# Effective Field Theory and the Nuclear Many-Body Problem

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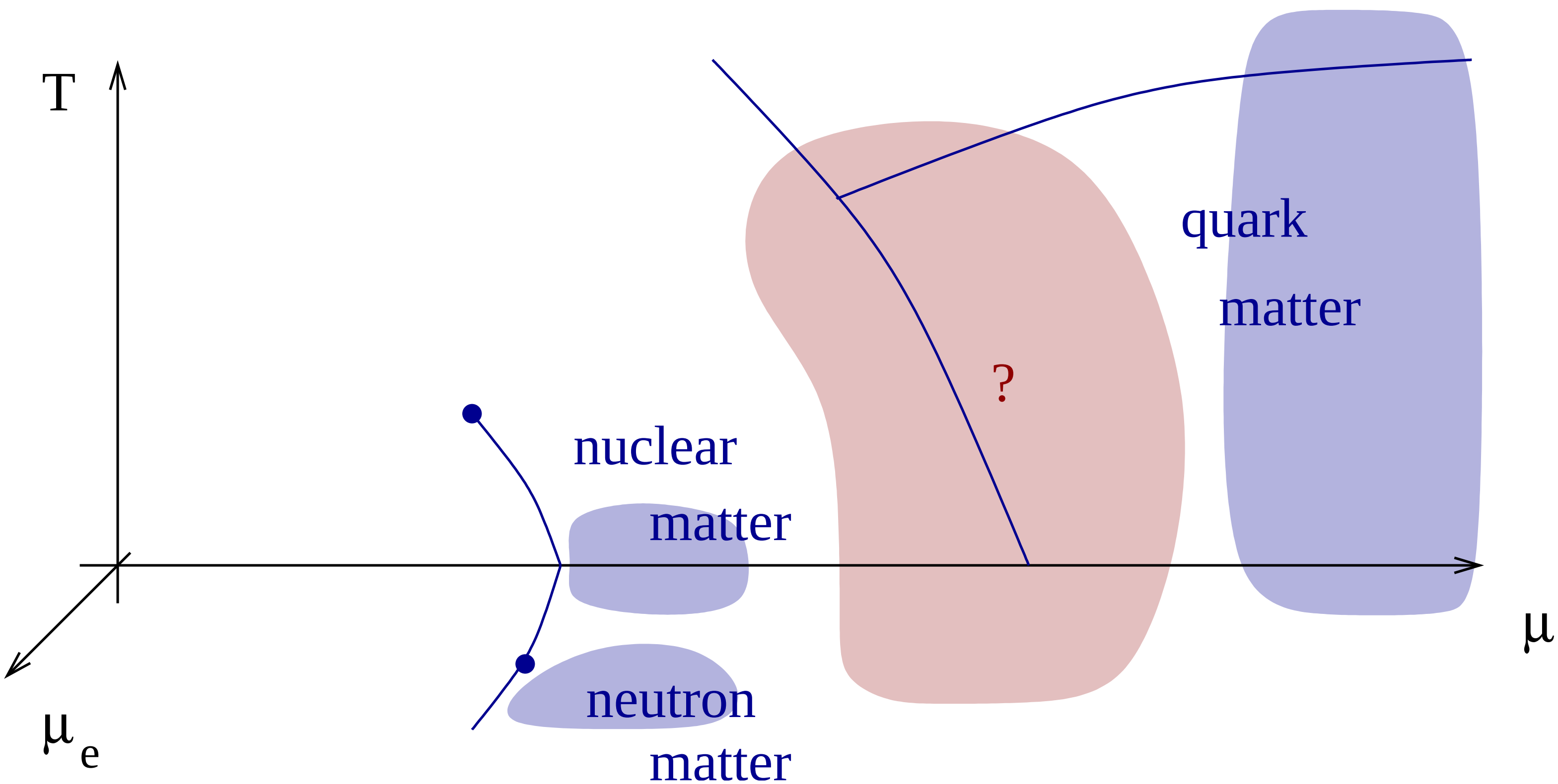
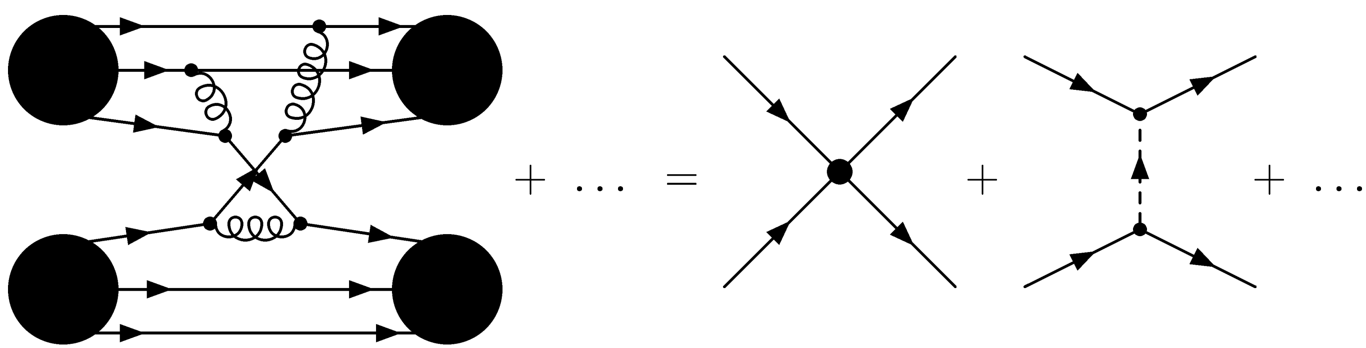
# Schematic Phase Diagram of Dense Matter



