Simulating stochastic fluids

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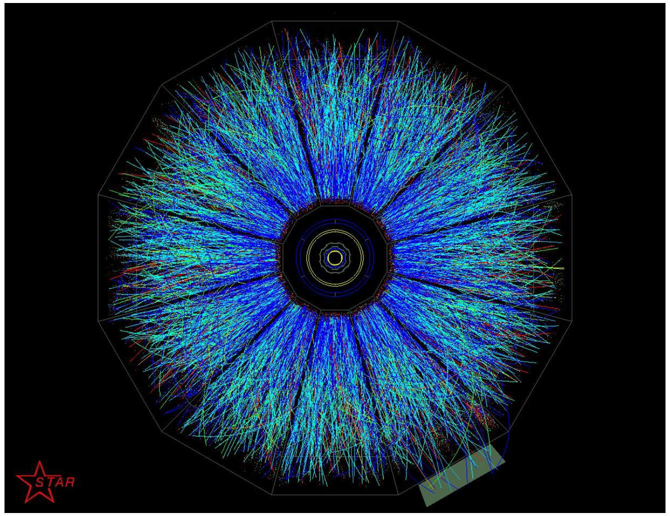
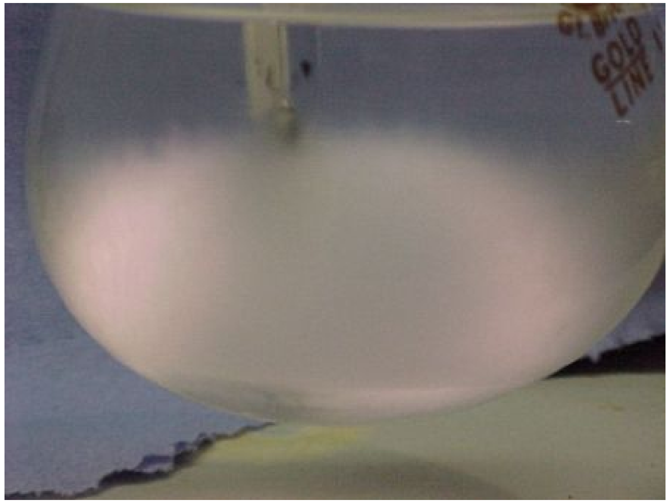
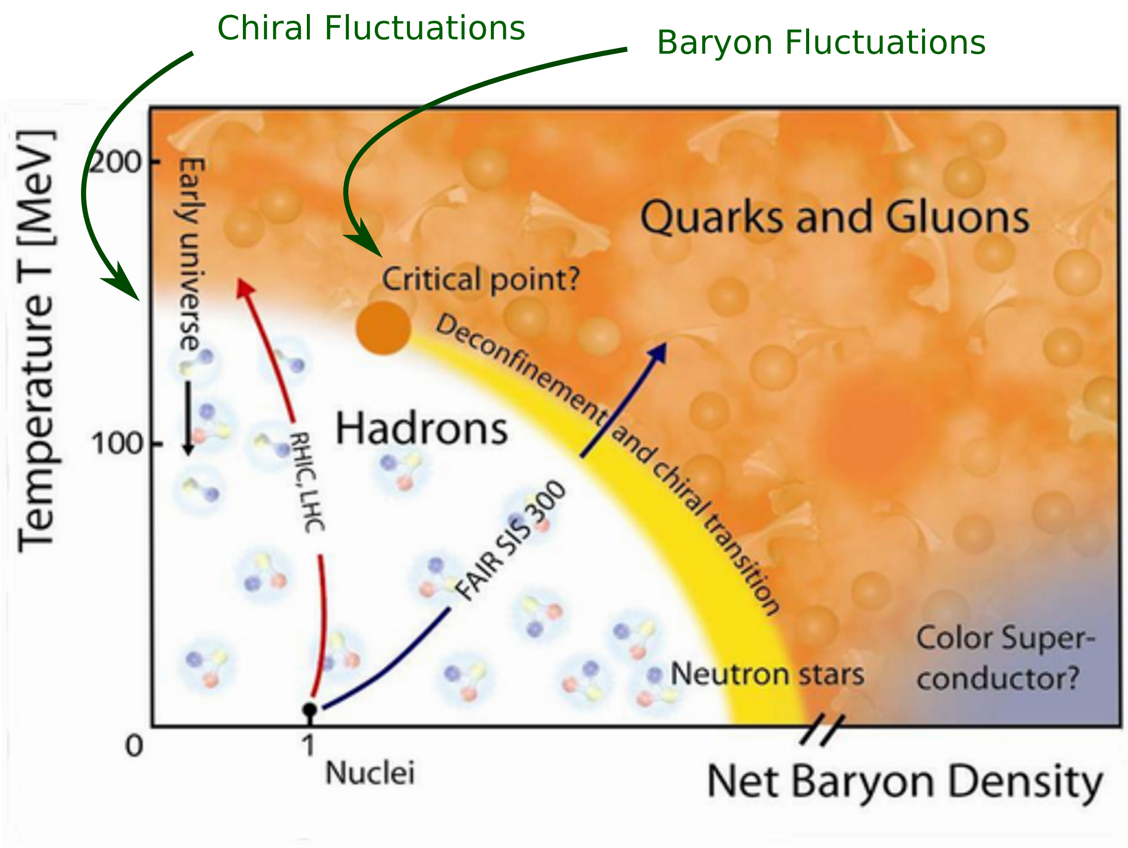




With C. Chattopadhyay, J. Ott, V. Skokov

References: 2403.10608 (PRL 2024), 2411.15994 (PRD 2025).

