Anisotropic fluid dynamics

Thomas Schaefer, North Carolina State University

Outline

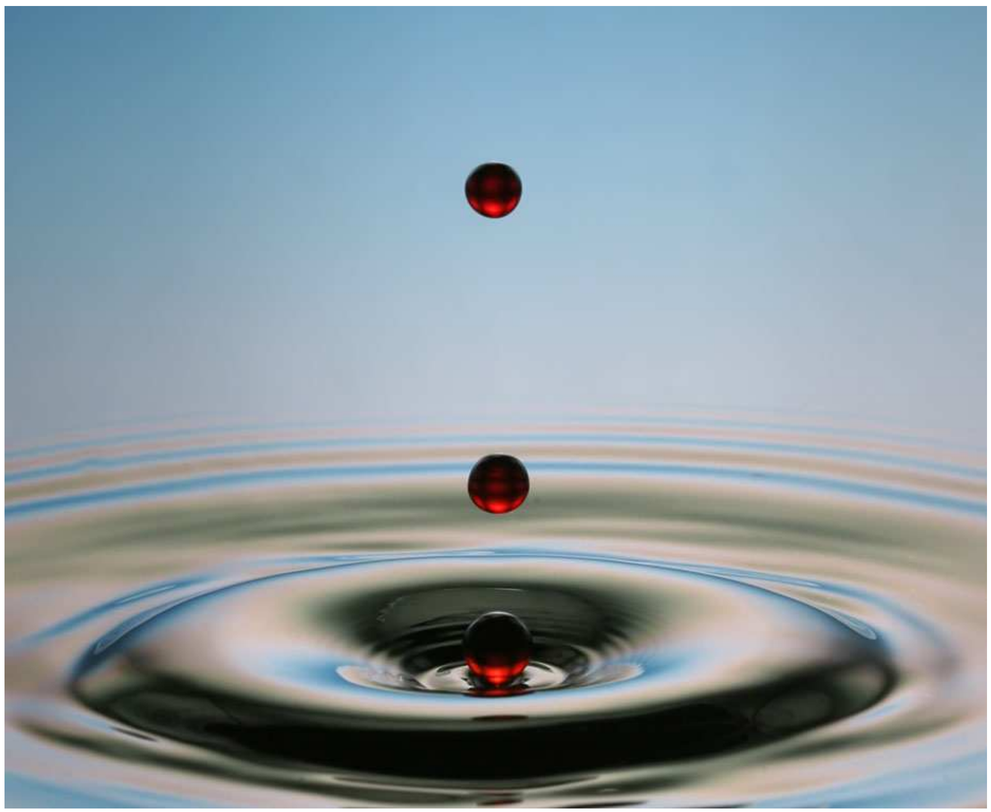
We wish to extract the properties of nearly perfect (low viscosity) fluids from experiments with trapped gases, colliding nuclei, etc.

The natural tool for these studies is the Navier-Stokes equation, which describes the macroscopic motion of a fluid in which viscous corrections are small.

The problem is that this is not the case flor the entire system. There is a dilute corona in which fluid dynamics is not applicable.

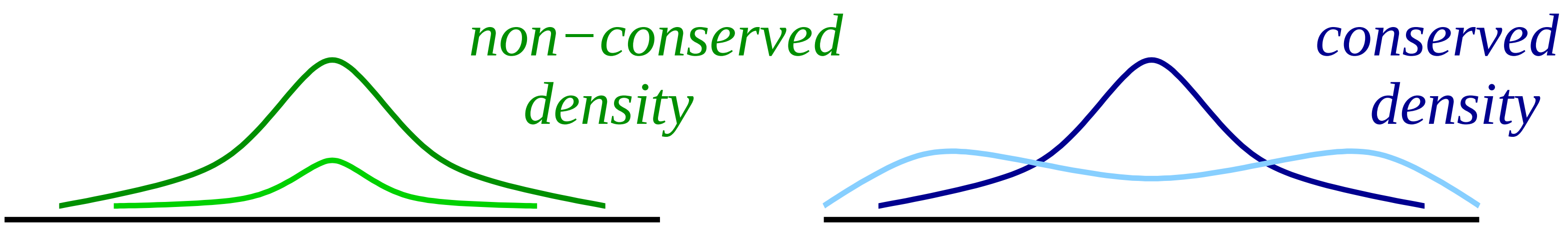
Hydroynamics

Hydrodynamics (undergraduate version): Newton's law for continuous, deformable media.



Fluids: Gases, liquids, plasmas, . . .

Hydrodynamics (postmodern): Effective theory of non-equilibrium long-wavelength, low-frequency dynamics of any many-body system.



