy} \} \rangle_{\omega,k} \simeq \rho_0^2 \langle \{ v_x v_y, v_x v_y \} \rangle_{\omega,k}$$



Match to response function in $\omega \to 0$ (Kubo) limit

$$G_R^{xyxy} = P + \delta P - i\omega[\eta + \delta\eta] + \omega^2 \left[\eta\tau_\pi + \delta(\eta\tau_\pi)\right]$$

with

$$\delta P \sim T\Lambda^3 \qquad \delta\eta \sim \frac{T\rho\Lambda}{\eta} \qquad \delta(\eta\tau_\pi) \sim \frac{1}{\sqrt{\omega}} \frac{T\rho^{3/2}}{\eta^{3/2}}$$

Hydro Loops: RG and "breakdown" of 2nd order hydro

Cutoff dependence can be absorbed into bare parameters. Non-analytic terms are cutoff independent.

Fluid dynamics is a "renormalizable" effective theory.

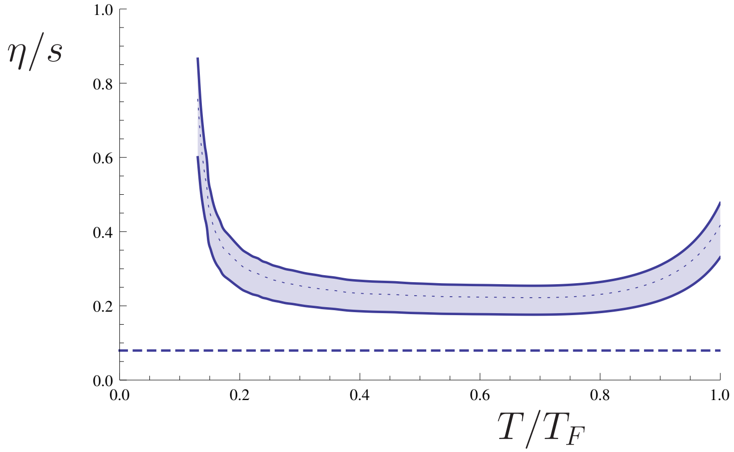
Small η enhances fluctuation corrections: $\delta\eta \sim T\left(\frac{\rho}{\eta}\right)^2\left(\frac{P}{\rho}\right)^{1/2}$

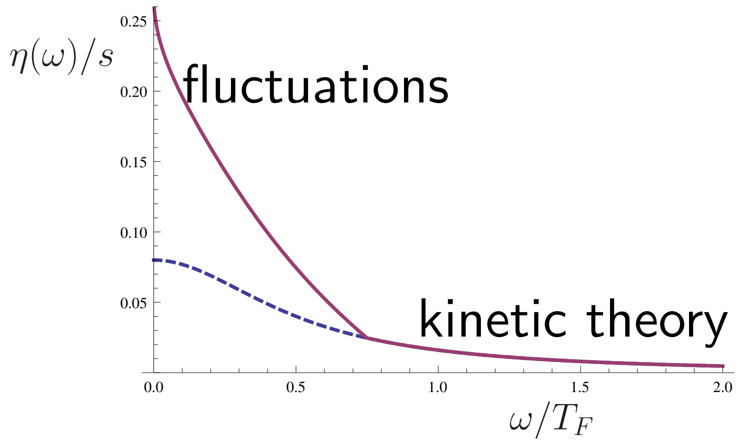
Small η leads to large $\delta\eta$: There must be a bound on η/n.

Relaxation time diverges: $\delta(\eta\tau_\pi) \sim \frac{1}{\sqrt{\omega}}\left(\frac{\rho}{\eta}\right)^{3/2}$

2nd order hydro without fluctuations inconsistent.

Fluctuation induced bound on η/s





$(\eta/s)_{min} \simeq 0.2$

spectral function

non-analytic $\sqrt{\omega}$ term

Schaefer, Chafin (2012), see also Kovtun, Moore, Romatschke (2011)

IIIa. Kinetic theory

Microscopic picture: Quasi-particle distribution function $f_p(x,t)$

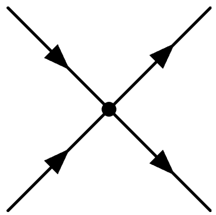
$$\rho(x,t) = \int d\Gamma_p \sqrt{g} m f_p(x,t) \qquad \pi_i(x,t) = \int d\Gamma_p \sqrt{g} p_i f_p(x,t)$$

$$\Pi_{ij}(x,t) = \int d\Gamma_p \sqrt{g} p_i v_j f_p(x,t)$$

Boltzmann equation

$$\left(\frac{\partial}{\partial t} + \frac{p^i}{m} \frac{\partial}{\partial x^i} - \left(g^{il} \dot{g}_{lj} p^j + \Gamma^i_{jk} \frac{p^j p^k}{m} \right) \frac{\partial}{\partial p^i} \right) f_p(t,x,) = C[f]$$

$$C[f] =$$



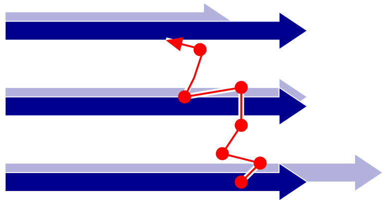
Solve order-by-order in Knudsen number $Kn = l_{mfp}/L$

Kinetic theory: Knudsen expansion

Chapman-Enskog expansion $f = f_0 + \delta f_1 + \delta f_2 + \ldots$

Gradient exp. $\delta f_n = O(\nabla^n)$

$\equiv$ Knudsen exp. $\delta f_n = O(Kn^n)$



First order result

Bruun, Smith (2005)

$$\delta^{(1)}\Pi^{ij} = -\eta\sigma^{ij} \qquad\qquad \eta = \frac{15}{32\sqrt{\pi}}(mT)^{3/2}$$