# Nuclear Effective Field Theory

Low Energy Nucleons:
- Nucleons are point particles
- Interactions are local
- Long range part: pions



Advantages:
- Systematically improvable
- Symmetries manifest (Chiral, gauge, ...)
- Connection to lattice QCD

# Effective Field Theory

Effective field theory for pointlike, non-relativistic neutrons

$$\mathcal{L}_{\text{eff}} = \psi^\dagger \left( i\partial_0 + \frac{\nabla^2}{2M} \right) \psi - \frac{C_0}{2} (\psi^\dagger \psi)^2 + \frac{C_2}{16} \left[ (\psi\psi)^\dagger (\psi \overleftrightarrow{\nabla}^2 \psi) + h.c. \right] + \ldots$$

Simplifications: neutrons only, no pions (very low energy)

Match to effective range expansion

$$p \cot \delta_0 = -\frac{1}{a} + \frac{1}{2} \Lambda^2 \sum_n r_n \left( \frac{p^2}{\Lambda^2} \right)^{n+1}$$

Coupling constants

$$C_0 = \frac{4\pi a}{M} \qquad\qquad C_2 = \frac{4\pi a^2}{M} \frac{r}{2}$$

# Few Body Physics: Successes

NN scattering: N$^3$LO potentials

External currents: $np \to d\gamma$ etc.

Three body systems: Efimov effect, Phillips line

Four body physics, ...

# The Nuclear Matter Problem is Hard: Traditional View

NN Potential has a very strong hard core

3-body forces, isobars, relativity, . . . important

Saturation density too small

# The Nuclear Matter Problem is Hard: EFT View

NN Potential has a very strong hard core

Short distance behavior not relevant

3-body forces, isobars, relativity, . . . important

3-body: Yes;    Isobars, relativity: Absorbed in counterterms

Saturation density too small

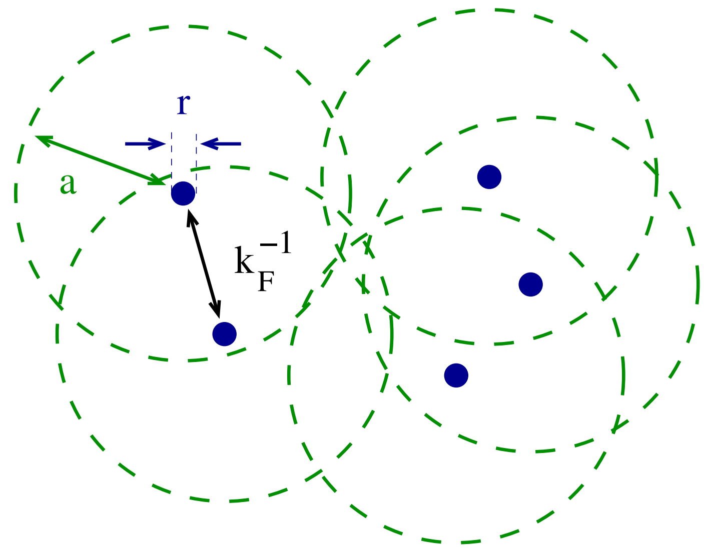
Yes: NN system and nuclear matter (?) are fine tuned

# Toy Problem (Neutron Matter)

Limiting case ("Bertsch" problem)

$$(k_F a) \to \infty$$

$$(k_F r) \to 0$$




No Expansion Parameters!

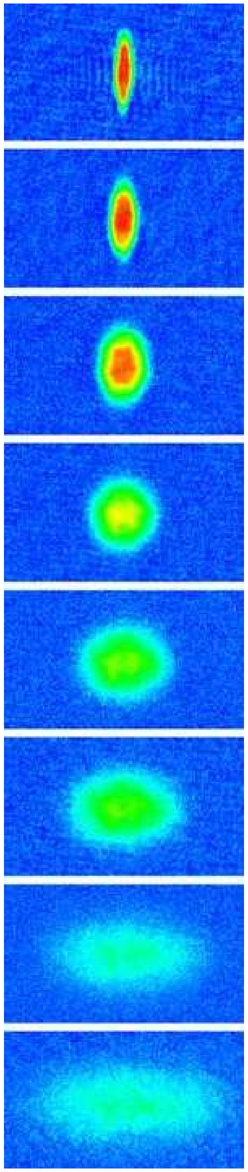
Universal properties $[E_F = k_F^2/(2m)$, $n_f = (2m\mu)^{3/2}/(3\pi^2)]$

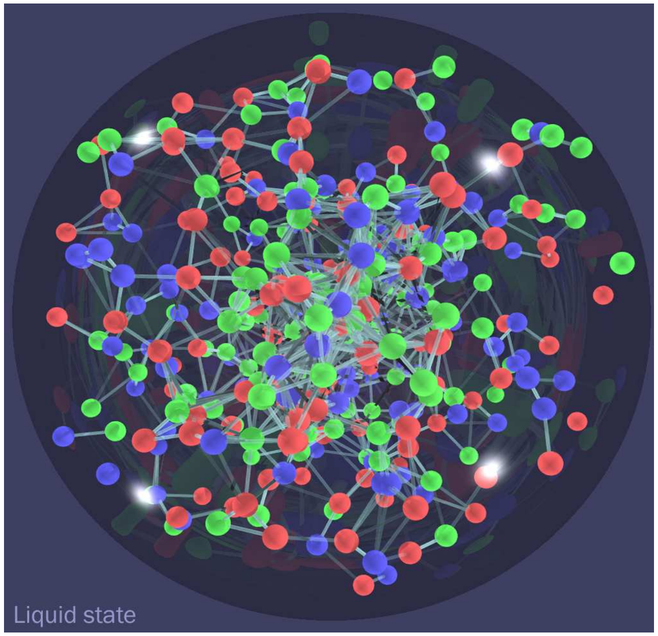
$$(E/A)|_{T=0} = \xi(E^{(0)}/A) = \xi \frac{3}{5} E_F$$

