Dynamical Modeling near the QCD Critical Point

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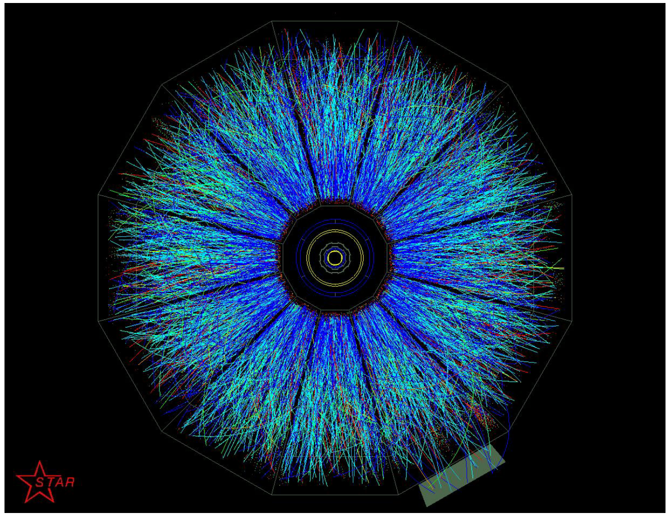
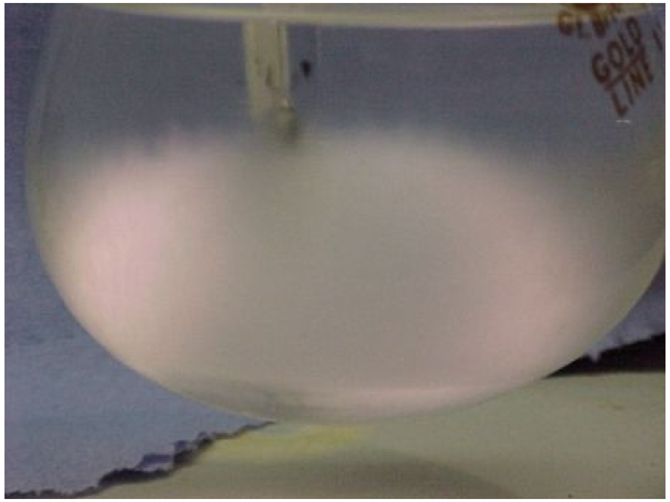
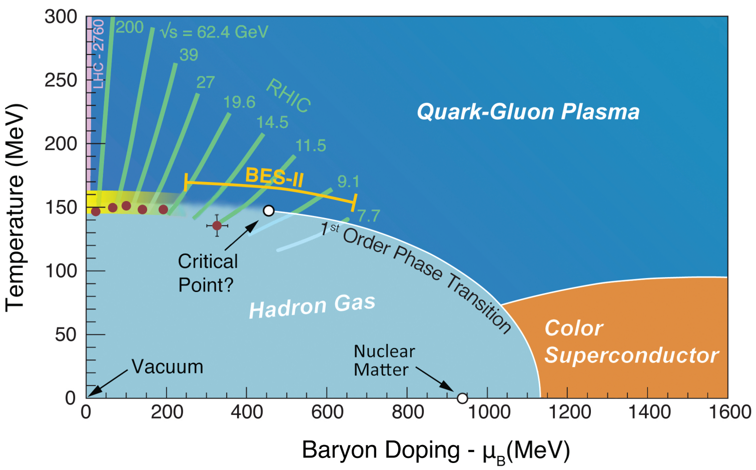


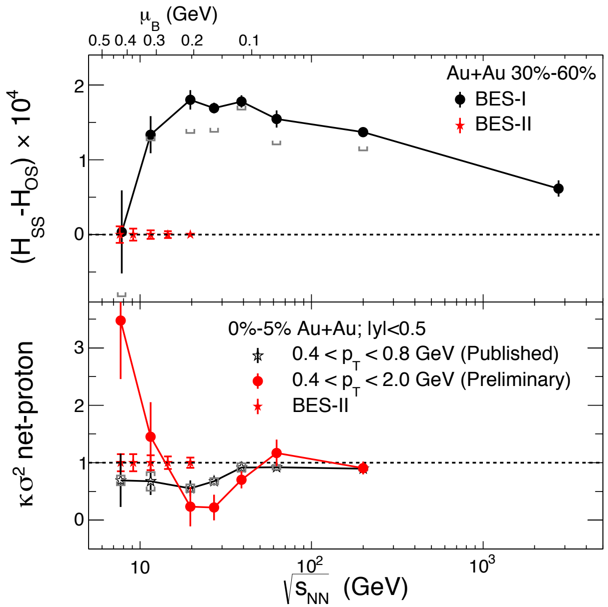


thanks to M. Bluhm, J. Chao, M. Martinez, M. Nahrgang, V. Skokov

RHIC beam energy scan

Can we locate the phase transition itself, either by locating a critical point, or identifying a first order transition?





BEST COLLABORATION

Why dynamical modeling?

It is possible that RHIC/BES (or a future program at Fair or Nica) will discover a critical point without the help of theorists, for example by observing non-monotonic variation of a large set of observables in a narrow range of beam energies, and demonstrating data collapse as a function of energy/centrality/rapidity.

It is also possible that the extracted fluctuation observables can be matched directly to QCD equilibrium susceptibilities near a critical point, computed by evading the sign problem with the help of a quantum computer.

[However, it is also conceivable that there is no QCD critical point in the regime of the phase diagram that can be probed in relativistic heavy ion collisions.]

Why dynamical modeling?

In this talk we will attempt to avoid excessive optimism or pessimism: There is a QCD critical point (and a first order phase transition) that can be discovered by HIC experiments. However, given the small system size and rapid evolution the interpretation of observables is not straightforward.