Second order result

Chao, Schaefer (2012), Schaefer (2014)

$$\begin{aligned}\delta^{(2)}\Pi^{ij} \;&=\; \frac{\eta^2}{P}\left[\langle D\sigma^{ij}\rangle + \frac{2}{3}\sigma^{ij}(\nabla\cdot v)\right] \\ &\quad + \frac{\eta^2}{P}\left[\frac{15}{14}\sigma^{\langle i}{}_k\sigma^{j\rangle k} - \sigma^{\langle i}{}_k\Omega^{j\rangle k}\right] + O(\kappa\eta\nabla^i\nabla^j T)\end{aligned}$$

relaxation time $\tau_\pi = \eta/P$

Frequency dependence, breakdown of kinetic theory

Consider harmonic perturbation $h_{xy}e^{-i\omega t+ikx}$. Use schematic collision term $C[f_p^0 + \delta f_p] = -\delta f_p/\tau$.

$$\delta f_p(\omega, k) = \frac{1}{2T} \frac{-i\omega p_x v_y}{-i\omega + i\vec{v}\cdot\vec{k} + \tau_0^{-1}}\, f_p^0\, h_{xy}\,.$$

Leads to Lorentzian line shape of transport peak

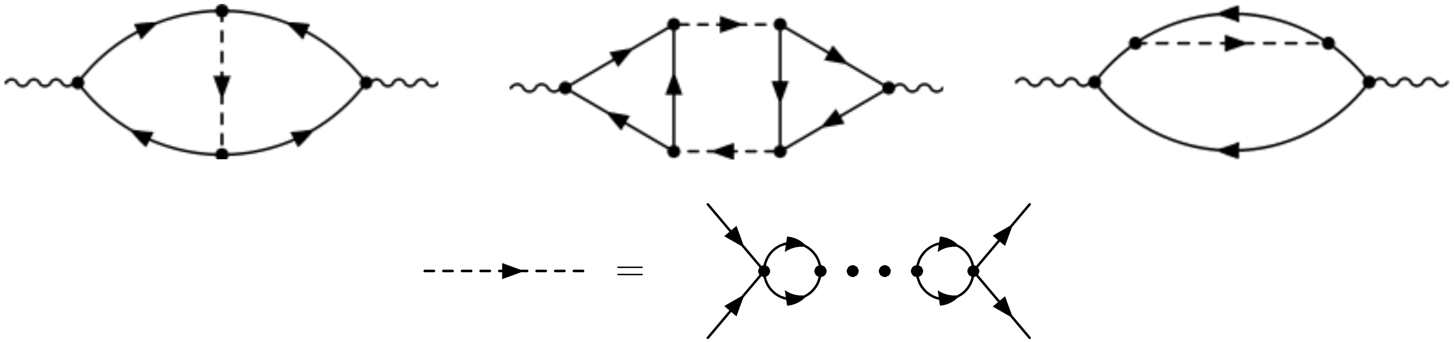
$$\eta(\omega) = \frac{\eta(0)}{1+\omega^2\tau_0^2}$$

Pole at $\omega = i\tau_0^{-1}$ $(\tau_0 = \eta/(sT))$ controls range of convergence of gradient expansion.

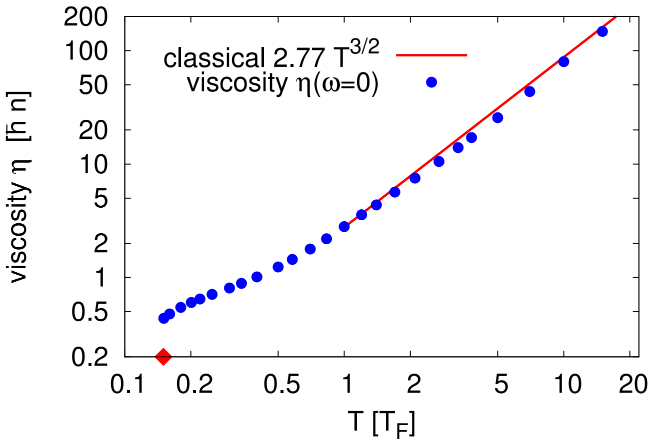
High frequency behavior misses short range correlations for $\omega > T$.

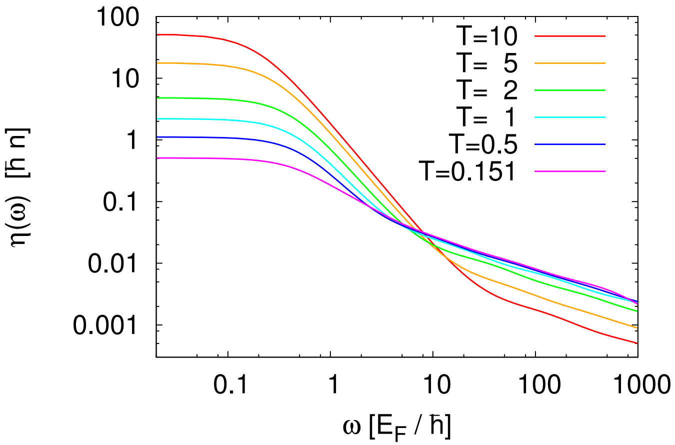
IIIb. Quantum Field Theory

The diagrammatic content of the Boltzmann equation is known: Kubo formula with “Maki-Thompson” + “Azlamov-Larkin” +“Self-energy”



Can be used to extrapolate Boltzmann result to $T \sim T_F$





Enss, Zwerger (2011), see also Levin (2014)

Short time behavior: OPE

Operator product expansion (OPE)

$$\eta(\omega) = \sum_n \frac{\langle \mathcal{O}_n \rangle}{\omega^{(\Delta_n - d)/2}} \qquad \mathcal{O}_n(\lambda^2 t, \lambda x) = \lambda^{-\Delta_n} \mathcal{O}_n(t, x)$$

Leading operator: Contact density (Tan)

$$\mathcal{O}_{\mathcal{C}} = C_0^2 \psi\psi\psi^\dagger\psi^\dagger = \Phi\Phi^\dagger \qquad \Delta_{\mathcal{C}} = 4$$