$\tau \sim \tau_{micro}$

$\tau \sim \lambda$

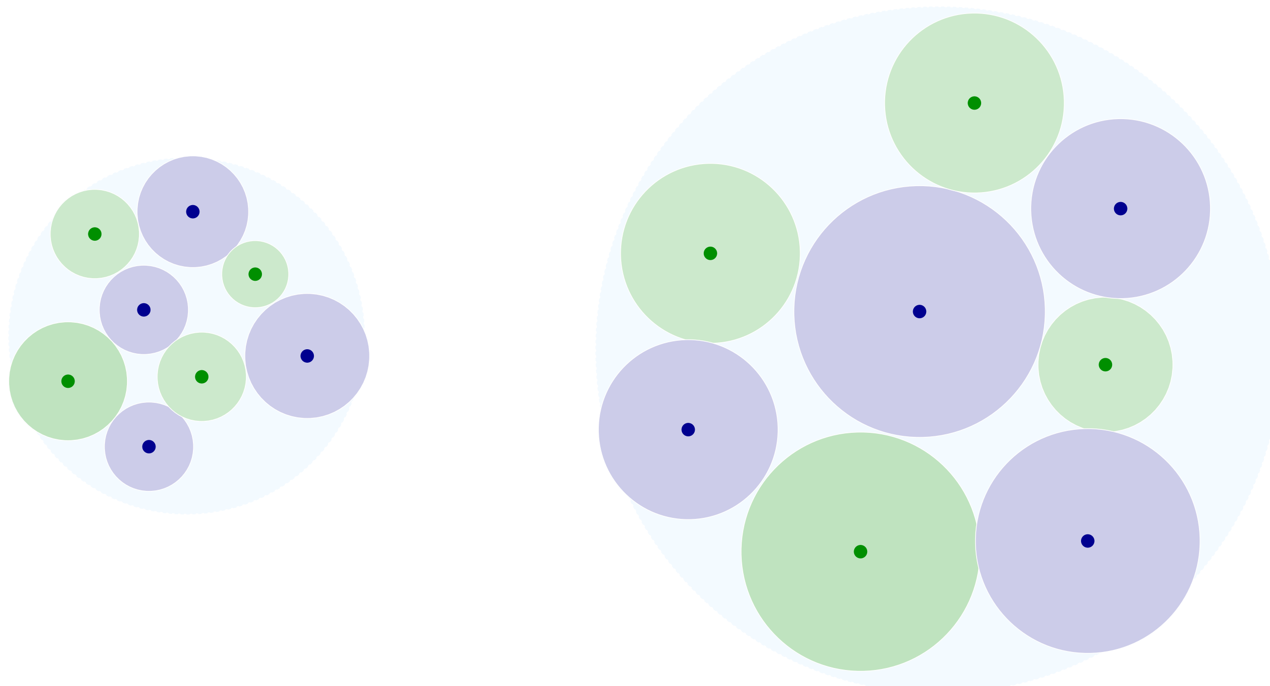
$\tau \gg \tau_{micro}$: Dynamics of conserved charges.

Water: $(\rho, \epsilon, \vec{\pi})$

Not your grandfathers fluid

Consider a many body system (unitary Fermi gas) with $\sigma \sim 1/k^2$

Can be made using Feshbach resonances in dilute atomic gases.



Systems remains hydrodynamic despite expansion

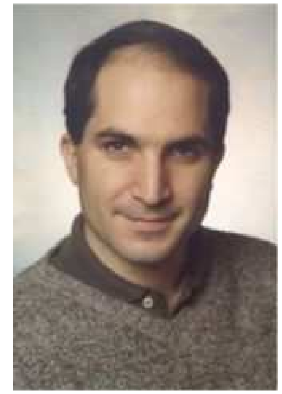
Effective theories for fluids (Unitary Fermi Gas, $T > T_F$)

$$\mathcal{L} = \psi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2M} \right) \psi - \frac{C_0}{2} (\psi^\dagger \psi)^2$$

$$\frac{\partial f_p}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f_p = C[f_p] \qquad \omega < T$$

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0 \quad \omega < T \frac{s}{\eta}$$

Effective theories (Strong coupling)



$$\mathcal{L} = \bar{\lambda}(i\sigma \cdot D)\lambda - \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} + \ldots \Leftrightarrow S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \mathcal{R} + \ldots$$

$$SO(d+2,2) \rightarrow Schr^2_d \qquad AdS_{d+3} \rightarrow Schr^2_d$$

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0 \quad (\omega < T)$$

Gradient expansion (simple non-relativistic fluid)

Simple fluid: Conservation laws for mass, energy, momentum

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \vec{j}^{\,\rho} = 0$$

$$\frac{\partial \epsilon}{\partial t} + \vec{\nabla} \vec{j}^{\,\epsilon} = 0$$

$$\frac{\partial \pi_i}{\partial t} + \frac{\partial}{\partial x_j} \Pi_{ij} = 0$$

Ward identity: mass current = momentum density

$$\vec{j}^{\,\rho} \equiv \rho \vec{v} = \vec{\pi}$$

Constitutive relations: Gradient expansion for currents

Energy momentum tensor

$$\Pi_{ij} = P\delta_{ij} + \rho v_i v_j + \eta \left(\partial_i v_j + \partial_j v_i - \frac{2}{3} \delta_{ij} \partial_k v_k \right) + O(\partial^2)$$