We cannot directly map fluctuation observables on equilibrium QCD observables, and we may have to look at more exclusive probes, such as correlation function in a specific kinematic range.

This means that we have to do dynamic modeling of critical fluctuations.

Outline:

1. Static universality: Realistic EOS with Ising universality
2. Dynamic universality: Model H in a static background
   (a) Bulk viscosity near the critical point
   (b) Multiplicative noise
3. Stochastic diffusion in an expanding medium
4. Hydrokinetics in an expanding background
5. Other approaches: Hydro+, etc

1. Equilibrium fluctuations

Consider an Ising-like system with order parameter ψ. Fluctuations governed by an entropy functional

$$Prob[\psi, \epsilon] \sim \exp(S[\psi, \epsilon]) \qquad S = \int d^3x \, s(\psi, \epsilon)$$

energy density ϵ, order parameter ψ

Conjugate variables

$$x^A = (\epsilon, \psi) \qquad X_A = -\frac{\partial s}{\partial x^A} = (r, h)$$

reduced temperature r, magnetic field h

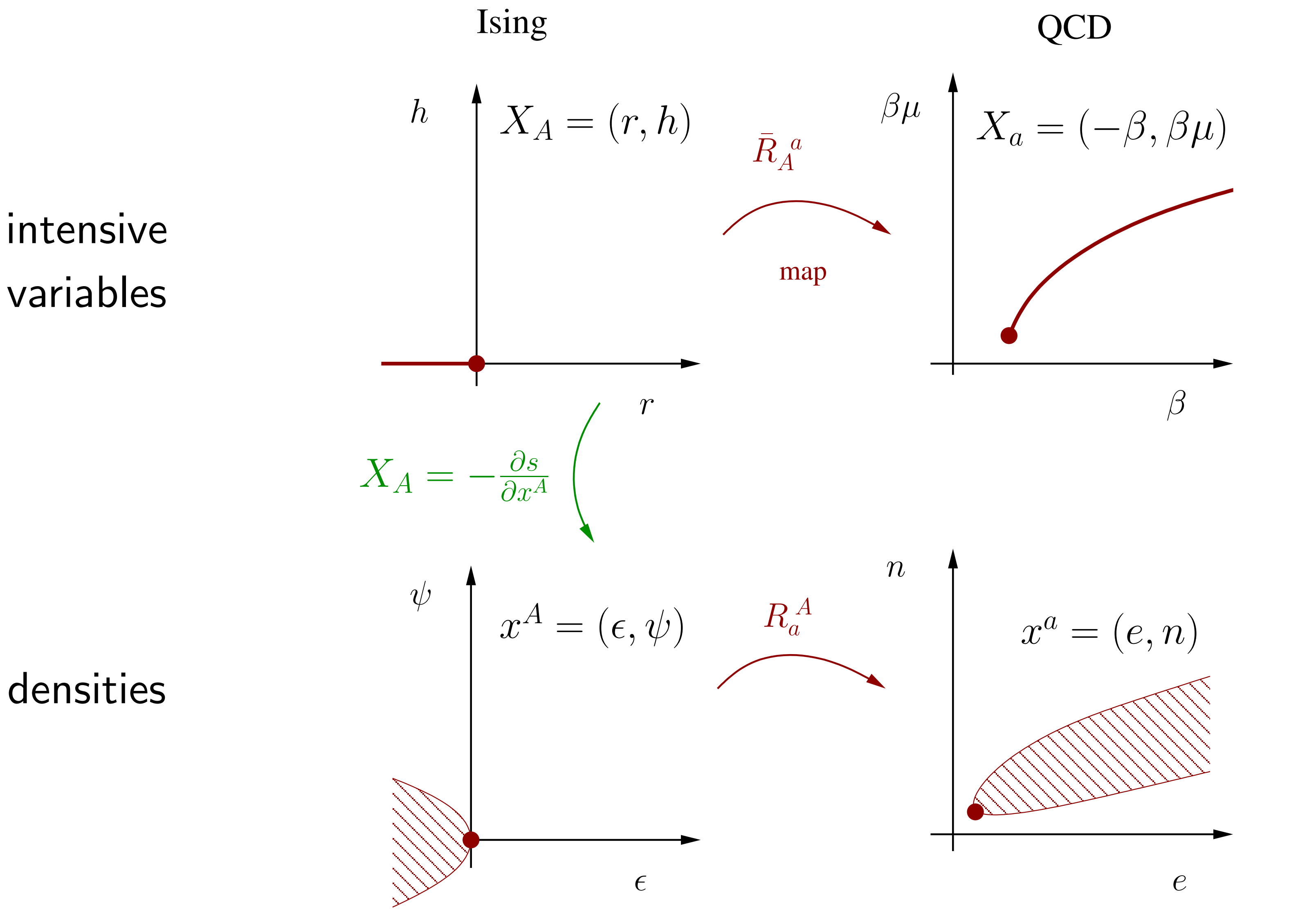
QCD: Canonical pair

$$x^a = (e, n) \qquad X_a = (-\beta, \beta\mu)$$

energy density e, baryon density n

inverse temperature β, chemical potential μ

Mapping the Ising EOS to QCD



$$R_b^A = (\partial x^A)/(\partial x^b) \qquad \bar{R}_A^b = (\partial X_A)/(\partial X_b) \qquad R_b^A \bar{R}_C^b = \delta_C^A$$