$\eta(\omega) \sim \langle \mathcal{O}_{\mathcal{C}} \rangle / \sqrt{\omega}$. Asymptotic behavior + analyticity gives sum rule

$$\frac{1}{\pi} \int dw \left[\eta(\omega) - \frac{\langle \mathcal{O}_{\mathcal{C}} \rangle}{15\pi\sqrt{m\omega}} \right] = \frac{\mathcal{E}}{3}$$

Randeria, Taylor (2010), Enss, Zwerger (2011), Hoffman (2013)

IV. Holography

DLCQ idea: Light cone compactification of relativistic theory in d+2

$$p_\mu p^\mu = 2p_+p_- - p_\perp^2 = 0 \qquad p_- = \frac{p_\perp^2}{2p_+} \qquad p_+ = \frac{2n+1}{L}$$

Galilean invariant theory in d+1 dimensions.

String theory embedding: Null Melvin Twist

$$AdS_{d+3} \xrightarrow{\mathrm{NMT}} Schr_d^2$$

$$Iso(AdS_{d+3}) = SO(d+2,2) \supset Schr(d)$$

Son (2008), Balasubramanian et al. (2008)

Other ideas: Horava-Lifshitz (Karch, 2013)

Schrödinger Metric

Fluctuations $\delta g_x^y = e^{-i\omega u}\chi(\omega, r)$ satisfy wave equation $(u = (r_+/r)^2)$

$$\chi''(\omega, u) - \frac{1+u^2}{f(u)u}\chi'(\omega, u) + \frac{u}{f(u)^2}\mathfrak{w}^2\chi(\omega, u) = 0$$

Retarded correlation function

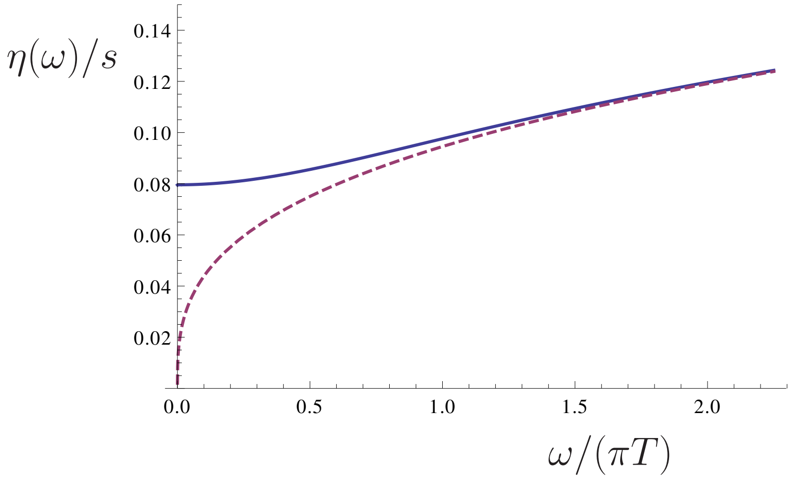
$$G_R(\omega) = \frac{\beta r_+^3 \Delta v}{4\pi G_5} \left. \frac{f(u)\chi'(\omega, u)}{u\chi(\omega, u)} \right|_{u\to 0} .$$

Viscosity from Kubo relation

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

Adams et al. (2008), Herzog et al. (2008)

Spectral function



$$\eta(0)/s = 1/(4\pi)$$

$$\eta(\omega \to \infty) \sim \omega^{1/3}$$

Kubo relation (incl. τ_π): $G_R(\omega) = P - i\eta\omega + \tau_\pi \eta \omega^2 + \kappa_R k^2$

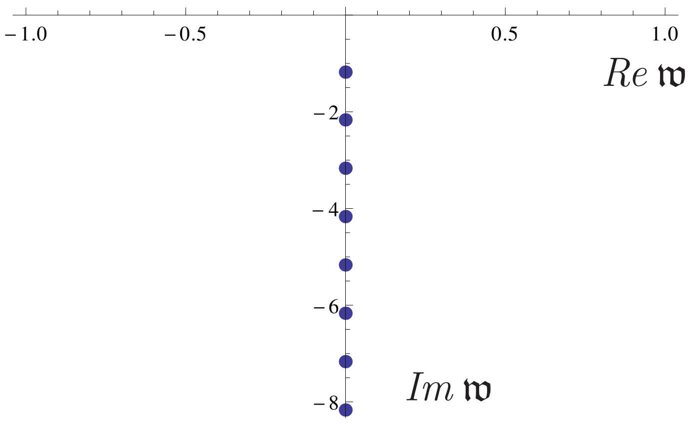
$$\tau_\pi T = -\frac{\log(2)}{2\pi} \qquad AdS_5 : \ \tau_\pi T = \frac{2 - \log(2)}{2\pi}$$

Range of validity of fluid dynamics: $\omega < T$

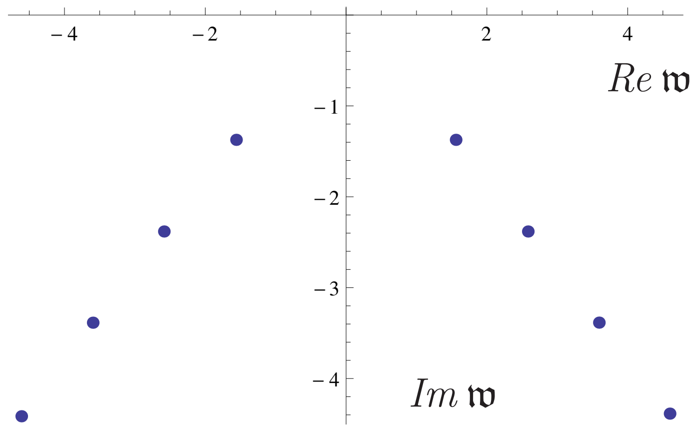
Sch_2: Cannot be matched to relaxation type hydro?