Gradient expansion, Kubo formula

Consider background metric $g_{ij}(t,x) = \delta_{ij} + h_{ij}(t,x)$. Linear response

$$\delta\Pi^{xy} = -\frac{1}{2} G_R^{xyxy} h_{xy}$$

Harmonic perturbation $h_{xy} = h_0 e^{-i\omega t}$

$$G_R^{xyxy} = P - i\eta\omega + \ldots$$

Kubo relation: $$\eta = -\lim_{\omega\to 0}\left[\frac{1}{\omega}\mathrm{Im} G_R^{xyxy}(\omega,0)\right]$$

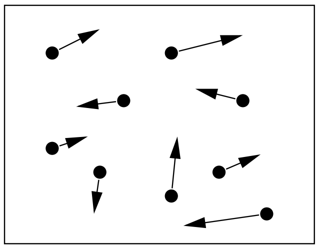
Gradient expansion: $\omega \leq \frac{P}{\eta} \simeq \frac{s}{\eta} T$.

Fluid dynamics from kinetic theory

Microscopic picture:

Quasi-particle distribution

function $f_p(x,t)$



$$\rho(x,t) = \int d\Gamma_p \, m f_p(x,t)$$

$$\pi_i(x,t) = \int d\Gamma_p \, p_i f_p(x,t)$$

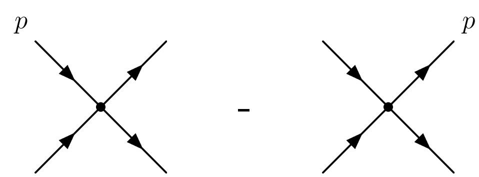
$$\Pi_{ij}(x,t) = \int d\Gamma_p \, p_i v_j f_p(x,t)$$

Boltzmann equation

$$\left(\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}_x + \vec{F} \cdot \vec{\nabla}_p \right) f_p(t,x,) = C[f_p]$$

Collision term

$$C[f_1] = \int d\Gamma_{234} (f_1 f_2 - f_3 f_4) w(12;34)$$



Fluid dynamics from kinetic theory

Conservation laws (collision term)

$$\int d\Gamma_p \, M_p \, C[f_p] = 0 \qquad\qquad M_p = \{1, p, E_p\}$$

Moments of Boltzmann equation imply fluid dynamic conservation laws

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \vec{\jmath}^{\,\rho} = 0$$

$$\frac{\partial \epsilon}{\partial t} + \vec{\nabla} \vec{\jmath}^{\,\epsilon} = 0$$

$$\frac{\partial \pi_i}{\partial t} + \frac{\partial}{\partial x_j} \Pi_{ij} = 0$$

Need constitutive equations (and equation of state)

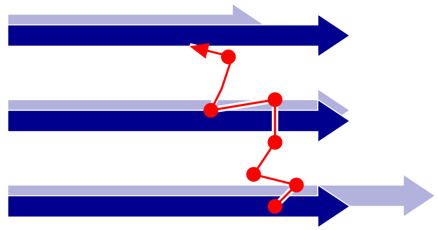
$$\vec{\jmath}^{\,\rho} = ? \qquad \vec{\jmath}^{\,\epsilon} = ? \qquad \Pi_{ij} = ?$$

Kinetic theory: Knudsen expansion

Chapman-Enskog expansion $f = f_0 + \delta f_1 + \delta f_2 + \ldots$

Gradient exp. $\delta f_n = O(\nabla^n)$

$\equiv$ Knudsen exp. $\delta f_n = O(Kn^n)$



Zeroth order result: $f_0 = \exp(-\beta(E_p - \vec{p}\cdot\vec{u} - \mu)) \quad \beta = 1/T$

$$\vec{\jmath}^{\,\rho} = \vec{\pi} = \rho\vec{u}$$

$$\vec{\jmath}^{\,\epsilon} = (\mathcal{E} + P)\vec{u} \qquad P = \frac{2}{3}\mathcal{E}$$

$$\Pi_{ij} = \rho u_i u_j + P\delta_{ij}$$

First order result: $\delta f_1 = -f_0 \frac{\eta}{PT} v^i v^j \sigma_{ij} + \ldots$

$$\delta^{(1)}\Pi_{ij} = -\eta\sigma_{ij}$$

$$\delta^{(1)}\jmath_i^{\,\epsilon} = -\eta u^j \sigma_{ij} - \kappa\nabla_i T$$

Kinetic theory: Knudsen expansion

For given $w(12;34)$ also obtain prediction for η, κ

$$\eta = \frac{15}{32\sqrt{\pi}}(mT)^{3/2} \qquad \kappa = \frac{225}{128\sqrt{\pi}m}(mT)^{3/2}$$