Motivation



Can we locate the chiral phase transition, or the endpoint of a first-order QGP-hadron gas transition?

Critical Dynamics

What is the dynamical theory near the critical point?

The basic logic of fluid dynamics still applies. Important modifications:

- Critical equation of state.
- Stochastic fluxes, fluctuation-dissipation relations.
- Possible Goldstone modes (chiral field in QCD)

Classified by Hohenberg & Halperin in 1977 (model A, B, . . .)

Chiral phase transition: Model G (Rajagopal & Wilczek, 1993)

Possible critical endpoint: Model H (Son & Stephanov, 2004)

Digression: Diffusion

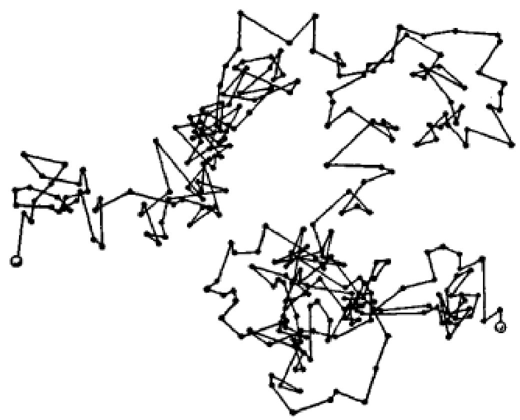
Consider a Brownian particle

$$\dot{p}(t) = -\gamma_D p(t) + \zeta(t) \qquad \langle \zeta(t)\zeta(t')\rangle = \kappa\delta(t-t')$$

drag (dissipation) white noise (fluctuations)

For the particle to eventually thermalize

$$\langle p^2 \rangle = 2mT$$



drag and noise must be related

$$\kappa = \frac{mT}{\gamma_D}$$

Einstein (Fluctuation-Dissipation)

Hydrodynamic equation for critical mode

Equation of motion for critical mode ϕ ("model H")

$$\frac{\partial\phi}{\partial t} = \kappa\,\nabla^2\frac{\delta\mathcal{F}}{\delta\phi} - g\left(\vec{\nabla}\phi\right)\cdot\frac{\delta\mathcal{F}}{\delta\vec{\pi}^T} + \zeta \qquad (g=1)$$

Diffusion   Advection   Noise

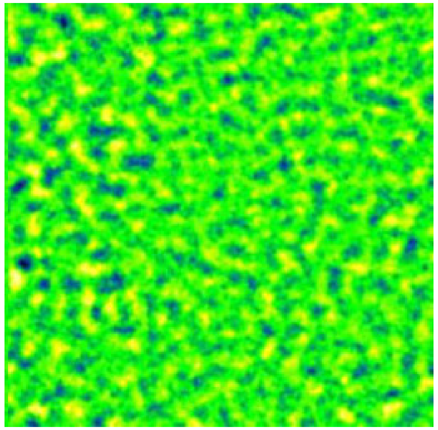
Equation of motion for momentum density π

$$\frac{\partial\vec{\pi}^T}{\partial t} = \eta\,\nabla^2\frac{\delta\mathcal{F}}{\delta\vec{\pi}^T} + g\left(\vec{\nabla}\phi\right)\cdot\frac{\delta\mathcal{F}}{\delta\phi} - g\left(\frac{\delta\mathcal{F}}{\delta\vec{\pi}^T}\cdot\vec{\nabla}\right)\vec{\pi}^T + \vec{\xi}$$

Free energy functional: Order parameter ϕ, momentum density $\vec{\pi} = w\vec{v}$

$$\mathcal{F} = \int d^3x\,\left[\frac{1}{2w}\vec{\pi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{m^2}{2}\phi^2 + \lambda\phi^4\right] \qquad D = m^2\kappa$$

Fluctuation-Dissipation relation



$$\langle \zeta(x,t)\zeta(x',t')\rangle = -2\kappa T \nabla^2 \delta(x-x')\delta(t-t')$$

$$\langle \xi_i(x,t)\xi_j(x',t')\rangle = -2\eta T \nabla^2 P^T_{ij}\delta(x-x')\delta(t-t')$$

ensures $P[\phi,\vec{\pi}] \sim \exp(-\mathcal{F}[\phi,\vec{\pi}]/T)$

Mode couplings governed by Poisson brackets: $\psi^a = (\phi, \vec{\pi})$

$$\partial_t \psi^a = \{\mathcal{H}, \psi^a\} = -\int d^3x \, \{\psi^a, \psi^b\} \, \frac{\delta \mathcal{H}}{\delta \psi^b} = -\int d^3x \, Q^{ab} \psi^b$$