BEST collaboration equation of state

Parotto et al. write $S(e,n) = S_{reg}(e,n) + AS_{crit}(e,n)$. Taylor expand regular part (constrained by lattice), linear map to Ising

$$\frac{T - T_c}{T_c} = \bar{w}\,(r\bar{\rho}\sin\alpha_1 + h\sin\alpha_2)$$

$$\frac{\mu - \mu_c}{T_c} = \bar{w}\,(-r\bar{\rho}\cos\alpha_1 - h\cos\alpha_2)$$

parameters

$$(\mu_c, T_c, \bar{w}, \bar{\rho}, \alpha_1, \alpha_2)$$

Critical Gibbs Free Energy (Zinn-Justin parameterization)

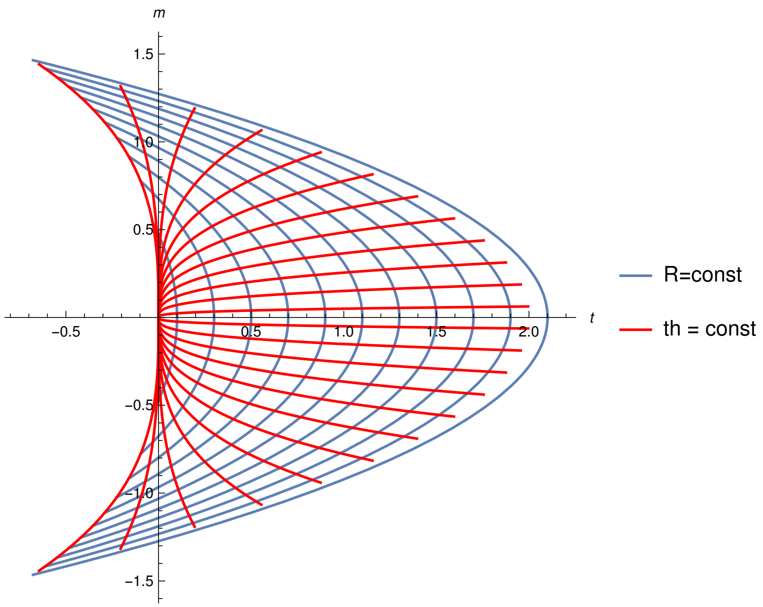
$$G[\psi, r] = h_0 M_0 R^{2-\alpha} g(\theta)$$

$$\psi = M_0 R^\beta \theta \quad r = R(1-\theta^2)$$

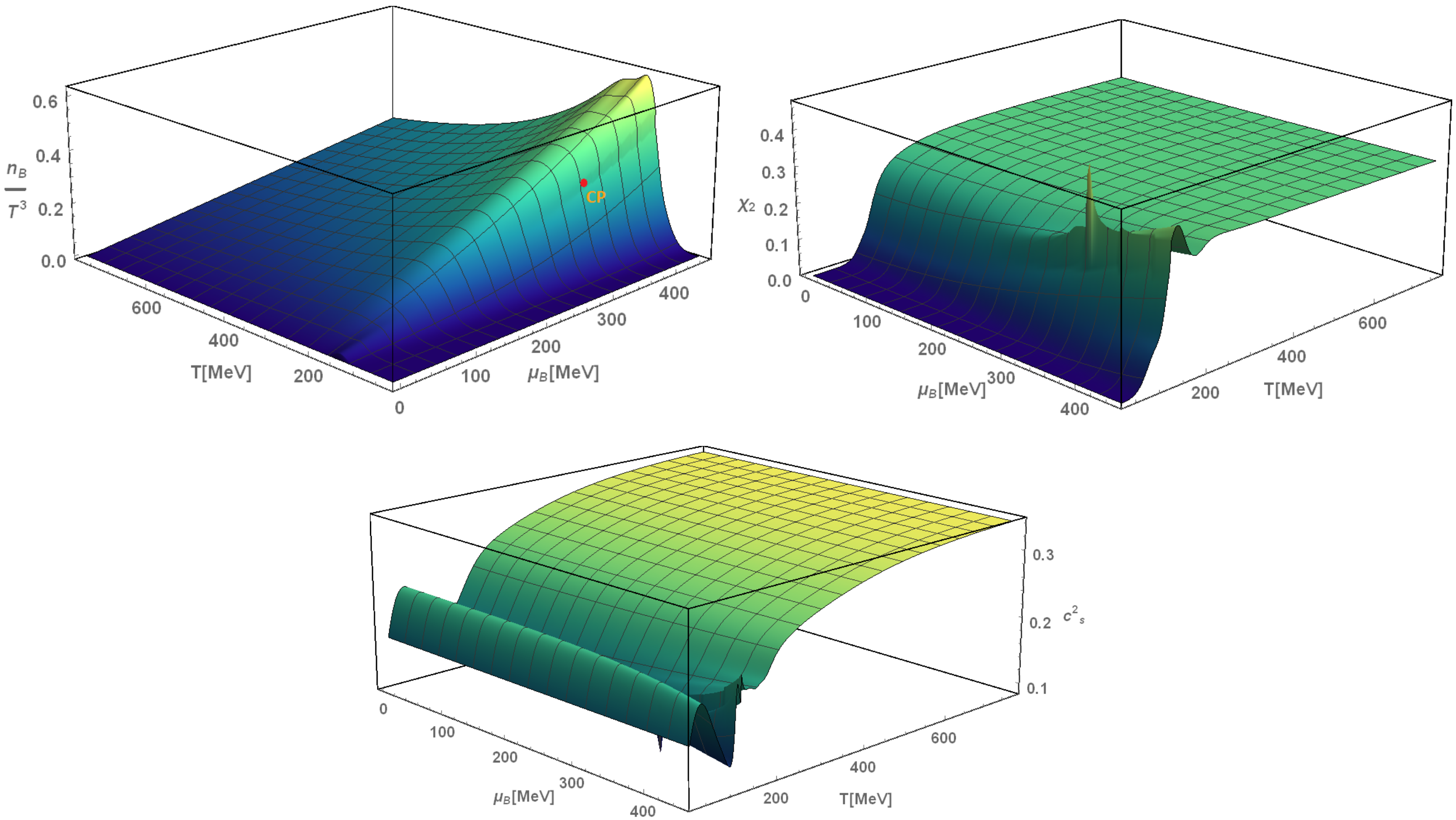
magnetic EOS

$$h = h_0 R^{\beta\delta} \tilde{h}(\theta)$$

$$\theta \in [-\theta_0, \theta_0] \quad \tilde{h}(\theta_0) = 0$$



A critical equation of state for QCD



Baryon density, compressibility, speed of sound.