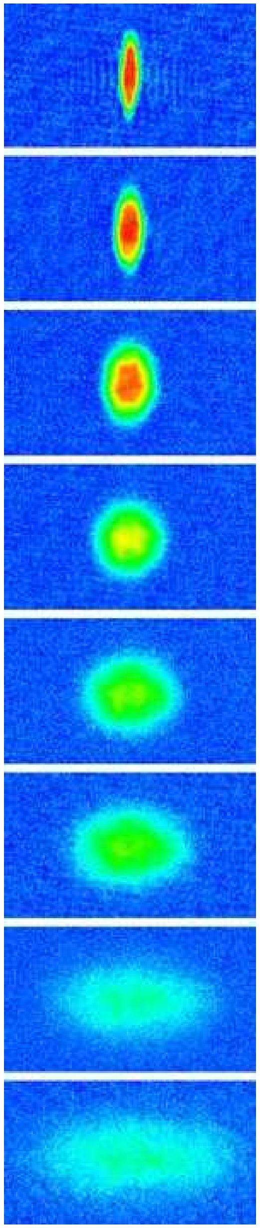
Second order result

Chao, Schaefer (2012), Schaefer (2014)

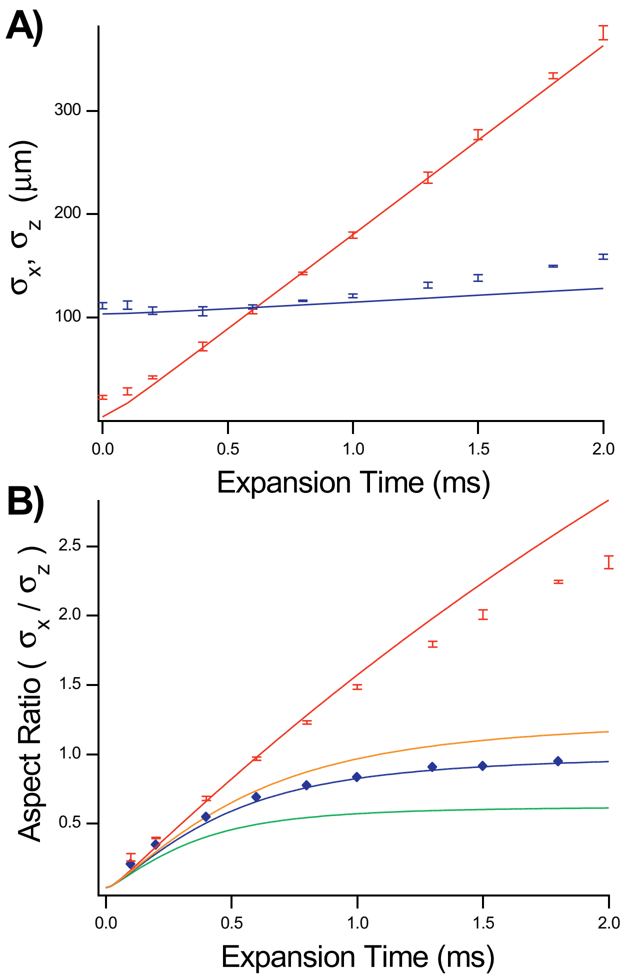
$$\begin{aligned} \delta^{(2)}\Pi^{ij} &= \frac{\eta^2}{P}\left[{}^{\langle}D\sigma^{ij\rangle} + \frac{2}{3}\sigma^{ij}(\nabla\cdot v)\right] \\ &\quad + \frac{\eta^2}{P}\left[\frac{15}{14}\sigma^{\langle i}{}_k\sigma^{j\rangle k} - \sigma^{\langle i}{}_k\Omega^{j\rangle k}\right] + O(\kappa\eta\nabla^i\nabla^j T) \end{aligned}$$

relaxation time $\quad \tau_\pi = \eta/P$

Experiments: Elliptic flow



O'Hara et al. (2002)