Provides mutual coupling between ϕ and $\vec{\pi}$

$$\frac{\partial \phi}{\partial t} = -\left(\vec{\nabla}\phi\right) \cdot \frac{\delta \mathcal{F}}{\delta \vec{\pi}^T} \qquad \frac{\partial \vec{\pi}^T}{\partial t} = +\left(\vec{\nabla}\phi\right) \cdot \frac{\delta \mathcal{F}}{\delta \phi}$$

as well as self coupling (self-advection) of $\vec{\pi}$

$$\frac{\partial \vec{\pi}^T}{\partial t} = -\left(\frac{\delta \mathcal{F}}{\delta \vec{\pi}^T} \cdot \vec{\nabla}\right) \vec{\pi}^T$$

Note: There is a consistent truncation ("model H0") in which the self-coupling of $\vec{\pi}$ is dropped. This model is claimed to be in the same dynamical universality class as model H. Hohenberg, Halperin, RMP (1977)

And: There is a generalization ("compressible model H") in which $\vec{\pi}^L$ is retained. This theory is needed to understand the critical behavior of the bulk viscosity. M. Martinez, T.S., V.S, arXiv:1906.11306

Numerical realization

Stochastic relaxation equation ("model A")

$$\partial_t \psi = -\Gamma \frac{\delta \mathcal{F}}{\delta \psi} + \zeta \qquad \langle \zeta(x,t)\zeta(x',t')\rangle = \Gamma T \delta(x-x')\delta(t-t')$$

Naive discretization

$$\psi(t+\Delta t) = \psi(t) + (\Delta t)\left[-\Gamma \frac{\delta \mathcal{F}}{\delta \psi} + \sqrt{\frac{\Gamma T}{(\Delta t) a^3}}\theta\right] \qquad \langle \theta^2 \rangle = 1$$

Noise dominates as $\Delta t \to 0$, leads to discretization ambiguities in the equilibrium distribution.

Idea: Use Metropolis update

$$\psi(t+\Delta t) = \psi(t) + \sqrt{2\Gamma(\Delta t)}\theta \qquad p = min(1, e^{-\beta \Delta \mathcal{F}})$$

Numerical realization

Central observation

$$\begin{aligned} \langle \psi(t+\Delta t,\vec{x}) - \psi(t,\vec{x}) \rangle &= -(\Delta t)\,\Gamma\,\frac{\delta \mathcal{F}}{\delta \psi} + O\left((\Delta t)^2\right) \\ \langle [\psi(t+\Delta t,\vec{x}) - \psi(t,\vec{x})]^2 \rangle &= 2(\Delta t)\,\Gamma\,T + O\left((\Delta t)^2\right) \,. \end{aligned}$$

Metropolis realizes both diffusive and stochastic step. Also

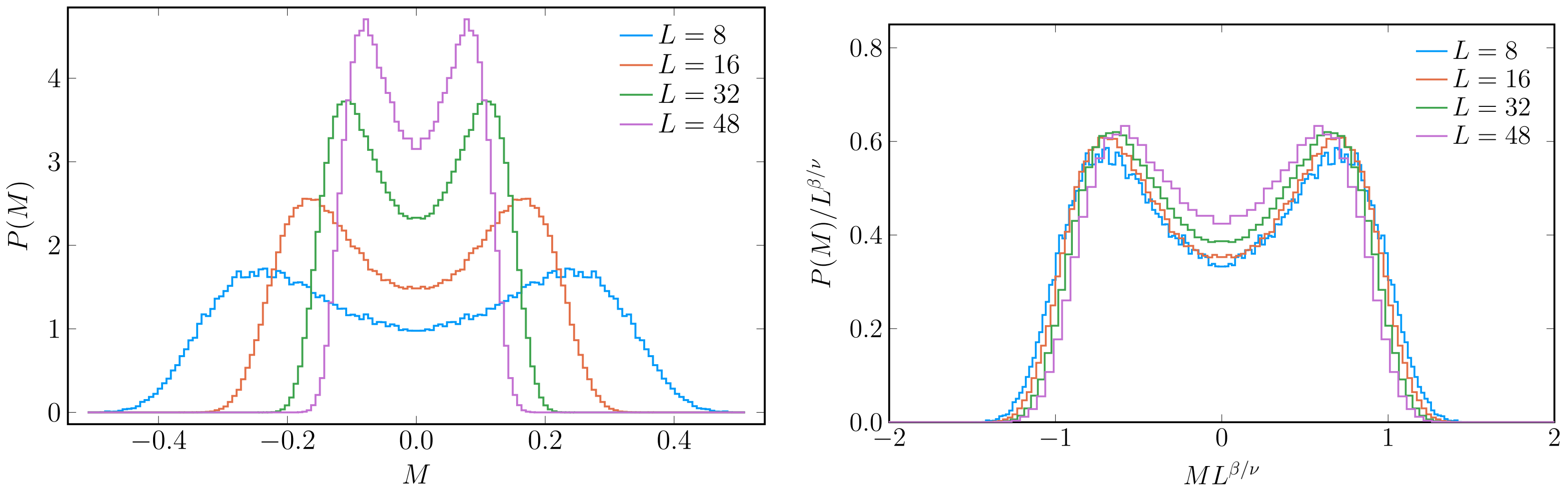
$$P[\psi] \sim \exp(-\beta \mathcal{F}[\psi])$$

Note: Still have short distance noise; need to adjust bare parameters such as Γ, m^2, λ to reproduce physical quantities.