Parotto et al. (2018)

2. Hydrodynamic equation for critical mode

Equation of motion for critical mode ψ coupled to momentum density $\vec{\pi}$ ("model H")

$$\frac{\partial\psi}{\partial t} = \lambda\nabla^2\frac{\delta\mathcal{F}}{\delta\psi} - g\vec{\nabla}\psi\cdot\frac{\delta\mathcal{F}}{\delta\vec{\pi}_T} + \zeta_\psi$$

Diffusion Advection Noise

Free energy functional: Order parameter ψ, momentum density $\vec{\pi} = w\vec{v}$

$$\mathcal{F} = \int d^dx \left[\frac{1}{2w}\vec{\pi}^2 + \frac{\kappa}{2}(\vec{\nabla}\psi)^2 + \frac{m^2}{2}\psi^2 + \frac{u}{4}\psi^4\right] \qquad D = m^2\lambda$$

Noise average (noise kernel $L = DT\nabla^2$)

$$\langle O\rangle = \frac{1}{Z}\int D\zeta_\psi\, O(\psi(x,t))\, \exp\left(-\frac{1}{4}\int d^3x\, \zeta_\psi L^{-1}\zeta_\psi\right)$$

Model H: Effective Action

MSRJD: Write noise average as an effective action

$$Z_{MSR} = \int D\psi D\tilde{\psi} D\pi D\tilde{\pi} \, \exp\left(- \int d^4x \mathcal{L} \right)$$

$$\mathcal{L} = \tilde{\psi}\left(\partial_t - D\nabla^2\right)\psi + \tilde{\pi}_T\left(\partial_t - \nu\nabla^2\right)\pi_T \qquad \text{Diffusion}$$

$$-\tilde{\psi}DT\nabla^2\tilde{\psi} - \tilde{\pi}_T\nu T\nabla^2\tilde{\pi}_T \qquad \text{Noise}$$

$$+\frac{1}{w}\tilde{\psi}\pi\cdot\nabla\psi + u\tilde{\psi}D\nabla^2\psi^3 + \ldots \qquad \text{Advection \& Interaction}$$

Consider background fluid at rest, $\psi_0 = const$, $\vec{\pi}_0 = 0$:

- Gaussian part: $(\tilde{\psi}, \psi)$ gives matrix propagators with analytic structure of Keldysh formalism
- $\mathcal{T}$-invariance: Detailed balance and Fluctuation-Dissipation relations.

Martin, Siggia, Rose, Phys. Rev. A 8, 423 (1973).

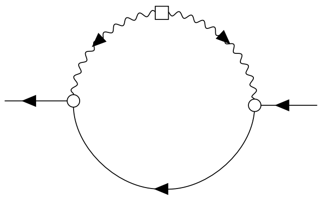
Model H: Critical Dynamics

Non-critical fluids: Gradient expansion $k\xi \ll 1$.

Critical fluids: RG analysis, but fixed point not weakly coupled.

Consider "mode coupling" approximation: Use bare shear viscosity, and static susceptibility χ_k

$$G^{-1}(\omega, k) = i\omega - Dk^2 - \Gamma_k$$



Order parameter relaxation rate ("Kawasaki function").

$$\Gamma_k = \frac{T\xi^{-3}}{6\pi\eta_0} K(k\xi) \qquad K(x) = \frac{3}{4}\left[1 + x^2 + \left(x^3 + x^{-1}\right)\arctan(x)\right] .$$

Order parameter susceptibility

$$\chi_k = \frac{\chi_0}{1 + (k\xi)^{2-\eta}} \qquad \chi_0 = \chi_0^2(\chi/\chi_0)^{2-\eta}$$

Application: Critical bulk viscosity

Express fluctuation in pressure in terms of Ising entropy

$$\delta P = \frac{e+P}{\beta}\, R_e^{\psi}\, \frac{\partial s^{Is}}{\partial \psi} - \frac{n}{\beta}\, R_n^{\epsilon}\, \frac{\partial s^{Is}}{\partial \epsilon}$$

$$\text{Main term :} \quad R_n^{\epsilon} = \frac{\partial \epsilon}{\partial n} \qquad \frac{\partial s^{Is}}{\partial \epsilon} \sim \gamma \psi^2 \qquad \gamma = \gamma_{\pm} r^{1-2\beta}$$

This coupling generates (Kubo relation)

$$\zeta \sim \beta V \int dt\, \langle \delta P(0) \delta P(t) \rangle \sim \beta V (\gamma n T R_n^{\epsilon})^2 \int dt\, \langle \psi^2(0) \psi^2(t) \rangle$$

Slow order parameter relaxation $\rightarrow$ large bulk viscosity

Application: Critical bulk viscosity

Bulk viscosity from order parameter relaxation

$$\zeta \sim (\gamma n T R_n^\epsilon)^2 \int d^3k \, \left. \frac{2T\chi_k^2}{-i\omega + 2\Gamma_k} \right|_{\omega\to 0} \sim (\gamma n T R_n^\epsilon)^2 \xi^4$$

Critical bulk viscosity

$$\frac{\zeta}{s} = \sin^2(\alpha_1) \left(\frac{4\pi}{s/\eta}\right) \left(\frac{\xi}{\xi_0}\right)^{2.8} \begin{cases} 3.4 \cdot 10^{-2} & r > 0 \\ 2.2 \cdot 10^{-1} & r < 0 \end{cases}$$

$z \simeq 3$ dynamical critical exponent.