Hydrodynamic

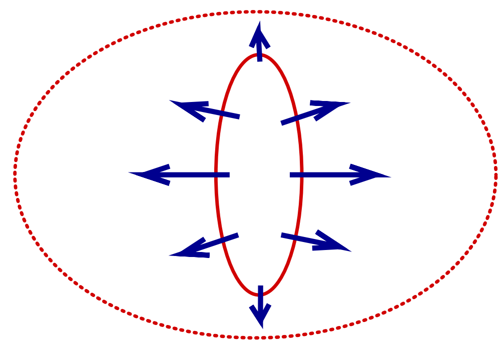
expansion converts

coordinate space

anisotropy

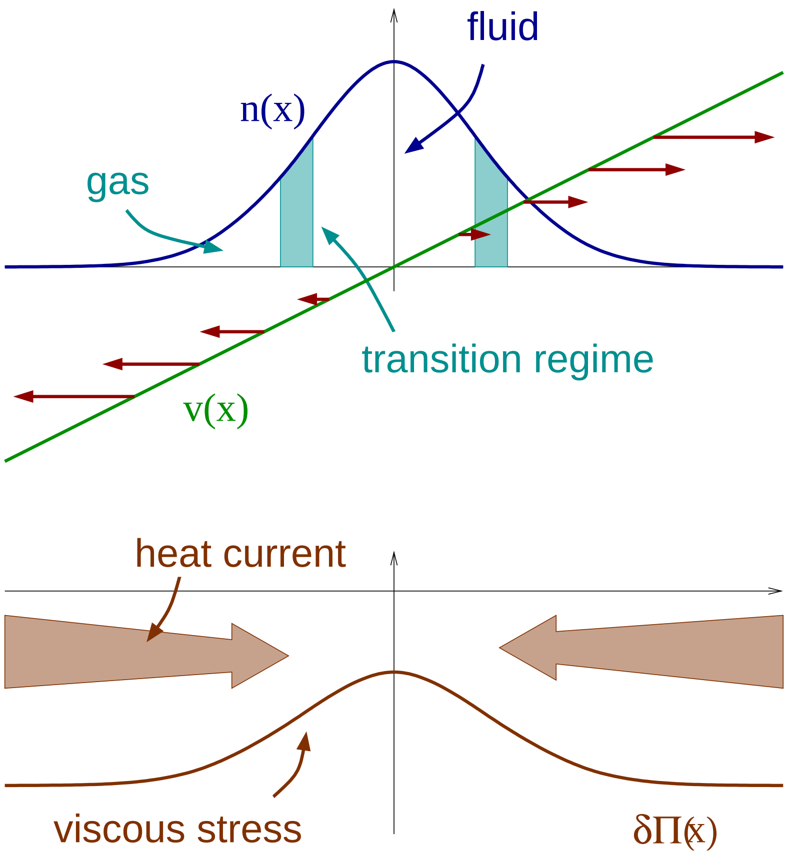
to momentum space

anisotropy



Determination of $\eta(n,T)$

Measurement of $A_R(t, E_0)$ determines $\eta(n,T)$. But:



The whole cloud is not a fluid.

Can we ignore this issue?

No. Hubble flow & low density viscosity $\eta \sim T^{3/2}$ lead to paradoxical fluid dynamics.

$$\dot{Q} = \int \sigma \cdot \delta\Pi = \infty$$

Possible Solutions

Combine hydrodynamics & Boltzmann equation. Not straightforward.

Hydrodynamics + non-hydro degrees of freedom ($\mathcal{E}_a$; $a = x, y, z$)

$$\frac{\partial \mathcal{E}_a}{\partial t} + \vec{\nabla} \cdot \vec{\jmath}^{\,\epsilon}_a = -\frac{\Delta P_a}{2\tau} \qquad \Delta P_a = P_a - P$$

$$\frac{\partial \mathcal{E}}{\partial t} + \vec{\nabla} \cdot \vec{\jmath}^{\,\epsilon} = 0 \qquad \mathcal{E} = \sum_a \mathcal{E}_a$$

τ small: Fast relaxation to Navier-Stokes with $\tau = \eta/P$

τ large: Additional conservation laws. Ballistic expansion.

Anisotropic hydro from kinetic theory

Consider modified expansion

$$f = f_A + \delta f_1' + \delta f_2' + \ldots$$

Anisotropic distribution function

$$f_A = \exp\left(-\frac{(p_a - mu_a)^2}{2mT_a} - \frac{\mu}{\bar{T}}\right) \qquad \bar{T} = (\prod T_a)^{1/3}$$

- f_A is an exact solution of the Boltzmann equation in the ballistic limit.
- The viscous stresses and dissipative corrections to the energy current have the same form as in the Chapman-Enskog theory.

Anisotropic Hydrodynamics from kinetic theory

Moments of the Boltzmann equation with $M_p = \{1, \vec{p}, E_P\}$.

Navier-Stokes with $\delta\Pi_{aa} = \Delta P_a$

Moments of the Boltzmann equation with p_a^2

$$\frac{\partial \mathcal{E}_a}{\partial t} + \vec{\nabla} \cdot \vec{\jmath}^{\,\epsilon}_a = -\frac{\Delta P_a}{2\tau} \qquad\qquad \Delta P_a = P_a - P$$

