Instantons and the mass of the η' revisited

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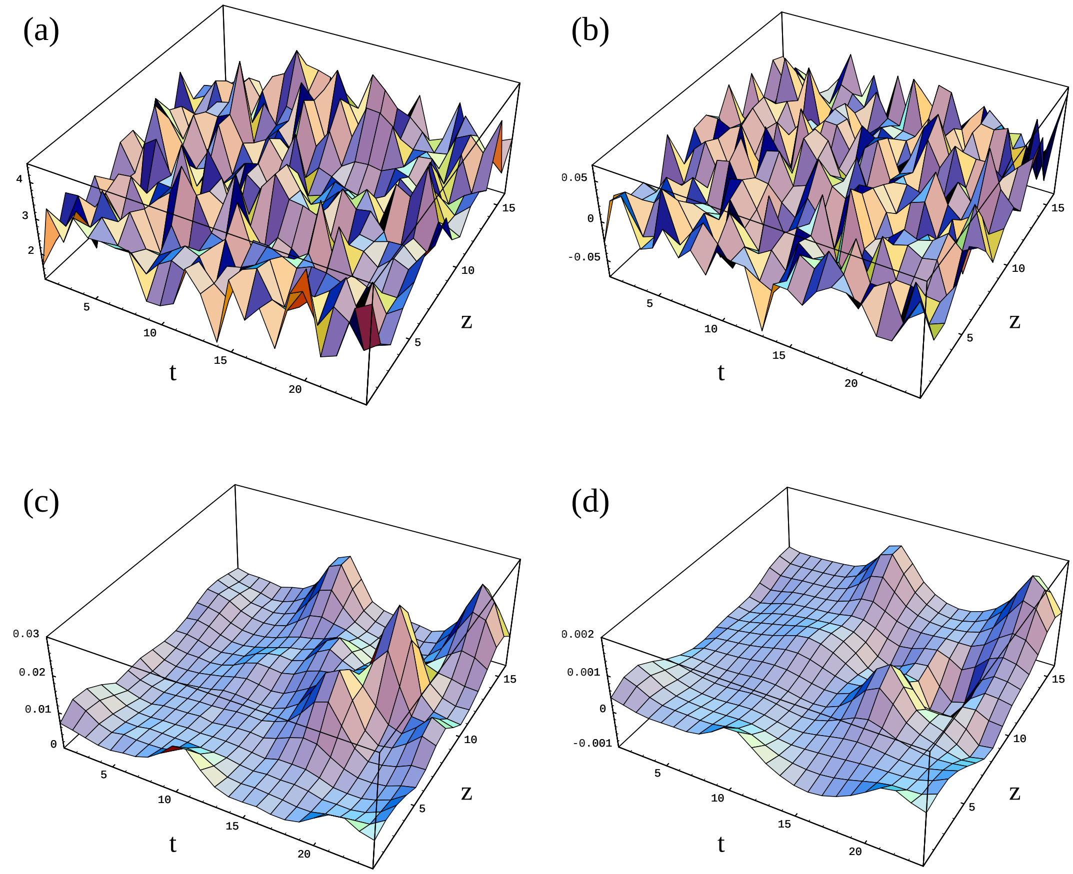


Outline

- Introduction: Why Instantons?

  correlators, effective interactions, zero modes

- Instantons and the η': large baryon density

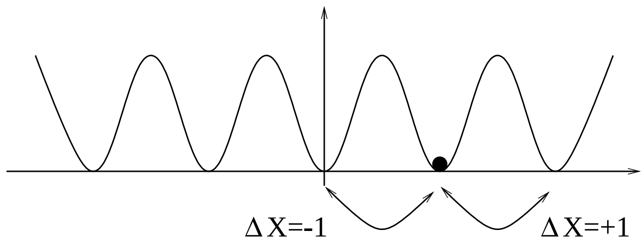
  $\rho \gg \Lambda_{QCD}^{-3}$: semi-classical approximation under control

- Instantons and the η': large N_c

  smooth large N_c limit? Witten-Veneziano relation?

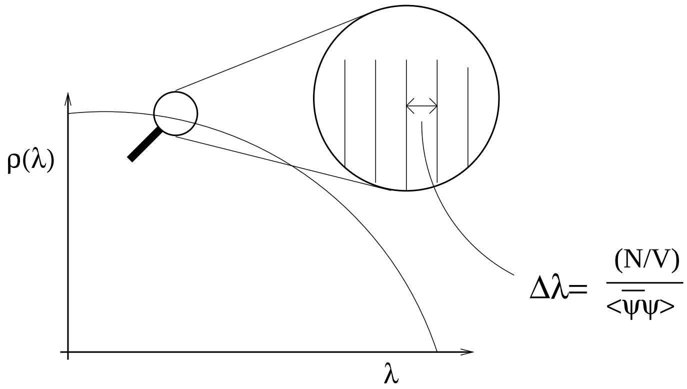
Why Instantons ?