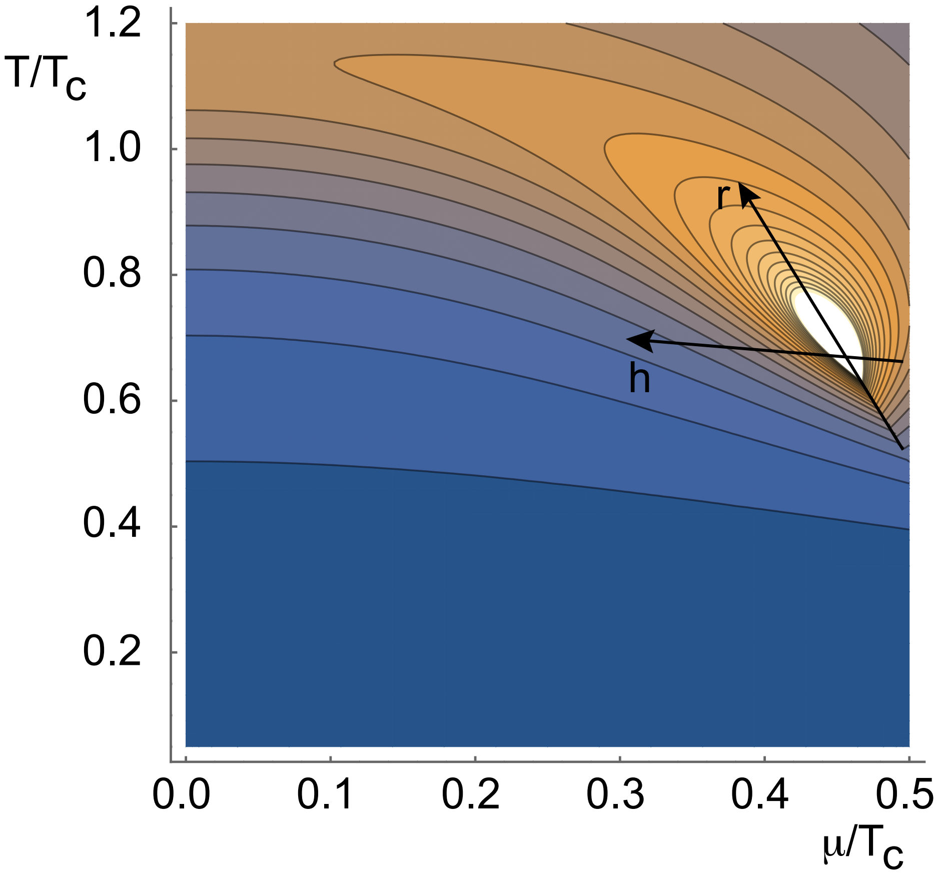
$\sin(\alpha_1)$: angle between Ising r and QCD temperature.

[Note: For $\sin(\alpha_1) \sim 0$ get $\zeta/s \sim (n/s)^2$.]

Amplitude ratio $(\gamma_-/\gamma_+)^2 \simeq 6$.

Martinez, T.S., Skokov [1906.11306]; see also Stephanov & Yin [1712.10305], An et al. [1912.13456], and Onuki, PRE 55 403 (1997).

The role of the Ising Map



Phase diagram in random matrix model, tuned to reproduce T_χ/T_{pc}

See also Pradeep & Stephanov 1905.13247

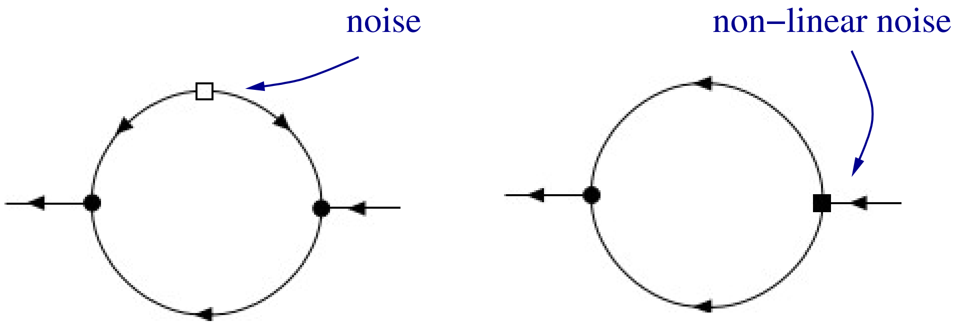
$$\frac{\zeta}{s} = \sin^2(\alpha_1)\left(\frac{4\pi}{s/\eta}\right)\left(\frac{\xi}{\xi_0}\right)^{2.8}\begin{cases} 3.4\cdot 10^{-2} & r>0 \\ 2.2\cdot 10^{-1} & r<0 \end{cases}$$

$$\sin^2(\alpha_1) \simeq 1/4$$

Study: The role of multiplicative noise

Other interactions: Field dependent diffusion/viscosity, $\lambda = \lambda_0(1 + \lambda_D \psi)$.

$$\mathcal{L}_{int} = -\frac{D_0 \lambda_D}{2} \left(\nabla^2 \tilde{\psi} \right) \psi^2 - \frac{D_0 \lambda_D}{m^2} \left(\vec{\nabla} \tilde{\psi} \right)^2 \psi$$



Coupling constant related by fluctuation-dissipation relation ($\mathcal{T}$-invariance)

Contribute to (non-critical) order parameter relaxation

$$\Sigma(\omega, k) = \frac{\lambda_D^2}{32\pi} \left(i\omega k^2 + \left[i\omega - Dk^2 \right] k^2 \right) \sqrt{k^2 - \frac{2i\omega}{D}}$$

(non-critical) Kawasaki function not modified.