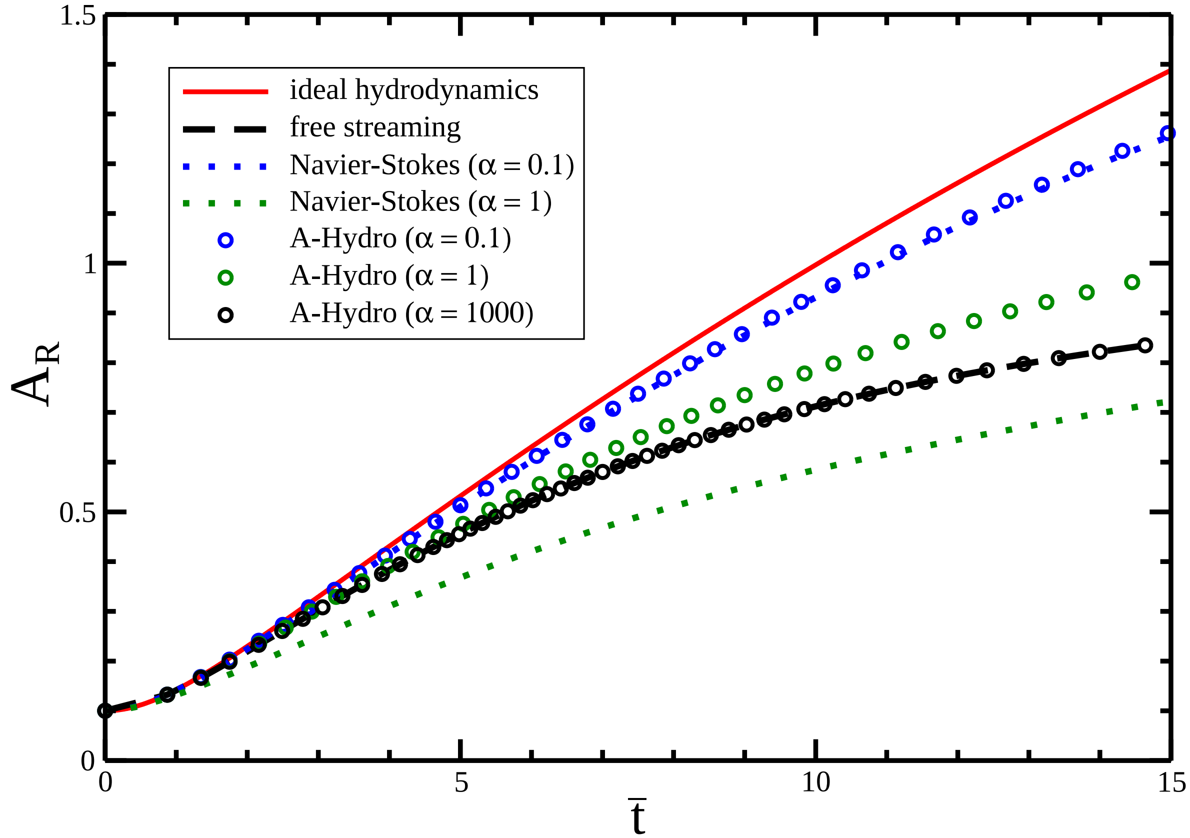
with $P_a = 2\mathcal{E}_a$ $(P = \frac{2}{3}\mathcal{E})$ and $\tau = \eta/P$.

Solve fluid dynamic equations for small τ

$$\delta\Pi_{aa} = \Delta P_a = -\eta\sigma_{aa}$$

Ballistic limit $\tau \to \infty$: Conservation law for $\mathcal{E}_a$.

Anisotropic Hydrodynamics: Aspect ratio



Consider $\eta = \alpha n$ and $\alpha \in [0, \infty)$

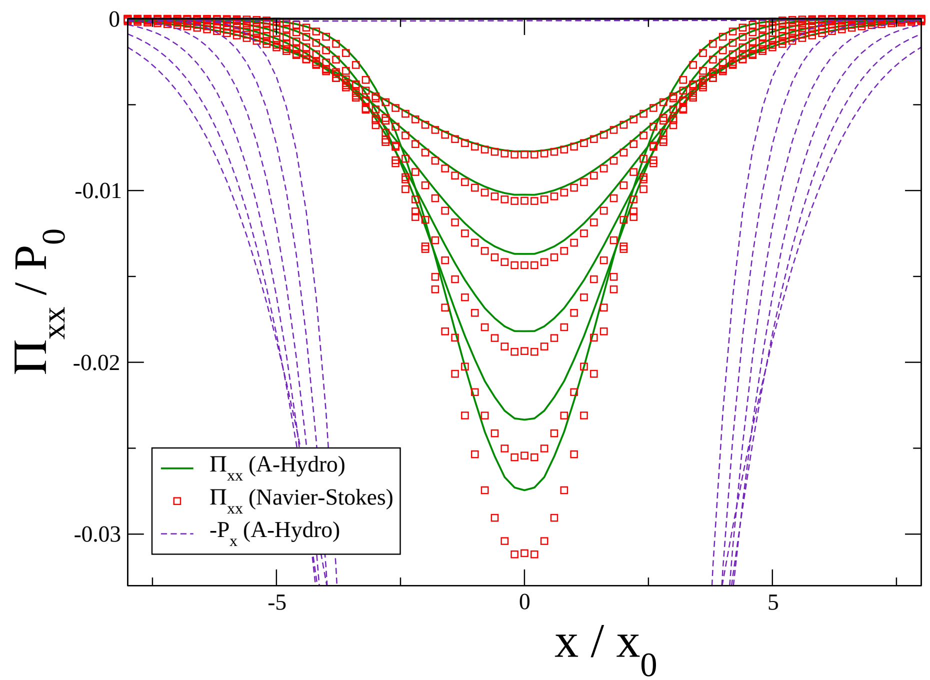
Navier-Stokes: Ideal hydro $\rightarrow$ very viscous hydro.