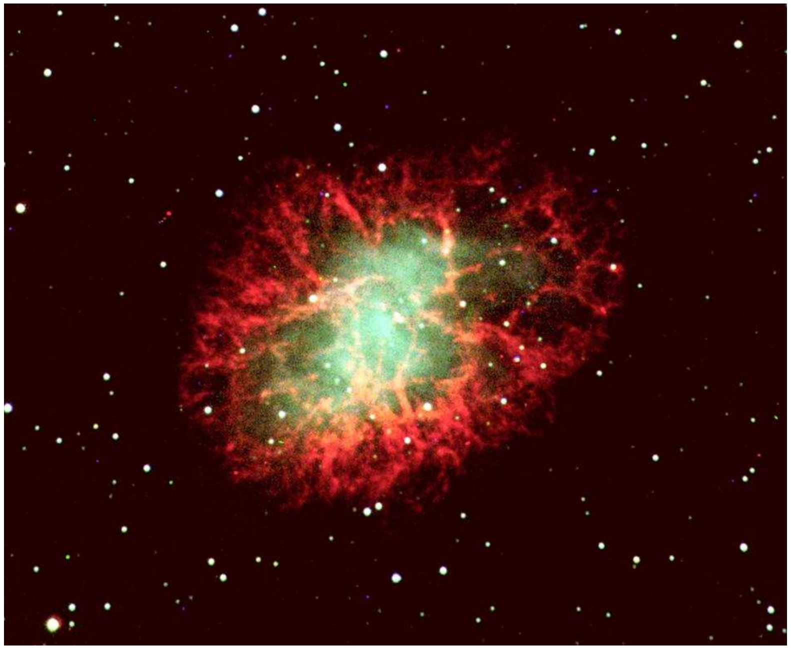
$$\Delta|_{T=0} = \zeta E_F \qquad [T_c = \zeta' T_F]$$

$$P(T,\mu) = \frac{2}{5}\mu n_f(\mu) f(T/T_F)$$

# Perfect Liquids





Neutron Matter (T=1 MeV)

sQGP (T=180 MeV)

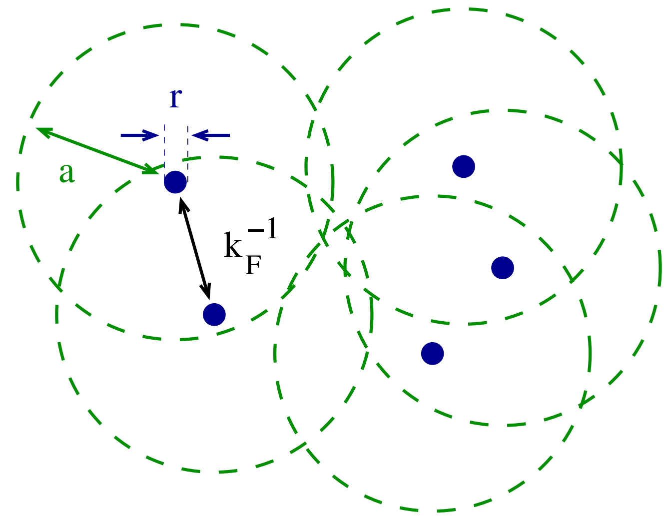
Trapped Atoms (T=1 neV)

# Universality

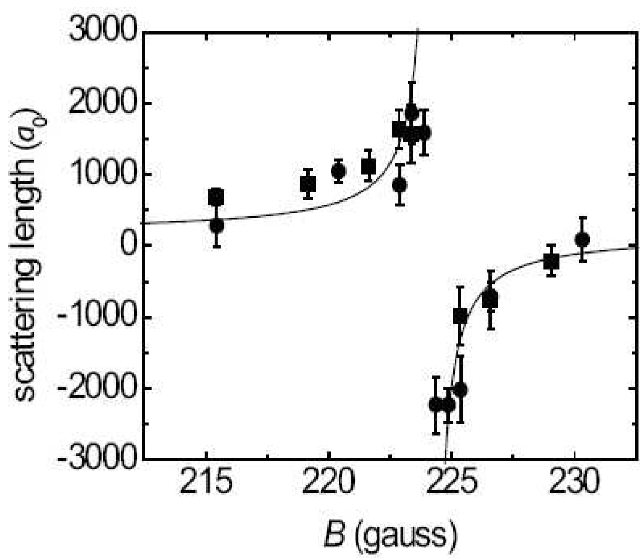
What do these systems have in common?