Schaefer (2014), BRSSS (2008)

Quasi-normal modes



Sch_2^2



AdS_5

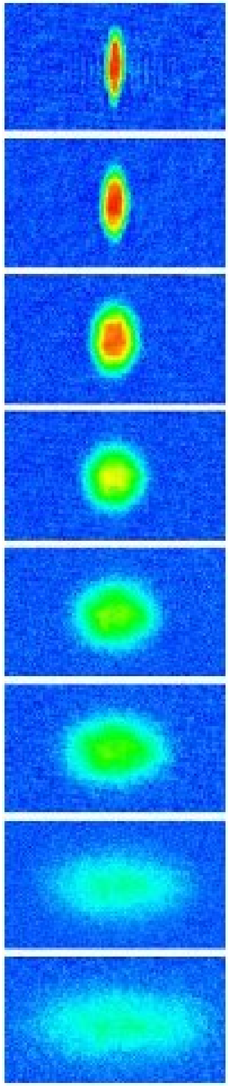
QNM's are stable, $\mathrm{Im}\,\lambda < 0$.

Pole at $\omega \sim iT$ limits convergence of fluid dynamics.

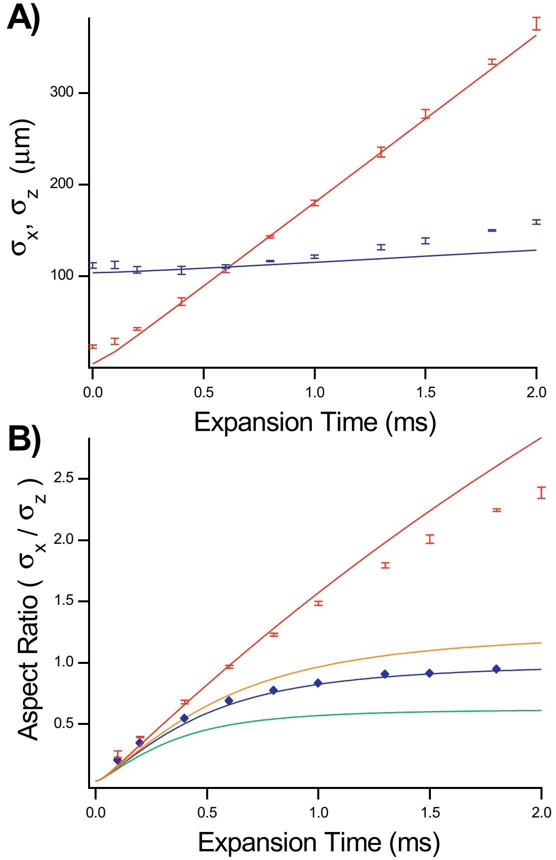
Modes overdamped in Sch_2^2.

Schaefer (2014), Starinets (2002), Heller (2012)

V. Experiments: Elliptic flow



O'Hara et al. (2002)



Hydrodynamic expansion

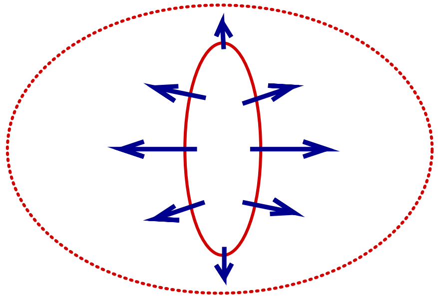
converts

coordinate space

anisotropy

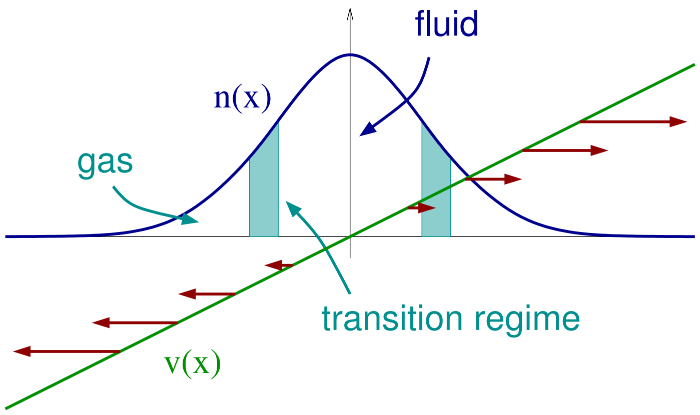
to momentum space

anisotropy



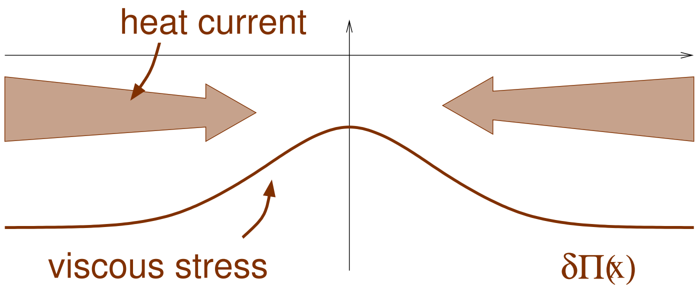
Determination of $\eta(n,T)$

Measurement of $A_R(t,E_0)$ determines $\eta(n,T)$. But:



The whole cloud is not a fluid.

Can we ignore this issue?



No. Hubble flow & low density viscosity $\eta \sim T^{3/2}$ lead to paradoxical fluid dynamics.

$$\dot{Q} = \int \sigma \cdot \delta\Pi = \infty$$

Possible Solutions

Combine hydrodynamics & Boltzmann equation. Not straightforward.