Chao, T.S. [2008.01269], see also Chen-Lin et al. [1811.12540]

Other results:

(Leading) MSRJD action in covariant form.

Kovtun, Moore, Romatschke [1405.3967]

Formulate effective action on Keldysh contour (KMS symmetry).
Study higher order (non-universal) corrections.

Glorioso, Liu [1805.09331]; Jain, Kovtun [2009.01356]

Fluctuation bounds, $\delta\eta \sim (s^2 T^3)/\eta_0^2$

Kovtun et al. [1104.1586]; Chafin, T.S. [1209.1006]; Akamatsu et al. [1708.05657]

Extend to non-trivial backgrounds.

3. Numerical Simulation: Stochastic Diffusion

Stochastic diffusion equation

$$\partial_t n_B(x,t) = \Gamma \nabla^2 \left(\frac{\delta \mathcal{F}}{\delta n_B} \right) + \nabla \cdot J(x,t)$$

$$\vec{J}(x,t) = \sqrt{2T\Gamma}\vec{\zeta}(x,t) \qquad \langle \zeta_i(x,t)\zeta_j(x',t')\rangle = \delta(x-x')\delta(t-t')\delta_{ij}$$

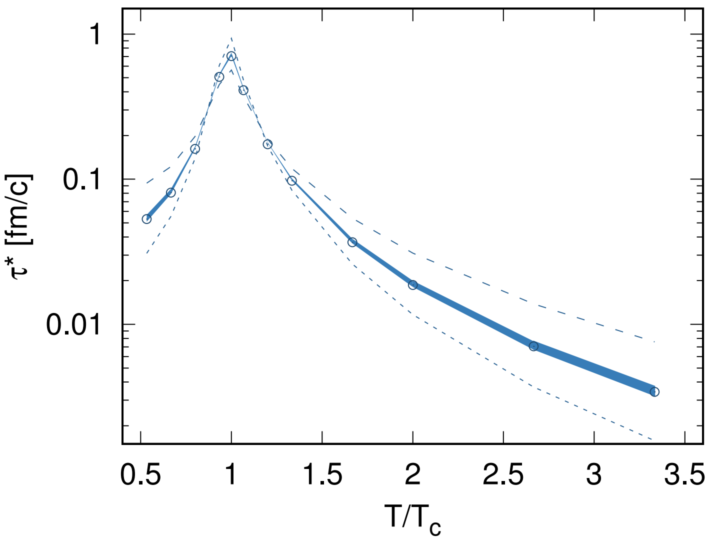
Free energy functional

$$\mathcal{F}[n_B] = T\int d^3x \left(\frac{m^2}{2n_c^2}(\Delta n_B)^2 + \frac{K}{2n_c^2}(\nabla n_B)^2 \right.$$

$$\left. + \frac{\lambda_3}{3n_c^3}(\Delta n_B)^3 + \frac{\lambda_4}{4n_c^4}(\Delta n_B)^4 + \frac{\lambda_6}{6n_c^6}(\Delta n_B)^6 \right)$$

Scale $m^2 \sim \xi^{-2}$, $\lambda_3 \sim \xi^{-3/2}$ etc., parameterize $\xi(r)$ with $r = \frac{T-T_c}{T_c}$.

Numerical results (diffusion in expanding critical fluid)



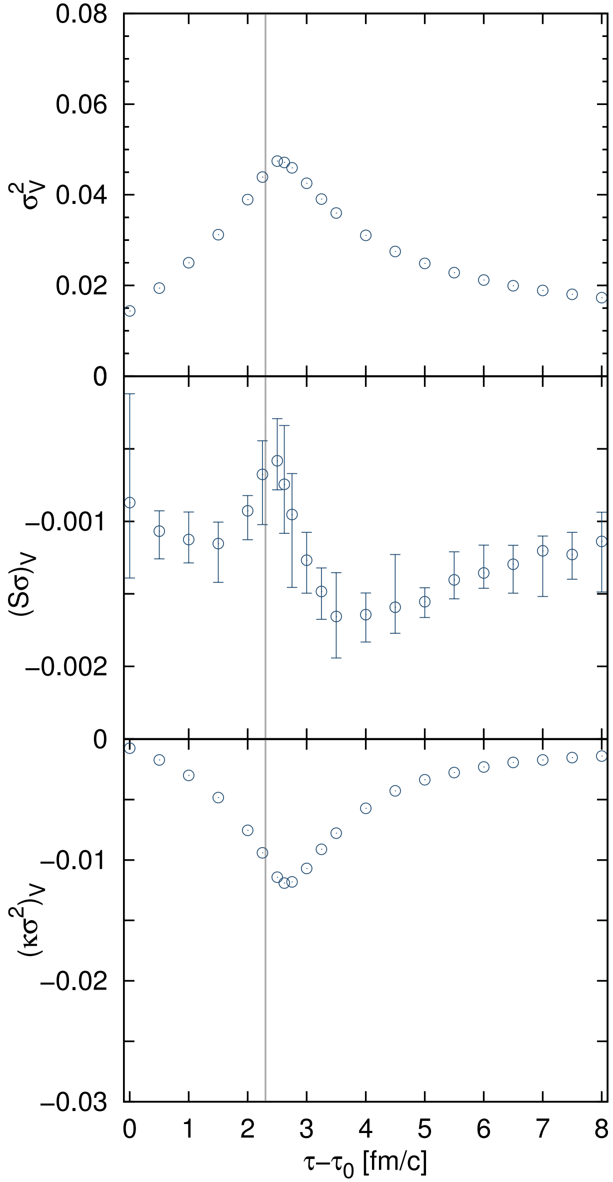
Dynamical scaling: Consider correlation function

$$C_2(t) = \langle \Delta n_B(k,0) \Delta n_B(-k,t) \rangle \text{ for } k = k^* \sim \xi^{-1}$$

Determine decay rate $C_2(t) \sim \exp(-t/\tau^*)$.