dilute: $r\rho^{1/3} \ll 1$

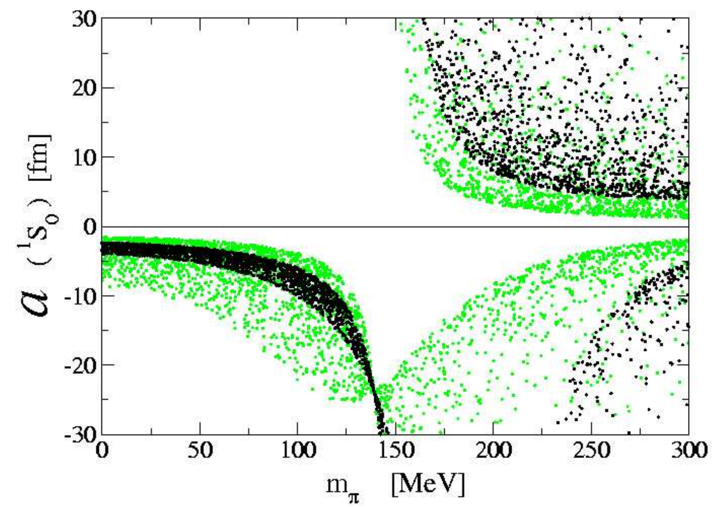
strongly correlated: $a\rho^{1/3} \gg 1$




## Feshbach Resonance in $^6$Li




## Neutron Matter



You have to work with the scattering length (and range parameters) you have, not with the scattering length you want!

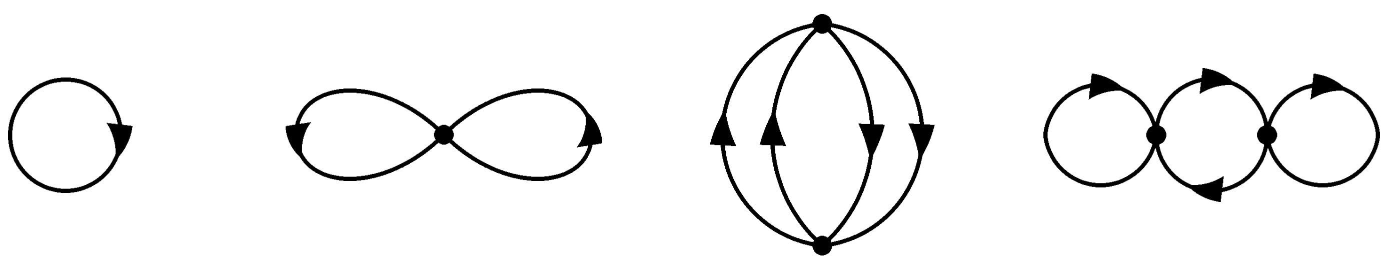
Donald H. Rumsfeld
(SecDef, retired)

# Warmup: Low Density Expansion

Finite density: $\mathcal{L} \to \mathcal{L} - \mu\psi^\dagger\psi \Rightarrow$ Modified propagator

$$G_0(k)_{\alpha\beta} = \delta_{\alpha\beta}\left(\frac{\theta(k-k_F)}{k_0 - k^2/2M + i\epsilon} + \frac{\theta(k_F-k)}{k_0 - k^2/2M - i\epsilon}\right)$$

Perturbative expansion

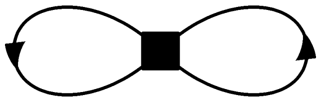


$\epsilon_F\rho$ $\qquad$ $\epsilon_F\rho\,(k_Fa)$ $\qquad$ $\epsilon_F\rho\,(k_Fa)^2$

$$\frac{E}{A} = \frac{k_F^2}{2M}\left[\frac{3}{5} + \left(\frac{2}{3\pi}(k_Fa) + \frac{4}{35\pi^2}(11 - 2\log(2))(k_Fa)^2\right) + \ldots\right]$$

# Low Density Expansion: Higher orders

Effective range corrections

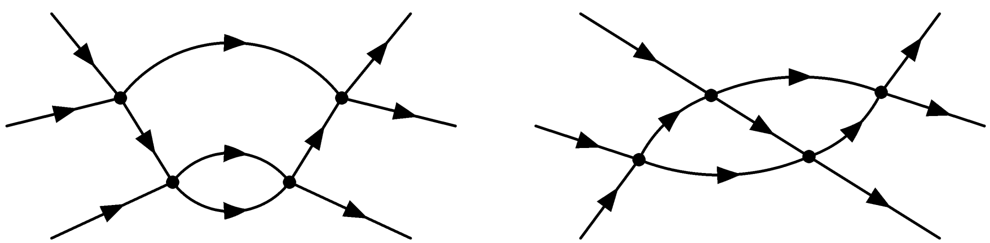


$$\frac{E}{A} = \frac{k_F^2}{2M}\frac{1}{10\pi}(k_F a)^2(k_F r)$$

Logarithmic terms

$$\frac{E}{A} = \frac{k_F^2}{2M}(g-1)(g-2)\frac{16}{27\pi^3}(4\pi - 3\sqrt{3})(k_F a)^4 \log(k_F a)$$