Hydrodynamics + non-hydro degrees of freedom ($\mathcal{E}_a$; $a = x, y, z$)

$$\frac{\partial \mathcal{E}_a}{\partial t} + \vec{\nabla} \cdot \vec{\jmath}^{\,\epsilon}_a = -\frac{\Delta P_a}{2\tau} \qquad \Delta P_a = P_a - P$$

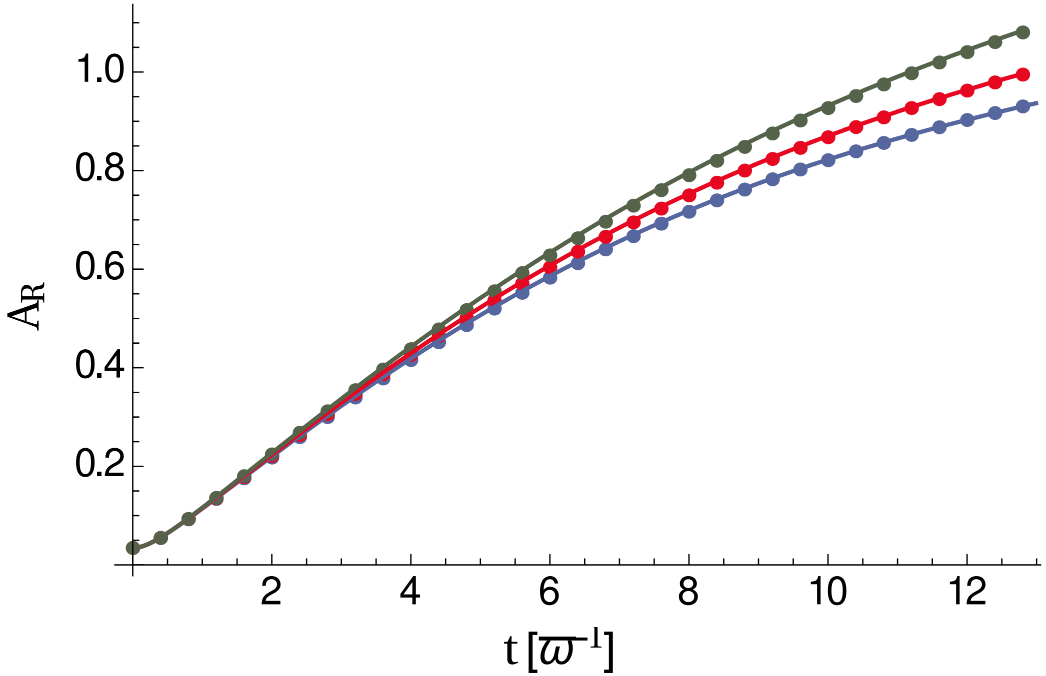
$$\frac{\partial \mathcal{E}}{\partial t} + \vec{\nabla} \cdot \vec{\jmath}^{\,\epsilon} = 0 \qquad \mathcal{E} = \sum_a \mathcal{E}_a$$

τ small: Fast relaxation to Navier-Stokes with $\tau = \eta / P$

τ large: Additional conservation laws. Ballistic expansion.

Anisotropic Hydrodynamics: Comparison with Boltzmann

Aspect ratio $A_R(t) = (\langle r_\perp^2 \rangle / \langle r_z^2 \rangle)^{1/2}$ $\qquad (T/T_F = 0.79, 1.11, 1.54)$



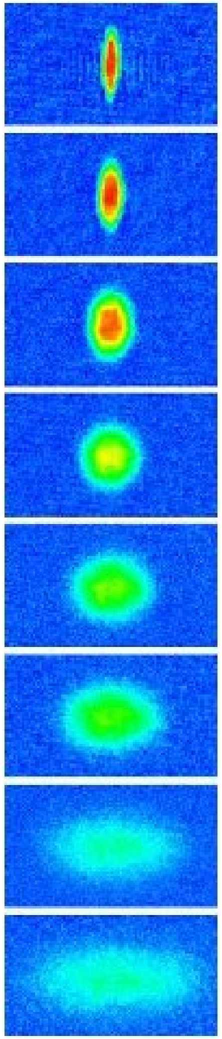
Dots: Two-body Boltzmann equation with full collision kernel

Lines: Anisotropic hydro with η fixed by Chapman-Enskog

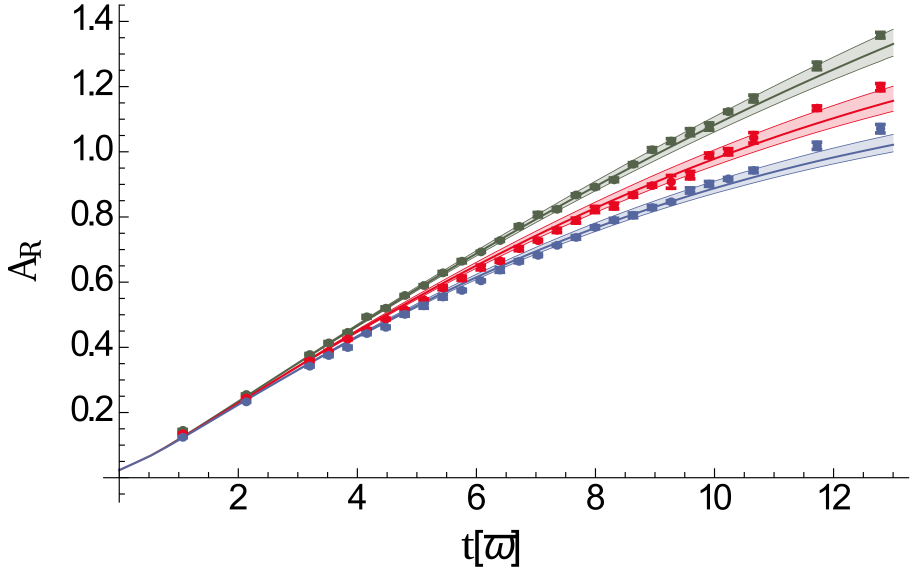
High temperature (dilute) limit: Perfect agreement!