A-hydro: Ideal hydro $\rightarrow$ ballistic expansion.

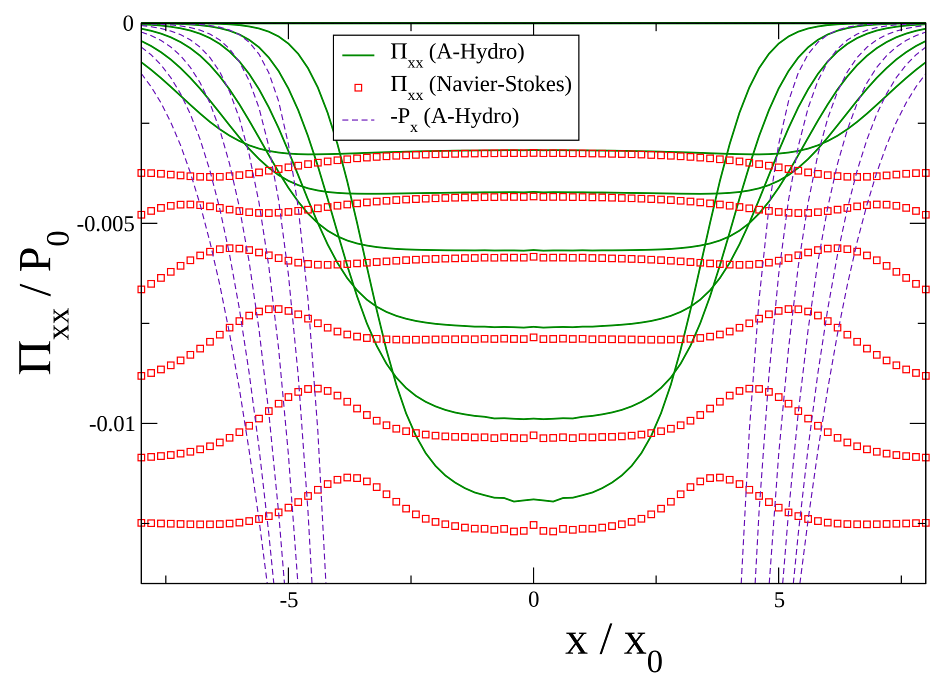
AVH1 hydro code, M. Bluhm & T.S. (2015)

Anisotropic Hydrodynamics: Evolution of $\delta\Pi_{aa}$

$\eta = \alpha_n n$



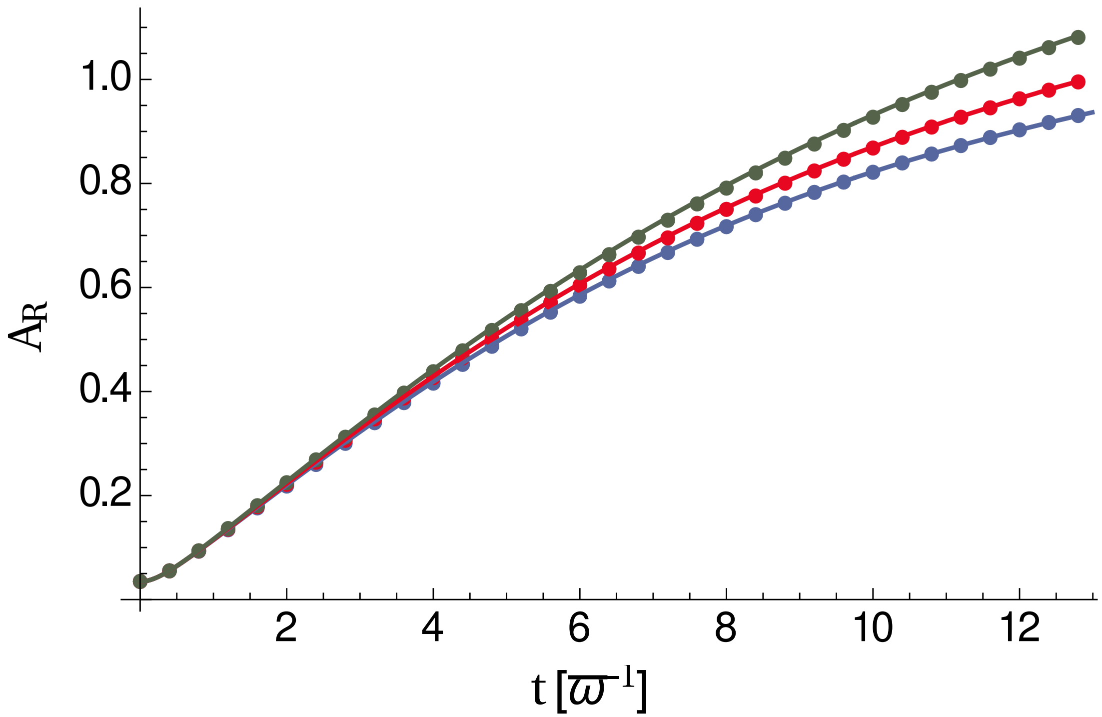
$\eta = \alpha_T (mT)^{3/2}$



$\delta\Pi_{xx}$(Navier-Stokes) $\delta\Pi_{xx}$(A-Hydro)