Model A: Static Distribution

Tune m^2 to critical point $m^2 = m_c^2$ (Ising critical point)

Check finite size scaling



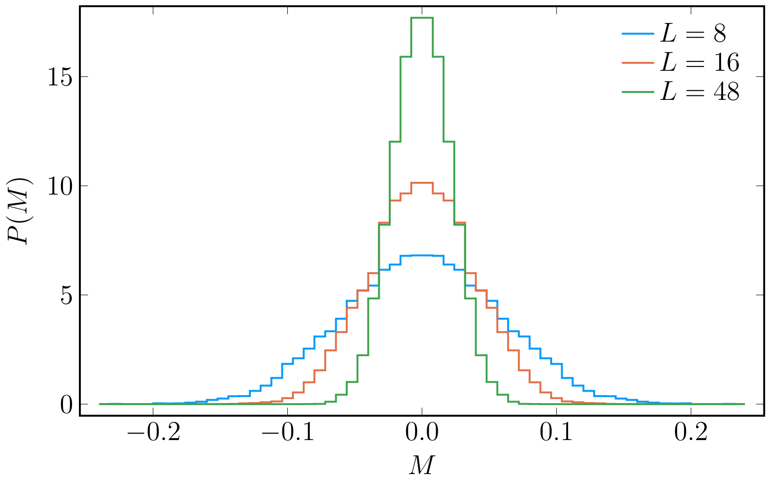
Critical exponents $\beta = 0.326$ and $\nu = 0.630$.

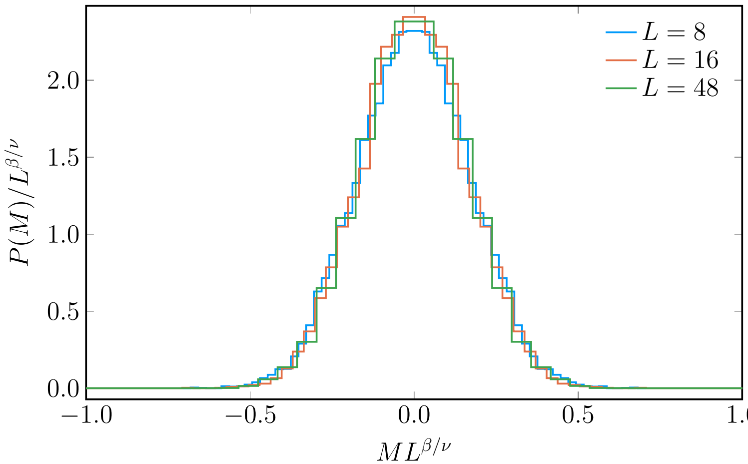
T.S., V. Skokov, arXiv:2204.02433

Model B: Static Distribution

Model B (conserving dynamics): Static distribution modified

but scaling exponent is the same





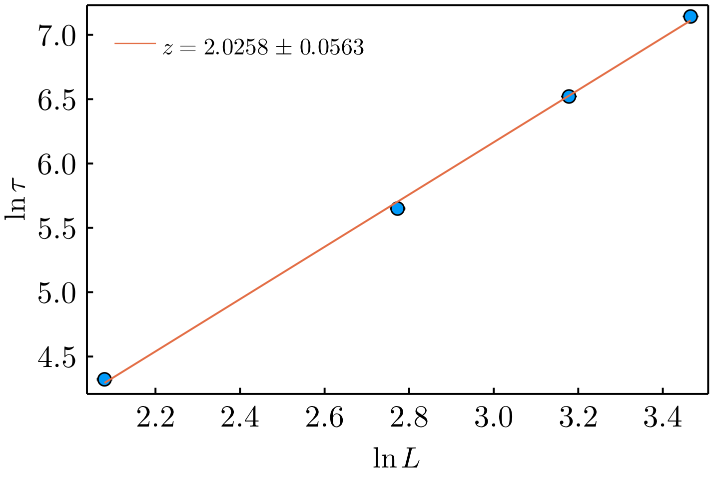
Critical exponents $\beta = 0.326$ and $\nu = 0.630$.