Boltzmann, Pantel et al (2015); AVH1 hydro code, M. Bluhm & T.S. (2015)

Elliptic flow: High T limit



Quantum viscosity $\eta = \eta_0 \frac{(mT)^{3/2}}{\hbar^2}$



$T/T_F = 0.79, 1.11, 1.54$

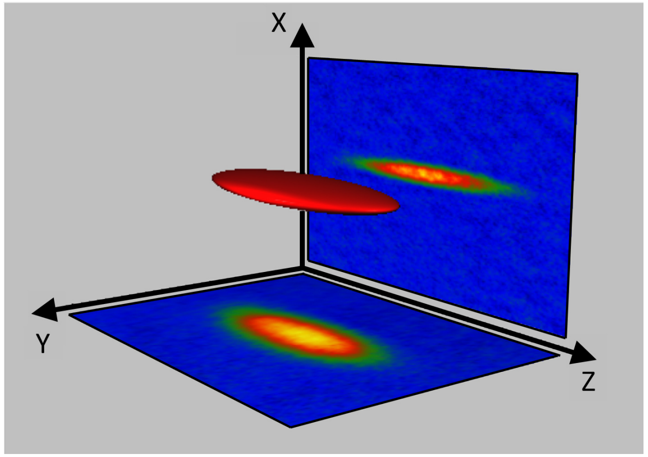
Cao et al., Science (2010)

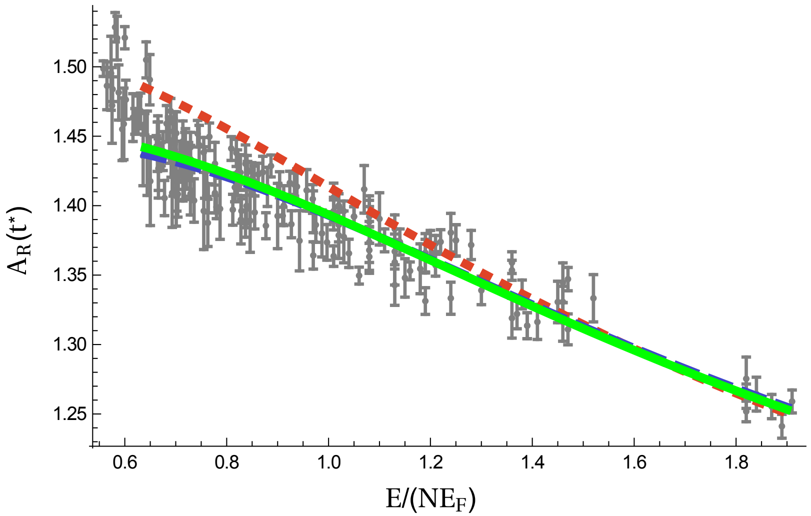
Bluhm et al., PRL (2016)

fit: $\eta_0 = 0.282 \pm 0.02$

theory: $\eta_0 = \frac{15}{32\sqrt{\pi}} = 0.269$

Anisotropic fluid dynamics analysis



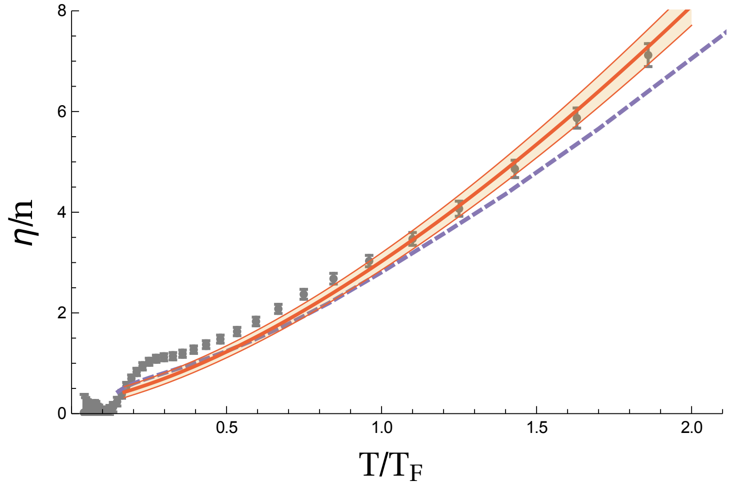


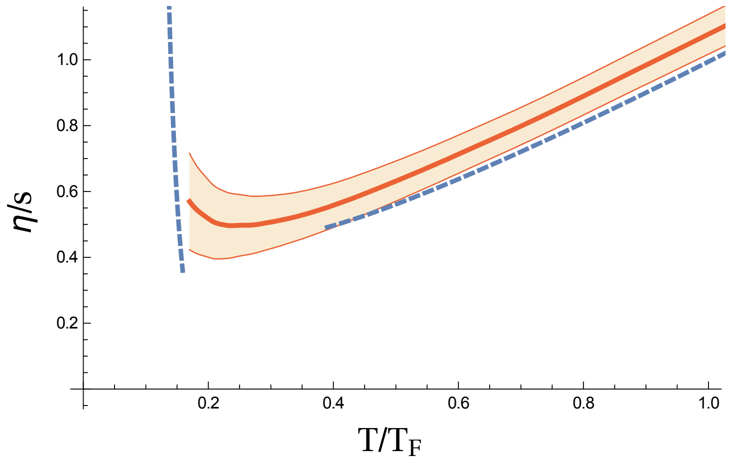
$A_R = \sigma_x/\sigma_y$ as function of total energy. Data: Joseph et al (2016). $E/(NE_F) \sim 0.6$ is the superfluid transition.

Grey, Blue, Green: LO, NLO, NNLO fit.

$$\eta = \eta_0 (mT)^{3/2} \left\{ 1 + \eta_2 n\lambda^3 + \eta_3 (n\lambda^3)^2 + \ldots \right\}$$

Reconstruct η/n and η/s





Left: η/n (Red band)

Right: η/s (Red band) $T_c \sim 0.17 T_F$.

Joseph et al. (Black points). Enss et al. (Dashed line).