- semi-classical expansion



$$\langle Q\!=\!k\pm 1 | Q\!=\!k\rangle \sim \exp\left(-\frac{8\pi^2}{g^2}\right)$$

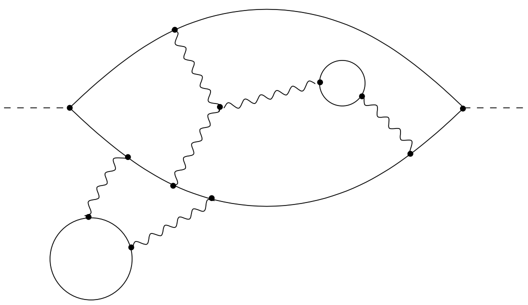
- quasi-zero modes and chiral symmetry breaking



$$\langle \bar{q}q\rangle = -\pi\rho(0)$$

$$\langle \bar{q}q\rangle \sim -\rho^{-1}\sqrt{\frac{N}{V}}$$

- hadronic correlators and effective quark-quark interaction



$$\Pi(x) = \langle \bar{q}\Gamma q(x)\bar{q}\Gamma q(0)\rangle = \Pi^0(x)+\delta\Pi(x)$$

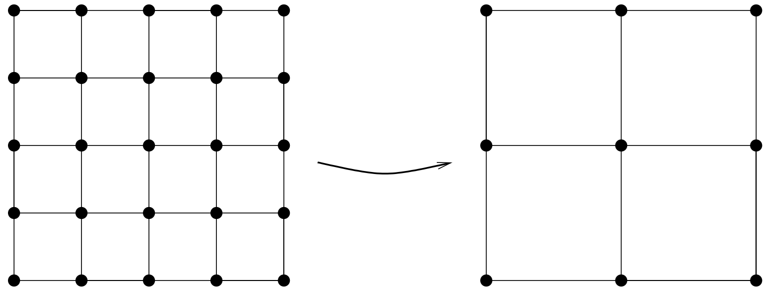
$$\delta\mathcal{L} = \sum_{\Gamma}\frac{g_\Gamma}{\Lambda_\chi^2}(\bar{q}\Gamma q)^2 + \ldots$$

Effective Interaction

- quark-quark intearction becomes non-perturbative at chiral symmetry breaking scale

$$x \sim \Lambda_{\chi SB}^{-1} \sim 0.2\,\mathrm{fm}$$

- integrate out short distance modes $x < \Lambda_{\chi SB}^{-1} \Rightarrow$ effective quark-quark interaction



$$\delta\mathcal{L} = \sum_\Gamma \frac{g_\Gamma}{\Lambda_\chi^2}(\bar{q}\Gamma q)^2 + \ldots$$