related to log divergence in $3 \to 3$ scattering amplitude



local counterterm $D(\psi^\dagger \psi)^3$ exists if $g \geq 3$

# Nonperturbative Methods

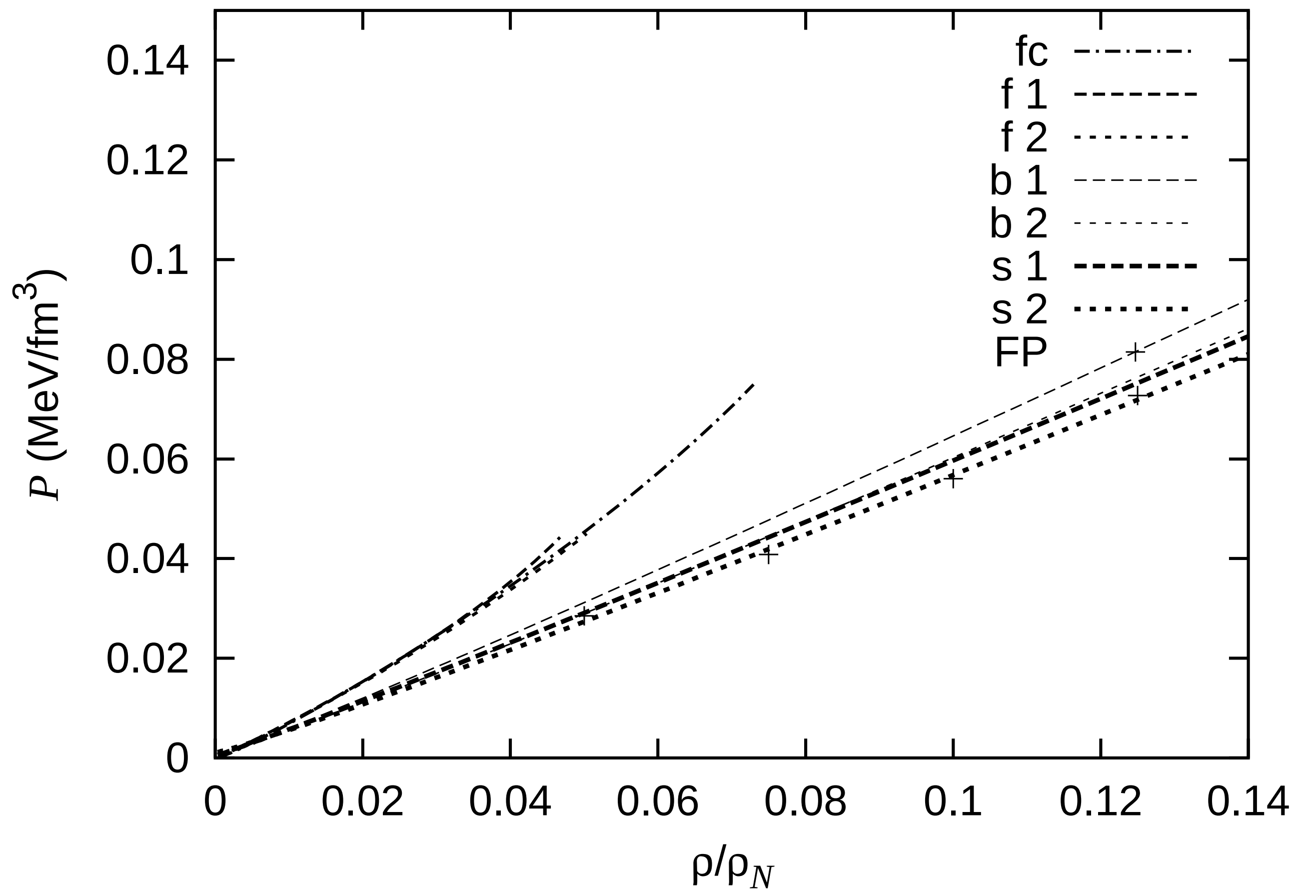
Lattice Field Theory

Other numerical methods: GFMC, VMC, . . .

Expansion in number of species (large N)

Expansion in dimensionality (large d, $\epsilon = 4 - d$)

# Lattice Field Theory




Lee, Schaefer (2004)

pure neutron matter, $T = 4$ MeV

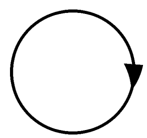
# Large N approximation(s)

Large N gives mean field dynamics. What mean field?

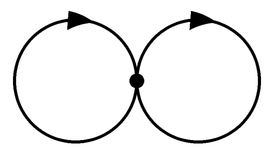
Determined by symmetries of the interaction

SU(2N) symmetric interaction

$$\mathcal{L} = C_0(\psi_f^\dagger \psi_f)^2$$



$N$

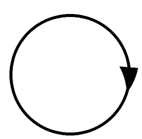


$N(C_0 N)$

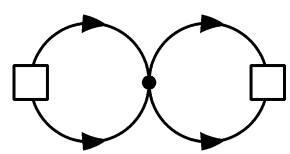
$$\rho = \frac{1}{N}\langle \psi^\dagger \psi \rangle$$

Sp(2N) symmetric interaction ($\mathcal{J} = (\sigma_2) \otimes \ldots \otimes (\sigma_2)$)

$$\mathcal{L} = C_0 \left| \psi_f \mathcal{J}^{fg} \psi_g \right|^2$$



$N$

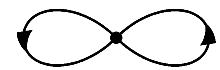


$N(C_0 N)$

$$\Phi = \frac{1}{N}\langle \psi_f \mathcal{J}^{fg} \psi_g \rangle$$

# Large N approximations

SU(2N): Hartree + ring diagrams ($x = N k_F a/\pi$)