C. Chattopadhyay et al., arXiv:2304.07279

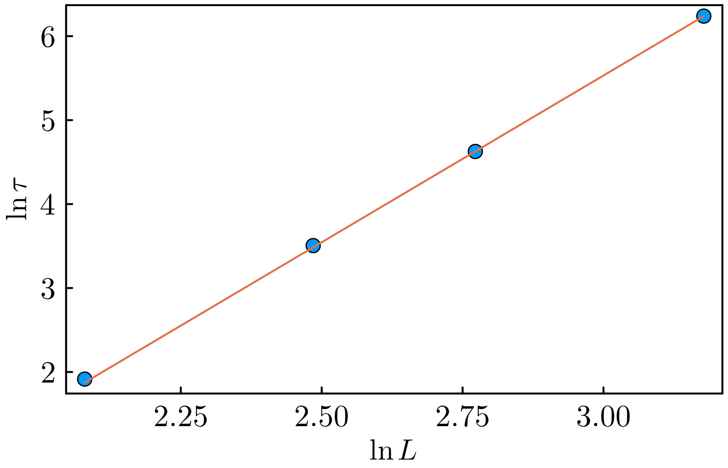
Model A/B: Dynamical Scaling

Finite size scaling of dynamic order parameter correlation function

$z_A \simeq 2.026$



$z_B \simeq 3.906$



$$G(t,k) = \int d^3x\, e^{i\vec{k}\cdot\vec{x}} \langle \psi(0,0)\psi(\vec{x},t)\rangle$$

Look for dynamic scaling $G(t,k,L) = \tilde{G}(t/L^z, kL)$

Numerical realization: Model H

Model H: Conserving update

$$\pi_\nu^{trial}(\vec{x}, t+\Delta t) = \pi_\nu(\vec{x}, t) + r_{\nu\mu}\,,$$
$$\pi_\nu^{trial}(\vec{x}+\hat{\mu}, t+\Delta t) = \pi_\nu(\vec{x}+\hat{\mu}, t) - r_{\nu\mu}\,,$$

$$r_{\nu\mu} = \sqrt{2\eta T(\Delta t)}\,\zeta_\nu\,.$$

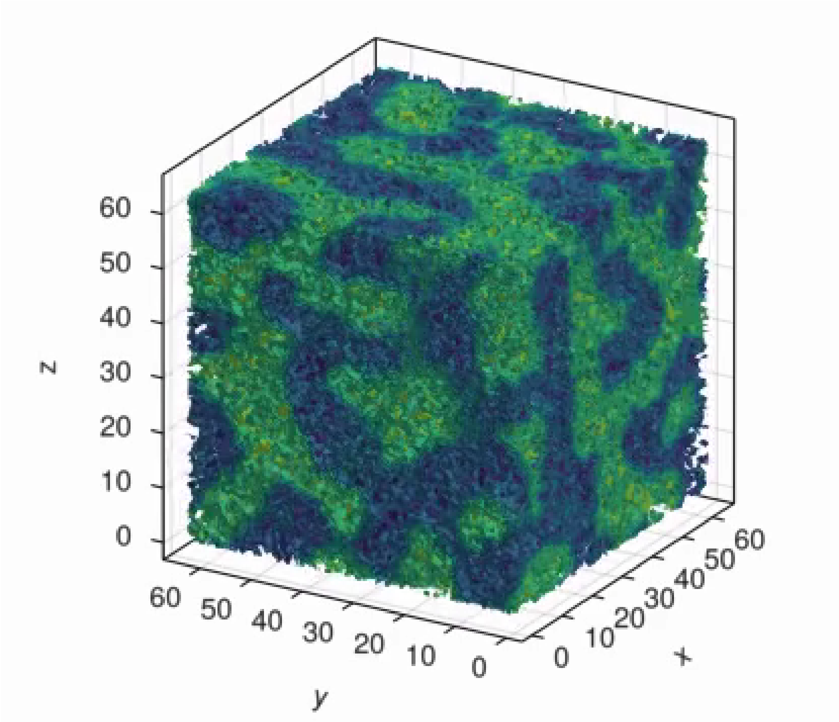
Advection (PB terms) conserves $\mathcal{H}$. On the lattice use "skew" discretized derivatives

$$\dot{\phi} = -\frac{1}{\rho}\,\pi_\mu^T\,\nabla_\mu^c \phi\,,$$
$$\dot{\pi}_\mu^T = -\left[\frac{1}{2}\nabla_\nu^c\left(\frac{1}{\rho}\pi_\nu^T \pi_\mu^T\right) + \frac{1}{2\rho}\pi_\nu^T\,\nabla_\nu^c \pi_\mu^T + \left(\nabla_\mu^c \phi\right)\left(\nabla_\nu^c \nabla_\nu^c \phi\right)\right]\,,$$

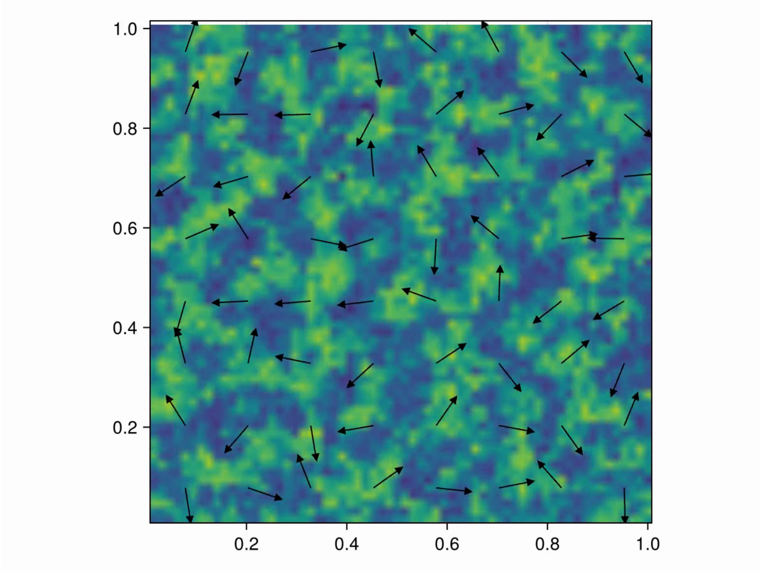
and project on π_μ^T using Fourier transforms.