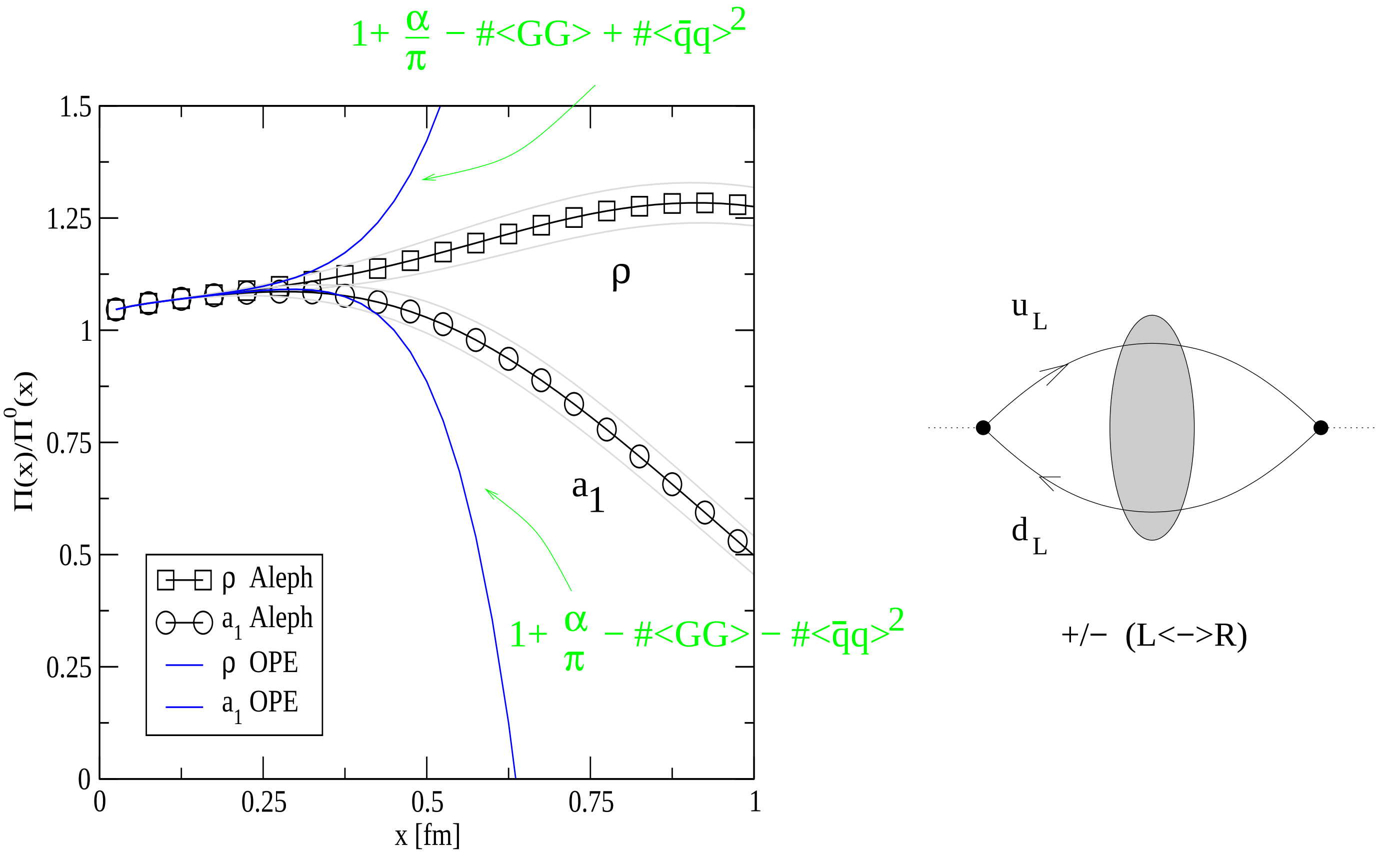
- Alternative: match hadronic correlation functions at distances $x \sim \Lambda_{\chi SB}^{-1}$

$$\Pi(x) = \langle \bar{q}\Gamma q(x)\bar{q}\Gamma q(0)\rangle$$

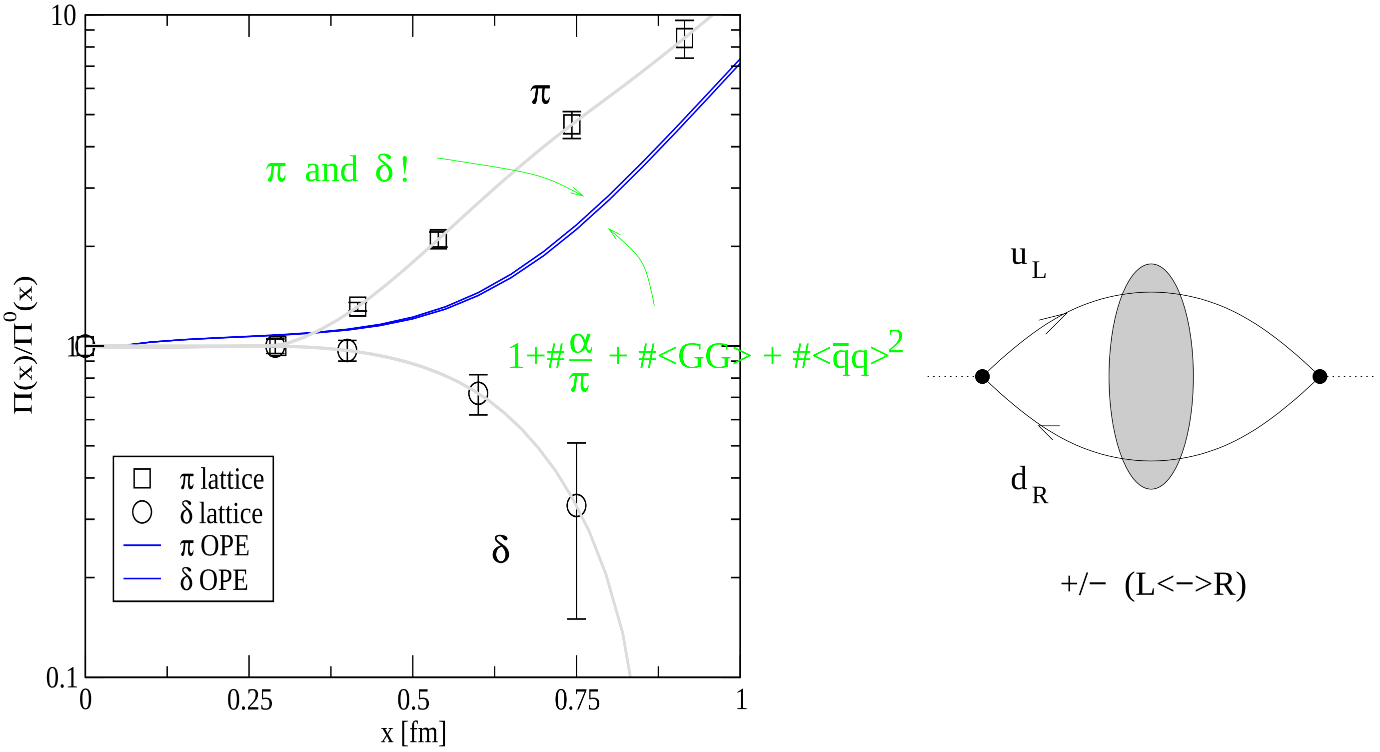
$$\Gamma_D = \{1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5\}$$