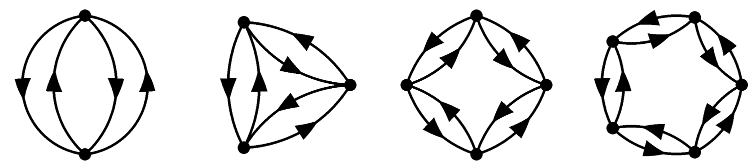


$N(C_0 N)$

$$\frac{E}{A} = \frac{k_F^2}{2M} \times \Big[ \Big( \frac{3}{5} + \frac{2x}{3} \Big)$$

$$+ \frac{1}{N} R(x) + \ldots \Big] \quad (\to \infty)$$

Furnstahl & Hammer (2002)



$(C_0 N)^k$

Sp(2N): BCS + fluctuations

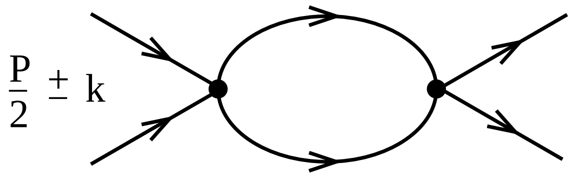
$$\frac{\Omega}{N} = - \int \frac{d^3p}{(2\pi)^3} \left\{ \sqrt{\epsilon_p^2 + \Phi^2} - \epsilon_p - \frac{m\Phi^2}{p^2} \right\} + O(1/N)$$

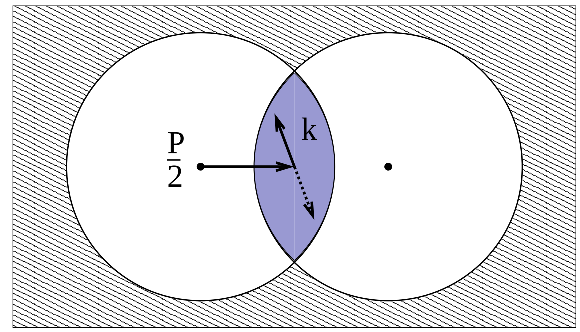
$$\xi = 0.591 - 0.312/N + \ldots = 0.279 \qquad (N = 1)$$

Sachdev (2006)

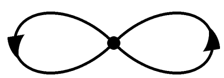
# Large d Limit

In medium scattering strongly restricted by phase space

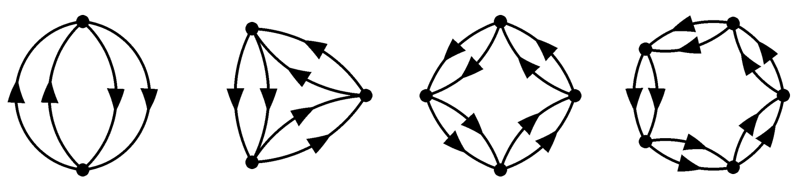



Find limit in which ladders are leading order



$(C_0/d) \cdot 1/d$



$(C_0/d)^k \cdot 1/d$

$$\lambda \equiv \left[ \frac{\Omega_d C_0 k_F^{d-2} M}{d(2\pi)^d} \right]$$

$$\lambda = const \quad (d \to \infty)$$

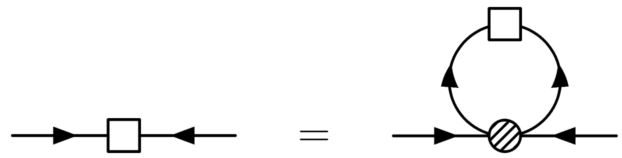
$$\xi = \frac{1}{2} + O(1/d)$$

Steele (1999), Schaefer et al (2003)

# Pairing in the Large d Limit

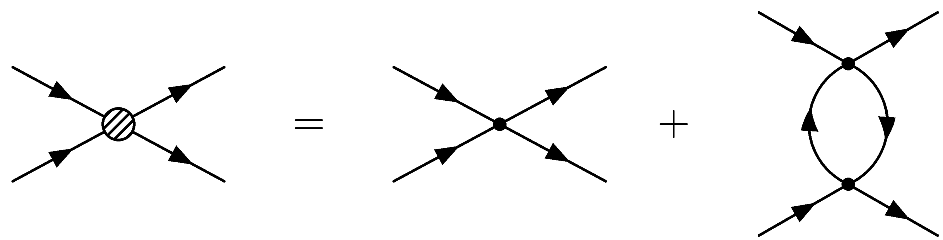
BCS gap equation

$$\Delta = \frac{|C_0|}{2} \int \frac{d^d p}{(2\pi)^d} \frac{\Delta}{\sqrt{\epsilon_p^2 + \Delta^2}}$$



Solution

$$\Delta = \frac{2e^{-\gamma} E_F}{d} \exp\left(-\frac{1}{d\lambda}\right)$$




$$\Delta = 0.375 E_F$$

Pairing energy (subleading in 1/d)

$$\frac{E}{A} = -\frac{d}{4} E_F \left(\frac{\Delta}{E_F}\right)^2 \sim \frac{1}{d}.$$

# Upper and lower critical dimension

Zero energy bound state for arbitrary d

$$\psi''(r) + \frac{d-1}{r}\psi'(r) = 0 \qquad (r > r_0)$$

d=2: Arbitrarily weak attractive potential has a bound state

$$\xi(d{=}2) = 1$$

d=4: Bound state wave function $\psi \sim 1/r^{d-2}$. Pairs do not overlap

$$\xi(d{=}4) = 0$$

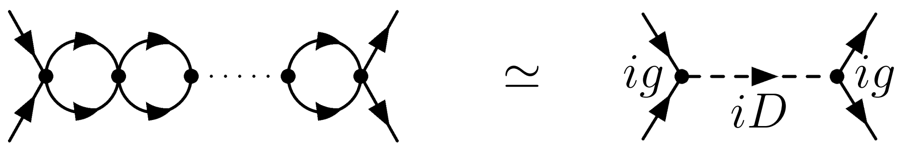
Conclude $\xi(d{=}3) \sim 1/2$?