Blue line: Expectation for $z = 4$.

M. Nahrgang, Bluhm, T.S., Bass [1804.05728]



Numerical results
(1d diffusion in expanding critical fluid)

Variance

Skewness

Kurtosis

M. Nahrgang et al. (2018)

Comments on stochastic fluid dynamics

Fluctuations depend on resolution scale, $(\Delta n_B/n_B) \sim 1/\sqrt{a^3}$. For typical resolution scales in relativistic heavy ion collisions, fluctuations are $O(1)$.

Fluctuations renormalize transport coefficients and the equation of state. In 3+1 dimensions, $\delta P \sim a^{-3}$ and $\delta D \sim \delta\eta \sim a^{-1}$.

This is a feature, not a bug, but it implies that the bare coefficients are different from the physical transport coefficients, and that critical fluctuations are small compared to high frequency noise.

In practice, may have to employ smoothing, filtering, or non-local noise kernels.

Alternative: Deterministic approaches.

4. Analytic study: Hydro tails in Bjorken geometry

Consider linearized stochastic dynamics about some fluid background (Bj)

Determine eigenmodes: two sound $\phi_{\pm}$, three diffusive modes $\phi_{\psi}, \phi_{\vec{\pi}_T}$.

Noise average: Deterministic equ for 2-point fct $W_{ab} = \langle \phi_a(\tau, x)\phi_b(\tau, x')\rangle$.

$$\partial_0 W + [\mathcal{A}, W] + \{\mathcal{D}, W\} = \mathcal{P}W + W\mathcal{P}^\dagger + \mathcal{N}$$

evolution+reactive + diffusive = sources + noise-correlator

Mixed representation: $W_{ab}(\tau, \vec{k})$.

Contains divergences, can be renormalized by subtraction in homogeneous system.

Akamatsu et al. (2016), Martinez, T.S. (2017).

Expanding System

Study transit of critical point: Consider $\hat{s} = s/n$ and follow "mode coupling" philosophy. Use static susceptibilty and critical relaxation rate $\Gamma_{\hat{s}}$.

$$\partial_t W_{\hat{s}\hat{s}}(t,k) = -2\Gamma_{\hat{s}}(t,k)\left[W_{\hat{s}\hat{s}}(t,k) - W^0_{\hat{s}\hat{s}}(t,k)\right] + \ldots,$$

$$\Gamma_{\hat{s}}(t,k) = \frac{\lambda_T}{C_p\xi^2}(k\xi)^2(1+(k\xi)^{2-\eta}), \qquad W^0_{\hat{s}\hat{s}}(t,k) = \frac{C_p(t)}{(1+(k\xi)^{2-\eta})},$$

Correlation length $\xi(t) = \xi(n(t), e(t)) = \xi_0 f_\xi(r(t), h(t))$

$$\mathrm{hydro}: \ \frac{\partial_t n}{n} \sim \frac{\partial_t e}{e} \sim \frac{1}{\tau_{exp}} \qquad \mathrm{Ising\,map}: \ (e,n) \to (r,h)$$

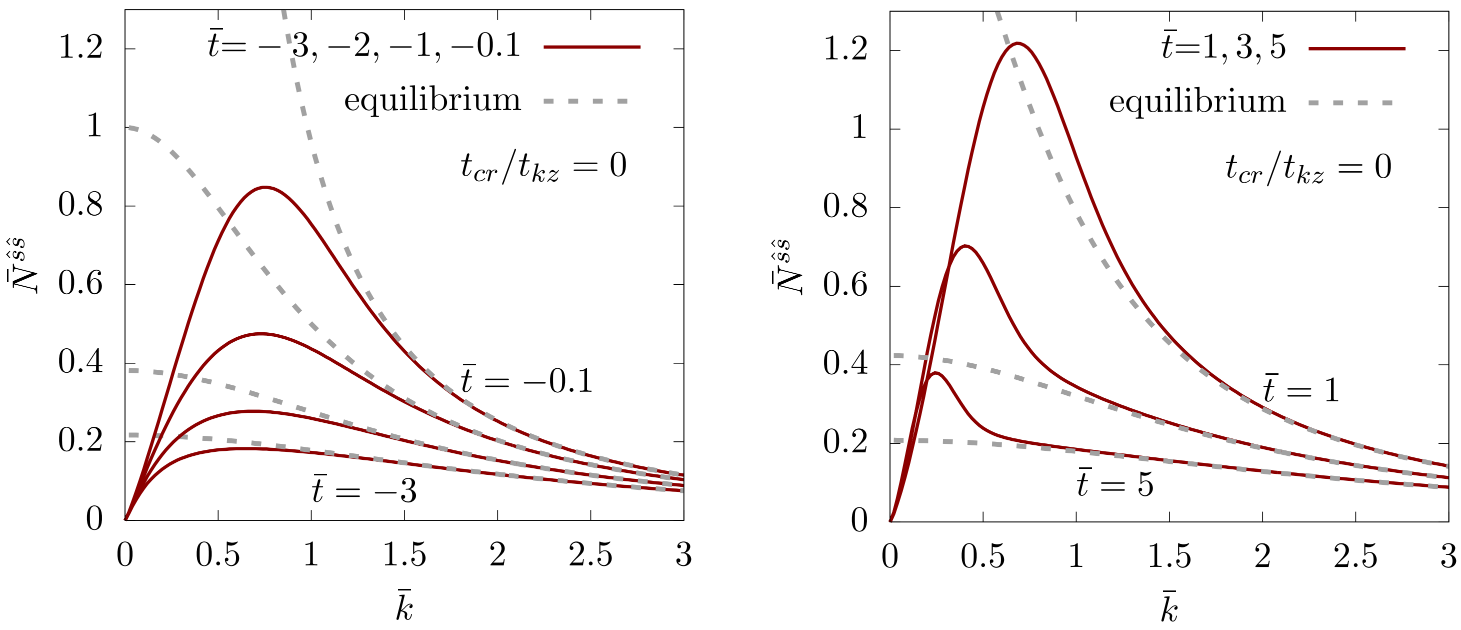
Emergent time scale t_{KZ}: Expansion rate matches relaxation time for modes with $k^* \sim \xi^{-1}$ (modes fall out of equilibrium).