Kinetic theory at low and high T (blue dashed)

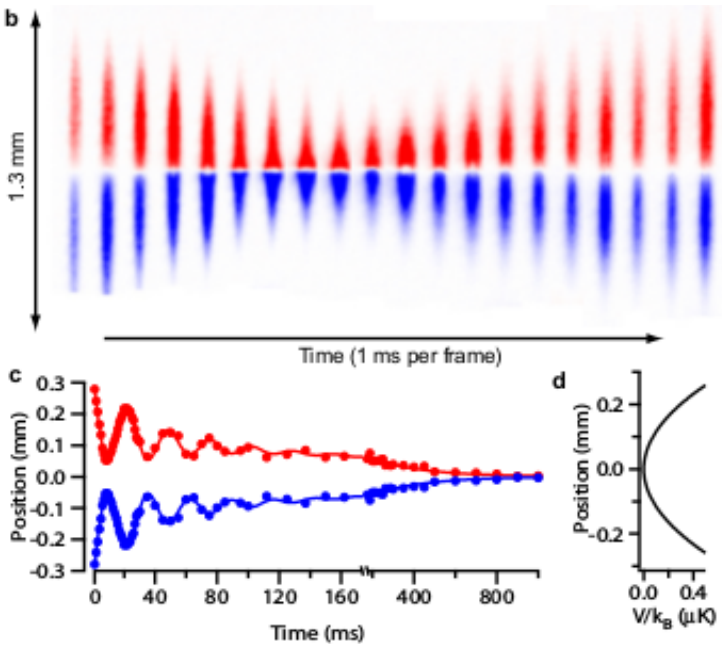
$$\eta(T \gg T_c) = (0.265 \pm 0.02)(mT)^{3/2}$$

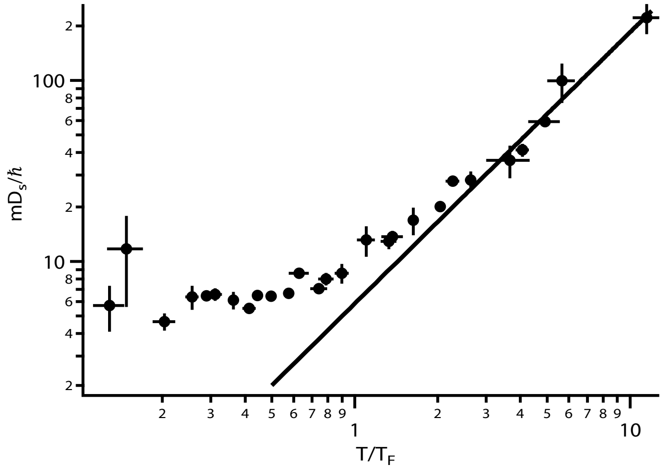
$$\eta_0(th) = \frac{15}{32\sqrt{\pi}} = 0.269$$

$$\eta/n|_{T_c} = 0.41 \pm 0.15$$

$$\eta/s|_{T_c} = 0.56 \pm 0.20$$

VI. Spin Diffusion





Simple scaling at high T

$$P \sim \exp(-\Gamma t) \quad \Gamma_s \equiv \frac{\omega_z^2}{\Gamma}$$

$$D_S = 6.3\frac{\hbar}{m}\left(\frac{T}{T_F}\right)^{3/2}$$

Sommer et al. (2011)

Theory predicts $D_s = 1.1(\hbar/m)(T/T_F)^{3/2}$, Bruun (2010), Sommer et al. (2011)

Spin Diffusion at high temperature

Kinetic theory $(T > T_f)$

$$D \sim 1/n$$

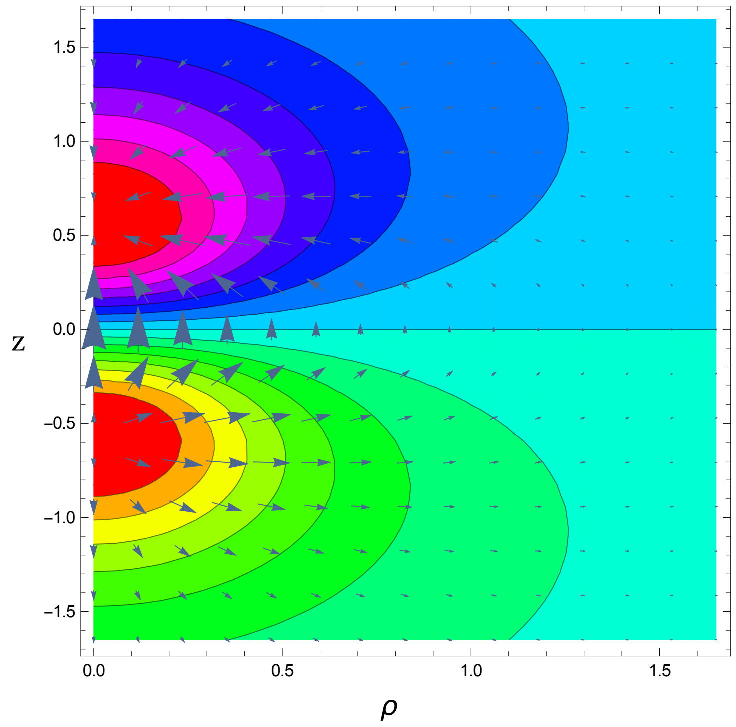
Get large spin current

$$\vec{j}_s \sim D\vec{\nabla} M$$

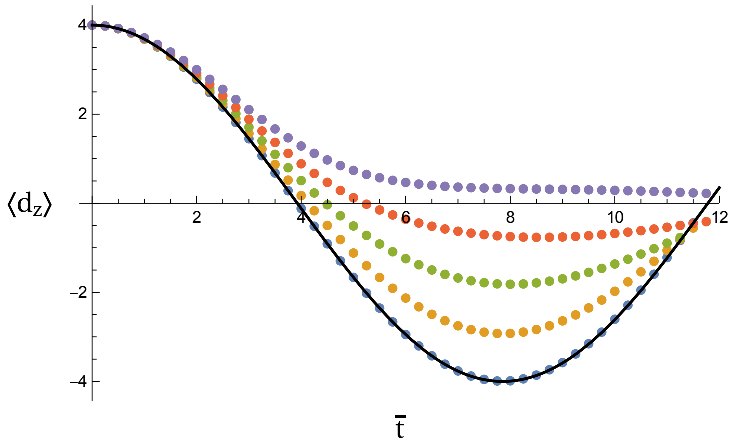
in dilute corona. Predict spin drag

$$\Gamma_s \sim \frac{1.8 E_F(0)}{\Gamma_{red}} \left(\frac{T_F}{T}\right)^{1/2}$$