Try expansion around $d = 4$ or $d = 2$?

Nussinov & Nussinov (2004)

# Epsilon Expansion

EFT version: Compute scattering amplitude ($d = 4 - \epsilon$)




$$T = \frac{1}{\Gamma\left(1 - \frac{d}{2}\right)} \left(\frac{m}{4\pi}\right)^{-d/2} \left(-p_0 + \frac{\epsilon_p}{2}\right)^{1-d/2} \simeq \frac{8\pi^2\epsilon}{m^2} \frac{i}{p_0 + \frac{\epsilon_p}{2} + i\delta}$$

$$g^2 \equiv \frac{8\pi^2\epsilon}{m^2} \qquad D(p_0, p) = \frac{i}{p_0 + \frac{\epsilon_p}{2} + i\delta}$$

Weakly interacting bosons and fermions

# Epsilon Expansion

Effective lagrangian for atoms $\Psi = (\psi_\uparrow, \psi_\downarrow^\dagger)$ and dimers $\phi$

$$\mathcal{L} = \Psi^\dagger \Big( i\partial_0 + \frac{\sigma_3 \nabla^2}{2m} \Big) \Psi + \mu \Psi^\dagger \sigma_3 \Psi + \Psi^\dagger \sigma_+ \Psi \phi + h.c.$$

Perturbative expansion: $\phi = \phi_0 + g\varphi$. Free part

$$\mathcal{L}_0 = \Psi^\dagger \Big[ i\partial_0 + \delta\mu + \sigma_3 \frac{\vec{\nabla}^2}{2m} + \phi_0(\sigma_+ + \sigma_-) \Big] \Psi + \varphi^\dagger \Big( i\partial_0 + \frac{\vec{\nabla}^2}{4m} \Big) \varphi \,.$$

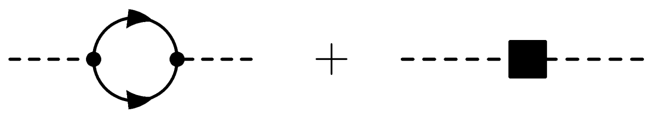
Interacting part $(g^2, \mu = O(\epsilon))$

$$\mathcal{L}_I = g\big(\Psi^\dagger \sigma_+ \Psi \varphi + h.c\big) + \mu \Psi^\dagger \sigma_3 \Psi - \varphi^\dagger \Big( i\partial_0 + \frac{\vec{\nabla}^2}{4m} \Big) \varphi \,.$$

Nishida & Son (2006)

# Epsilon Expansion

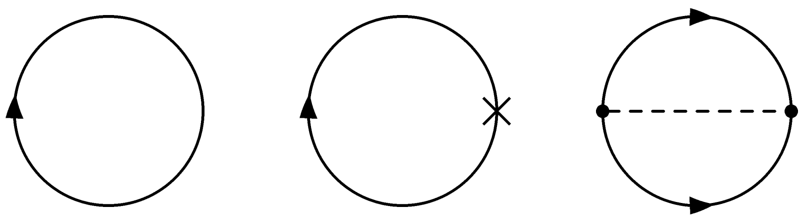
Consistency conditions



$= O(\epsilon)$

Also: tadpoles cancel

Effective potential



$O(1)$ $O(1)$ $O(\epsilon)$

$$\xi = \frac{1}{2}\epsilon^{3/2} + \frac{1}{16}\epsilon^{5/2}\ln\epsilon - 0.0246\,\epsilon^{5/2} + \ldots$$

$$\xi(\epsilon = 1) = 0.475$$

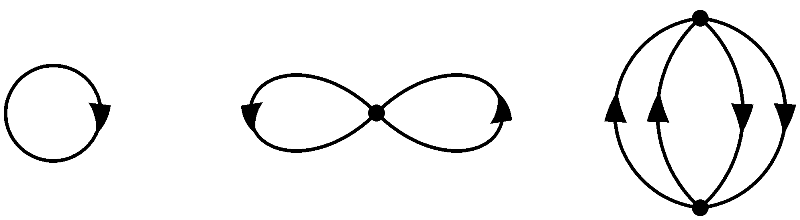
Problem: Higher order corrections large ($\sim 100$ %)!

# Near two dimensions

Scattering amplitude near d=2 ($\bar{\epsilon} = d - 2$)

$$\mathcal{A}(p_0, p) = i\frac{2\pi}{m}\,\bar{\epsilon} + O(\bar{\epsilon}^2) \qquad\qquad g^2 = \frac{2\pi\bar{\epsilon}}{m}$$

Effective potential (similar to $(k_F a)$ expansion)