Numerical results (critical Navier-Stokes)

Order parameter (3d)



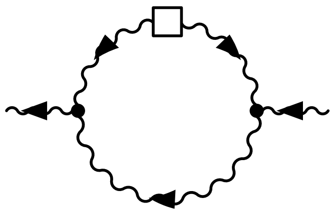
Order parameter/velocity field (2d)



Renormalized viscosity

Renormalization of η

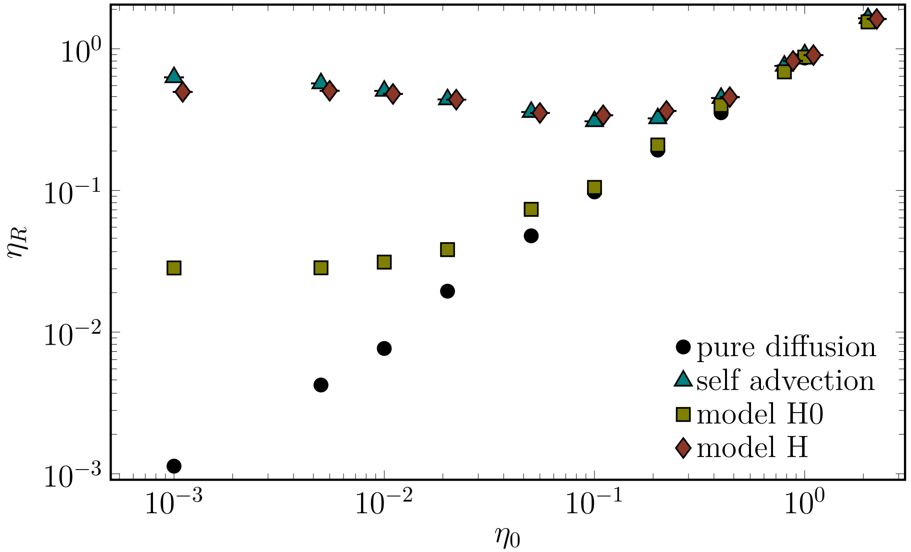
"Stickiness of shear waves"



$$\eta_R = \eta + \frac{7}{60\pi^2}\frac{\rho T \Lambda}{\eta}$$

Leads to minimum viscosity

Top: Model H

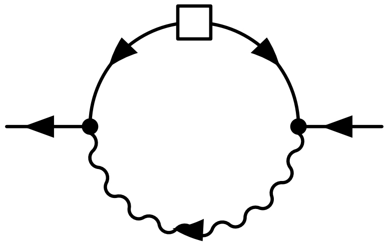


Middle: Model H0

Bottom: No advection

Relaxation Rate

Order parameter relaxation rate

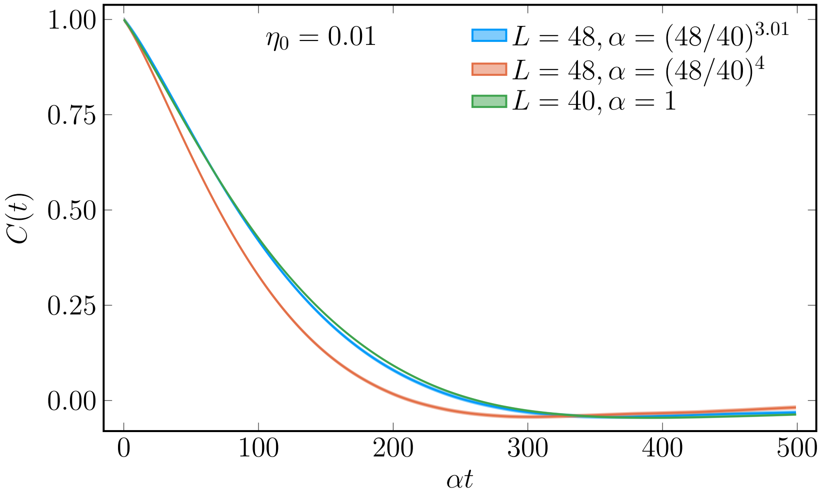


$$\Gamma_k = \frac{\Gamma}{\xi^4}(k\xi)^2\left(1+(k\xi)^2\right) + \frac{T}{6\pi\eta_R\xi^3}K(k\xi)$$