Emergent length scale l_{KZ}: $l_{KZ} = \xi(t_{KZ})$. $l_{KZ} \sim 1.6$ fm

Akamatsu, Teaney, Yan, Yi [1811.05081], see also Berdnikov, Rajagopal [hep-ph/9912274]

Expanding System: Numerical Results

$$\bar{k} = kl_{KZ},\ \bar{t} = t/t_{KZ}$$



before CP　　　　after CP

Akamatsu, Teaney, Yan, Yi [1811.05081]

5. Further developments, other approaches

HYDRO+: Study feedback of hydro fluctuations on hydro evolution. Central object: Non-equilibrium entropy $S \sim \frac{1}{2}\langle \log(\phi_Q/\phi_Q^{eq}) + (\phi_Q/\phi_Q^{eq}) - 1\rangle_Q$.

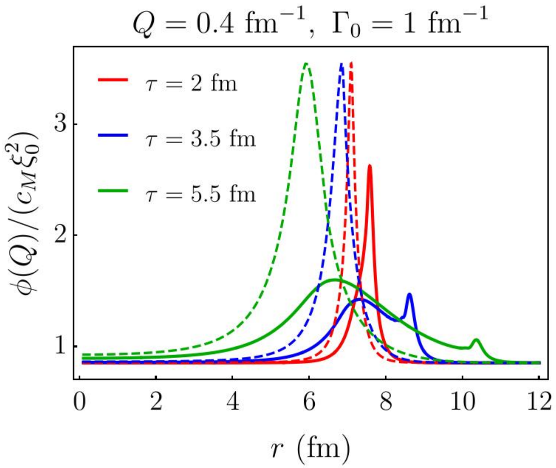
Fully covariant formulation of evolution equation for two-point function in a general background flow.

Re-interpret hydrokinetics in terms of particles (diffusons & phonons).

Evolution equation for higher moments (Fokker-Planck).

Further developments, other approaches

Hydro+ 2-pt function $\phi_Q(r,t)$



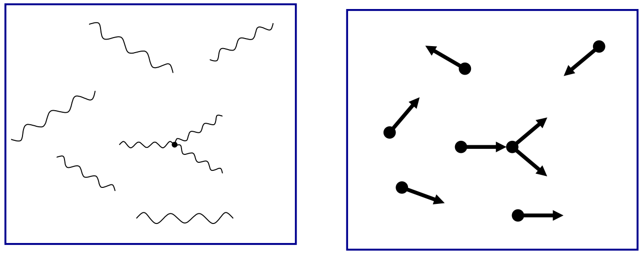
K. Rajagopal et al. [1908.08539]

Covariant Wigner-function for order parameter 2pt function

$$W(x,q) = \int d^4y\, \delta(u(x)\cdot y) e^{-iy\cdot q} \left\langle \psi\left(x+\frac{y}{2}\right)\psi\left(x-\frac{y}{2}\right)\right\rangle$$

An et al. [1912.13456]

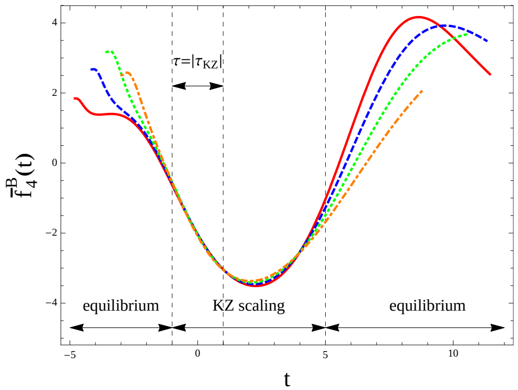
Kinetics of hydro fluctuations



$$W(x,q) = \sum_n w_n \delta(x-x_n)\delta(q-q_n)$$

An et al. [1912.13456]

Evolution of 4th order cumulant



S. Mukherjee et al. [1605.09341], An et al. [2009.10742]

Summary

Dynamical evolution of fluctuations is important.

Model H dynamics in local rest frame: New parameters related to embedding of Ising model, and background correlation length. New results on bulk viscosity and multiplicative noise. New ideas about effective actions on the Keldysh contour.

Dynamics in evolving background: Two basic approaches, "stochastic" or "deterministic", each with their own advantages and disadvantages. Backreaction of fluctuations likely not important. Studies of $C_2(p_\perp, \eta)$ important.

Not discussed: From conserved charges to particles.