AVH1 hydro code, M. Bluhm & T.S. (2015)

Anisotropic Hydrodynamics: Comparison with Boltzmann



$T/T_F =$

$0.79, 1.11, 1.54$

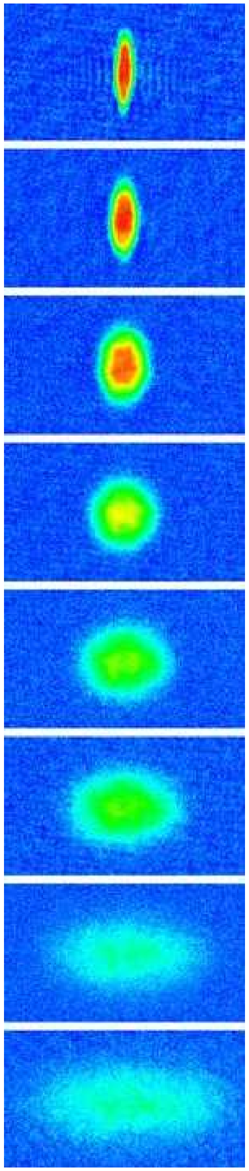
Dots: Two-body Boltzmann equation with full collision kernel

Lines: Anisotropic hydro with η fixed by Chapman-Enskog

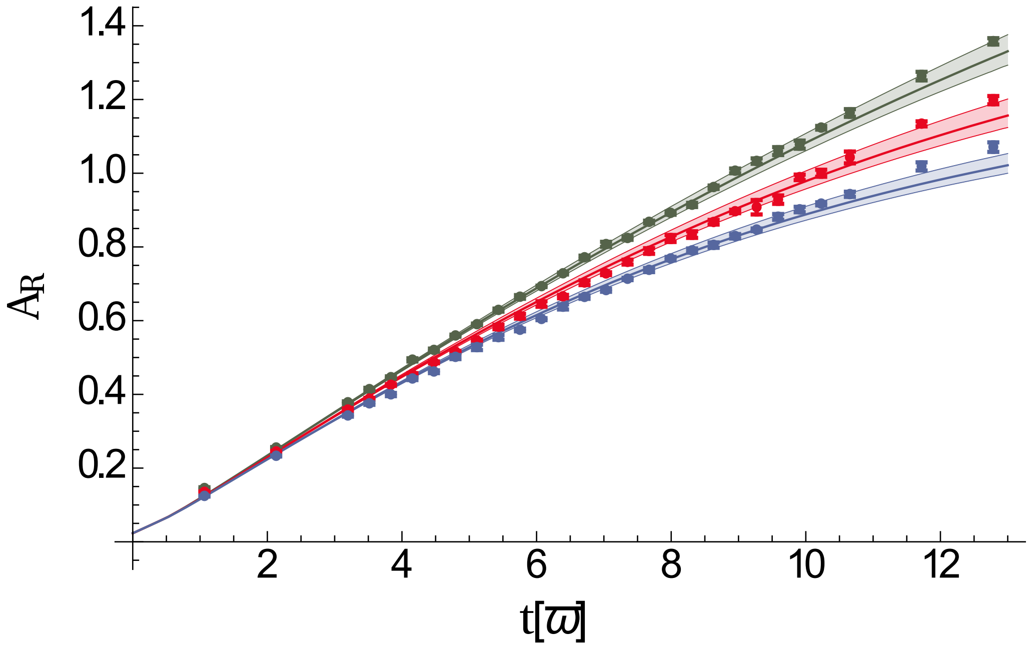
High temperature (dilute) limit: Perfect agreement!

AVH1 hydro code, M. Bluhm & T.S. (2015)

Elliptic flow: High T limit



Quantum viscosity $\eta = \eta_0 \frac{(mT)^{3/2}}{\hbar^2}$



Cao et al., Science (2010)

Bluhm et al., PRL (2016)

$T/T_F =$

$0.79, 1.11, 1.54$

fit: $\eta_0 = 0.28 \pm 0.02$

theory: $\eta_0 = \frac{15}{32\sqrt{\pi}} = 0.269$

Outlook

Fluid dynamics as an E(F)T?

Unfold temperature, density dependence of η/s.

Applications to other transport problems: Diffusion, superfluid hydrodynamics.