Crossover from $\tau_R \sim \xi^4$ at large η_R

to $\tau_R \sim \xi^3$ for small η_R

$$C(t) = \langle \phi_k(0)\phi_{-k}(t)\rangle$$

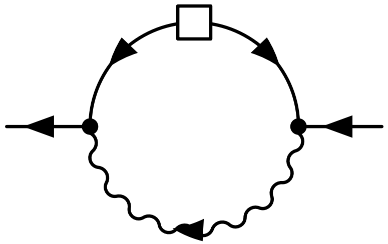


Dynamic Scaling:

$$z(\eta{=}0.01) = 3.07$$

Relaxation Rate

Order parameter relaxation rate



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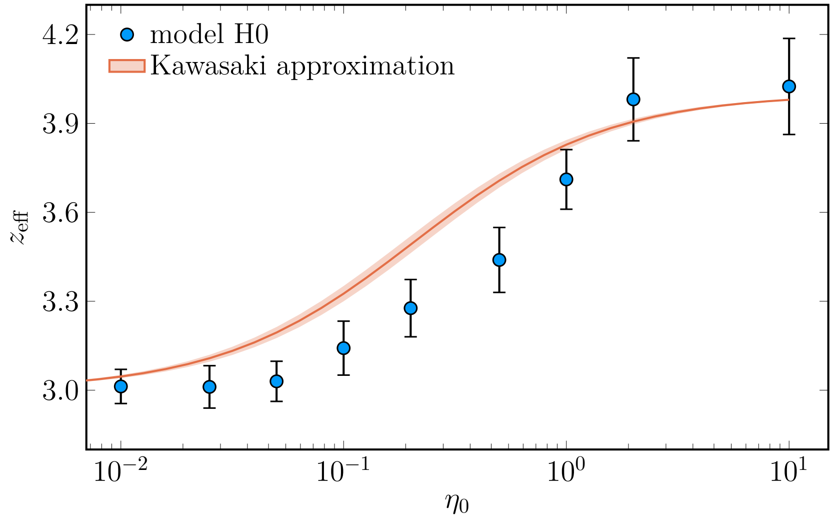
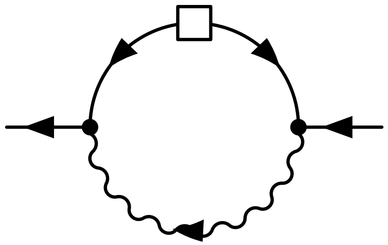


Crossover from $\tau_R \sim \xi^4$ at large η_R

to $\tau_R \sim \xi^3$ for small η_R

Universality: model H/H0

Order parameter relaxation rate



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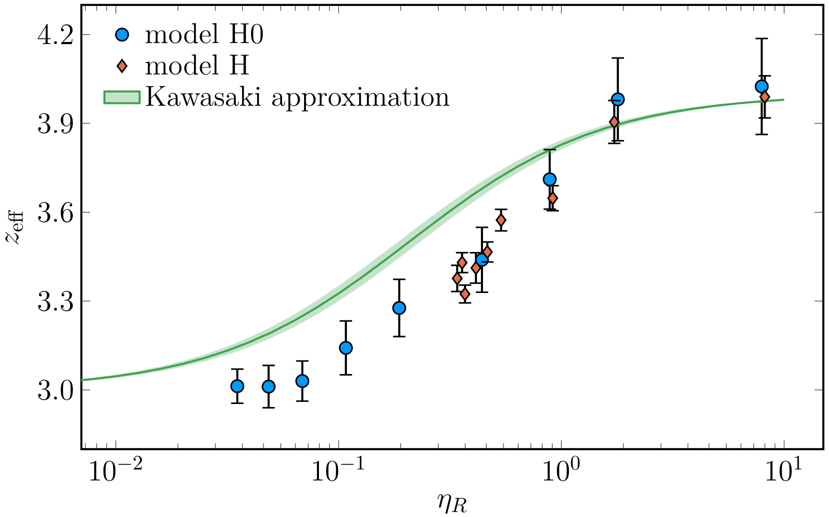


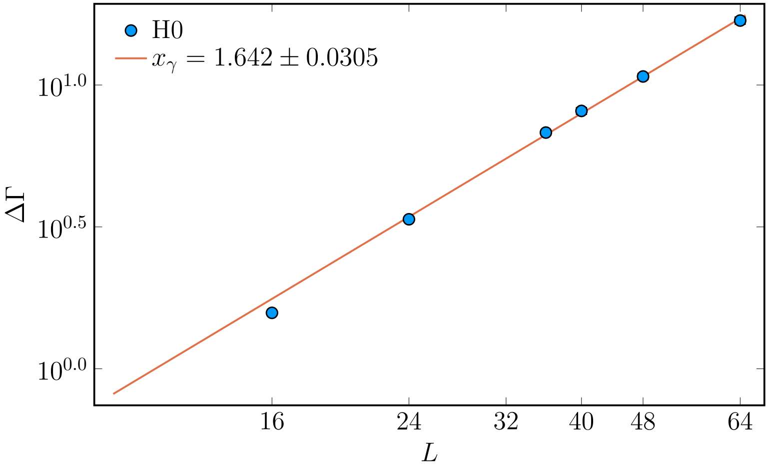
Crossover from $\tau_R \sim \xi^4$ at large η_R