with $\Gamma_{red} > 200$. Experiment ~ 11.3

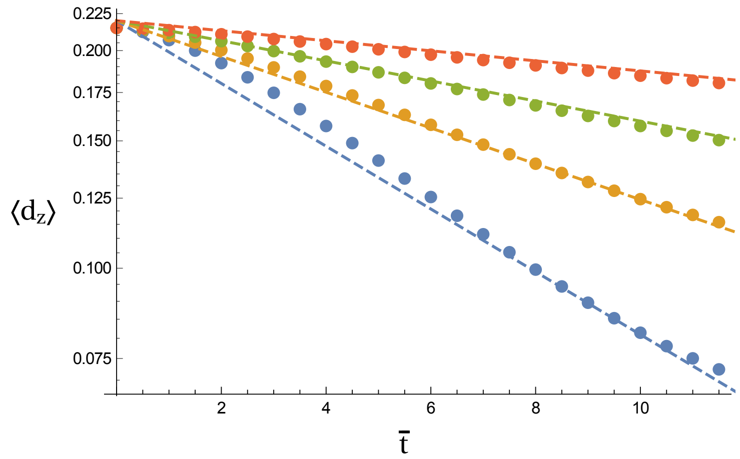


Spin Hydrodynamics



Crossover from ballistic to diffusive behavior for $D \sim const$

$D = \beta$, $\beta = (1000, 5, 2, 1, 0.05)$



Decay of spin dipole mode for diffusion constant $D \sim 1/n$

$D = \beta_T (mT)^{3/2}/n$, $\beta_T = (0.2, 0.1, 0.05, 0.02)$

Find $\Gamma_{red} \sim 11.0$, in agreement with experiment

Time to pour yourself a good fluid

Questions to ponder:

Fluid dynamics as an E(F)T?

Unfold temperature, density dependence of η/s, D_s and κ.

Best way to do hydro+? Ahydro, LBE, stoch hydro.

Quasi-particles or quasi-normal modes?

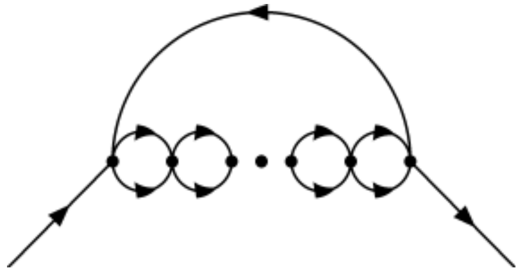
Appendix I: Beyond conformal symmetry

Bulk viscosity and conformal symmetry breaking

Conformal symmetry breaking (thermodynamics)

$$1-\frac{2\mathcal{E}}{3P}=\frac{\langle\mathcal{O}_{\mathcal{C}}\rangle}{12\pi maP}\sim\frac{1}{6\pi}n\lambda^3\frac{\lambda}{a}$$

How does this translate into $\zeta\neq 0$? Momentum dependent $m^*(p)$.



$$Im\,\Sigma(k)\sim zT\sqrt{\frac{T}{\epsilon_k}}Erf\left(\sqrt{\frac{\epsilon_k}{T}}\right)\ll T$$

$$Re\,\Sigma(k)\sim zT\frac{\lambda}{a}\sqrt{\frac{T}{\epsilon_k}}F_D\left(\sqrt{\frac{\epsilon_k}{T}}\right)$$

Bulk viscosity

$$\zeta=\frac{1}{24\sqrt{2}\pi}\lambda^{-3}\left(\frac{z\lambda}{a}\right)^2$$

$$\zeta\sim\left(1-\frac{2\mathcal{E}}{3P}\right)^2\eta$$

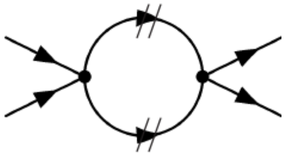
Shear viscosity and conformal symmetry breaking

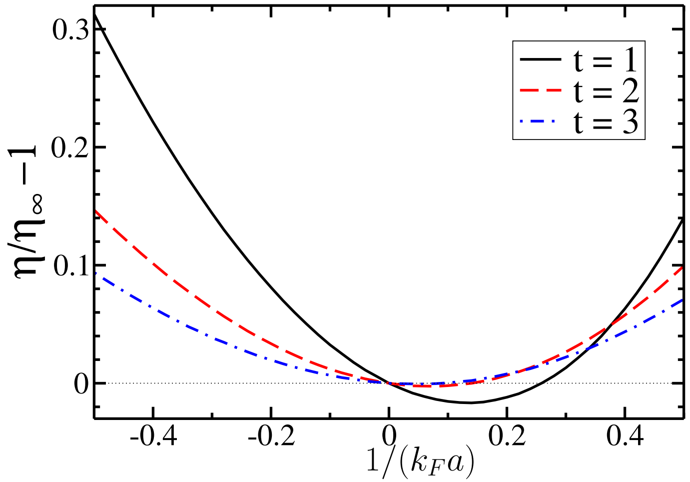
Consider shear viscosity at $a \neq \infty$

$$\eta = \eta_0 \left\{ 1 + O\left(\frac{\lambda^2}{a^2}\right) + O\left(\frac{z\lambda}{a}\right) + \ldots \right\}$$

Medium effects at $O(z\lambda/a)$: Self energy, in-medium scattering

$$\Pi(P,q) \; =$$





Minimum shear viscosity achieved on BEC side

Bluhm, Schaefer (2014)