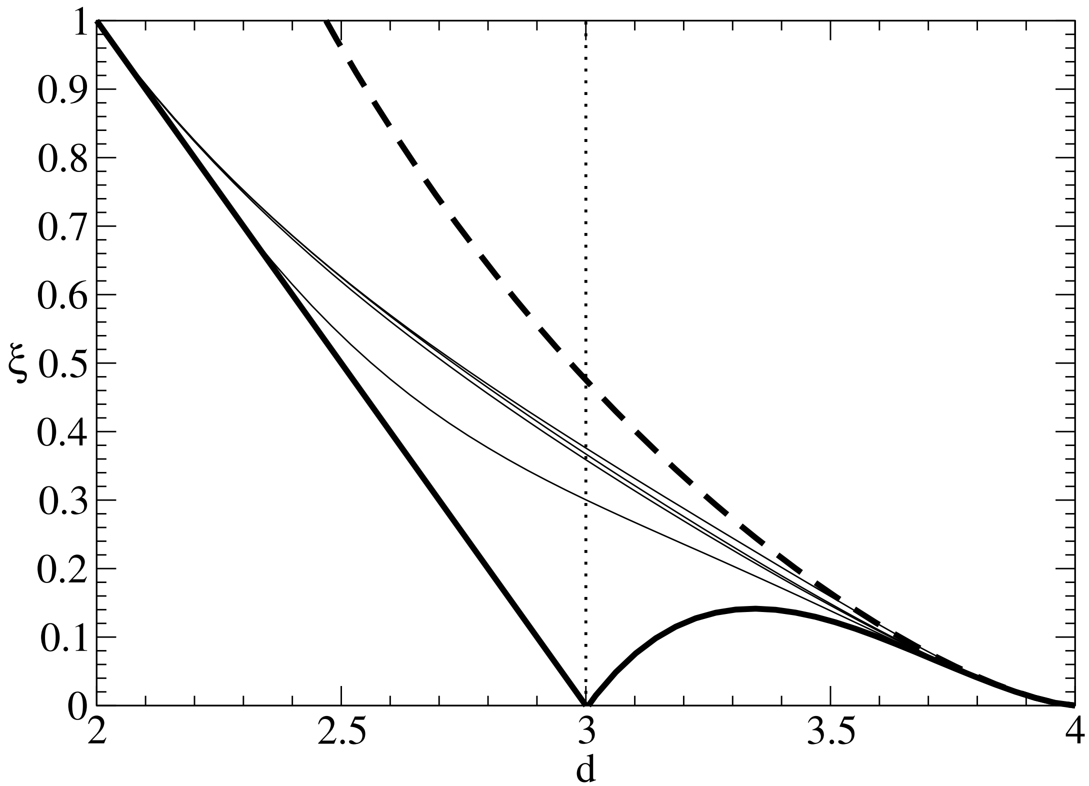
$O(1)$ $\qquad$ $O(\bar{\epsilon})$ $\qquad$ $O(\bar{\epsilon}^2)$

$$\xi = 1 - \bar{\epsilon} + O(\bar{\epsilon}^2)$$
$$= 0 \quad (\bar{\epsilon} = 1)$$

Superfluid gap (BCS + screening correction)

$$\Delta = \frac{2\mu}{e}\exp\left(-\frac{1}{\bar{\epsilon}}\right)$$

# Combine expansions near d=2 and d=4



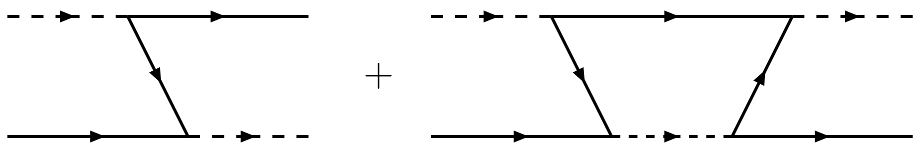

Conclude $\xi = (0.3 - 0.4)$

Arnold et al. (2006)

other appl.: Kryjevski,Rupak,Schaefer (2006)

# Few Body Physics

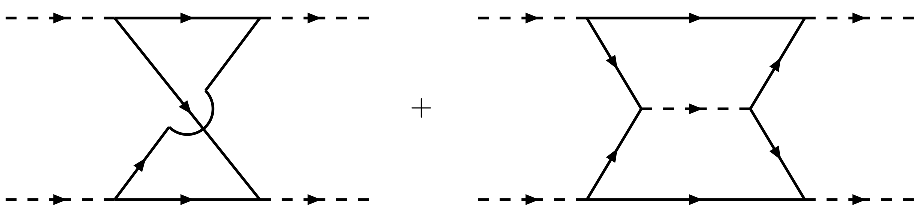
Few body physics perturbative despite large scattering length



$\mathcal{O}(\epsilon)$ $\qquad$ $\mathcal{O}(\epsilon^2)$

$$\frac{T_{ad}}{T_{aa}} = 1 - \frac{\epsilon}{6} + \ldots \simeq 0.83$$

$$a_{ad}/a = 1.11 \qquad (exact\, 1.18)$$



$\mathcal{O}(\epsilon^2)$ $\qquad$ $\mathcal{O}(\epsilon^3)$

$$\frac{T_{dd}}{T_{aa}} = \frac{1}{2}\epsilon - 0.172\epsilon^2 + \ldots \simeq 0.33$$

$$a_{dd}/a = 0.66 \qquad (exact\, 0.60)$$

$nD$ scattering (no range terms)

$$a_{nD}(s=3/2) \simeq 4.78\, fm \qquad a_{nD}^{\exp} = 6.35 \pm 0.02\, fm$$

Also: $a_{DD}(s=2) \simeq 3.15$ fm

Rupak (2006)

# Outlook

Several systemtic approaches available

None of them is perfect, emphasize different aspects

Can be combined in interesting ways

Real nuclear matter

More perturbative. Problem becomes easier?

$1/a$, range corrections have been studied

Explicit pions, three body clusters, ...

Real nuclei

Density functionals (LDA, KS, ...)