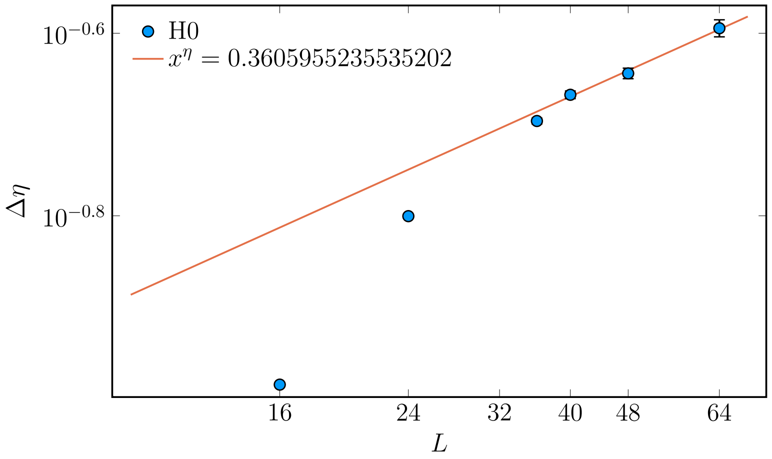
to $\tau_R \sim \xi^3$ for small η_R

Critical behavior of transport coefficients (2d, prelim)

$$\kappa = \frac{1}{3VT} \int dt\, d^3x_{1,2} \,\langle \vec{\jmath}(0,x_1)\vec{\jmath}(t,x_2)\rangle$$

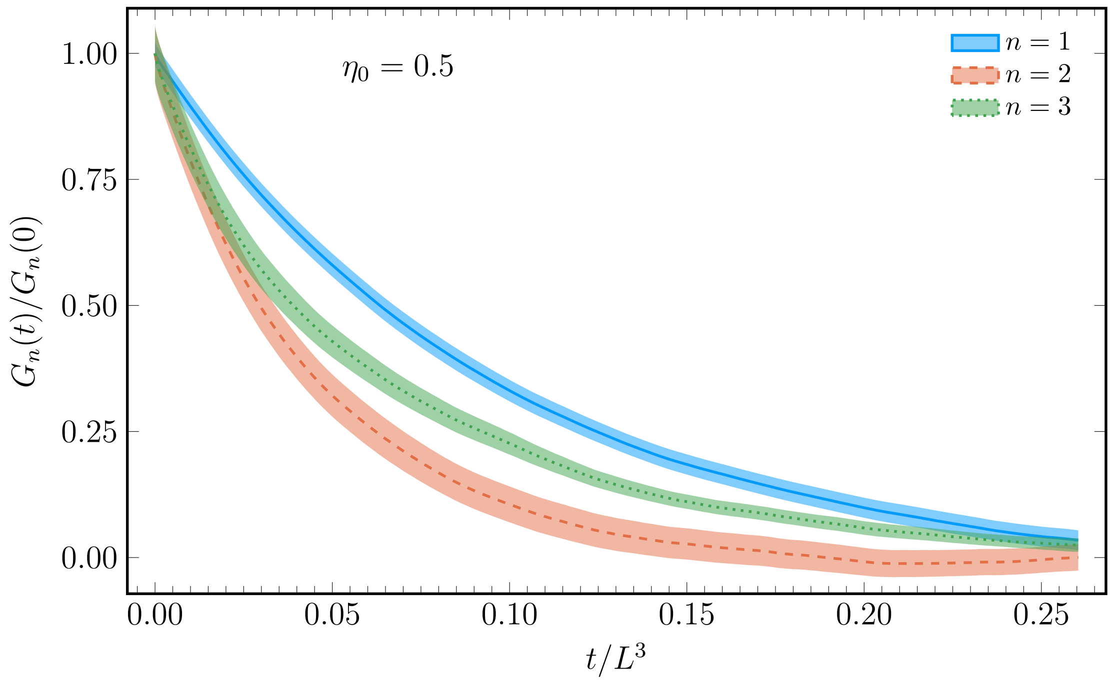
$$\eta = \frac{1}{VT} \int dt\, d^3x_{1,2} \,\langle \Pi_{xy}(0,x_1)\Pi_{xy}(t,x_2)\rangle$$





Consistent with $z = 4 - x_\kappa - \eta^*$ ($\eta^* = 0.25$ correlation function exponent)

Evolution of higher moments



$$G_n(t) = \langle M^n(t) M^n(0) \rangle \,, \qquad M(t) = \int_V d^3x \, \phi(\vec{x}, t)$$

Relaxation time $\tau_n = \tau_n^{(0)} L^z$ with exponent z independent of n

But: $\tau_n^{(0)}$ depends (non-trivially) on n

Summary and Outlook

Numerical simulation of stochastic fluid dynamics, observed renormalization of shear viscosity and dynamical scaling. Obtained $z \simeq 3.07$, in good agreement with the ϵ expansion.

Outlook: Extend the present framework to full (relativistic) fluid dynamics, or couple the simulations to fixed relativistic background flow (no backreaction). Density frame provides a promising approach.