$$\Gamma_I = \{1, \vec{\tau}\}$$

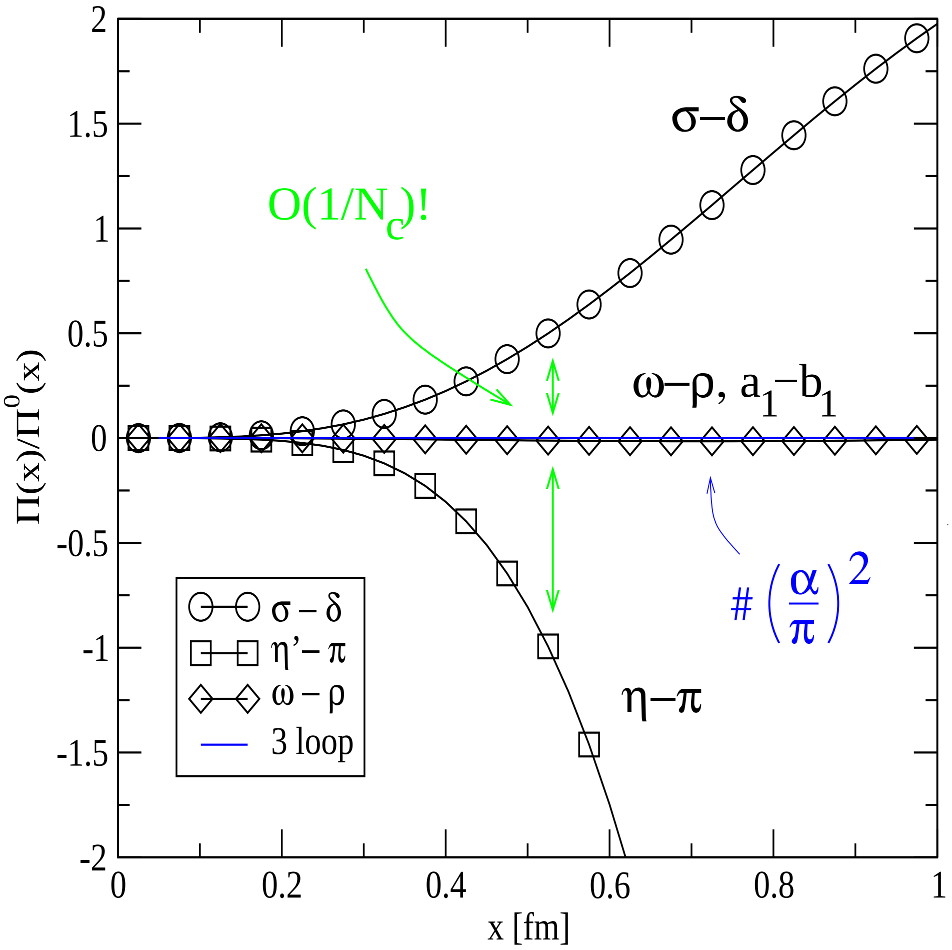
vector channels: ρ and a_1

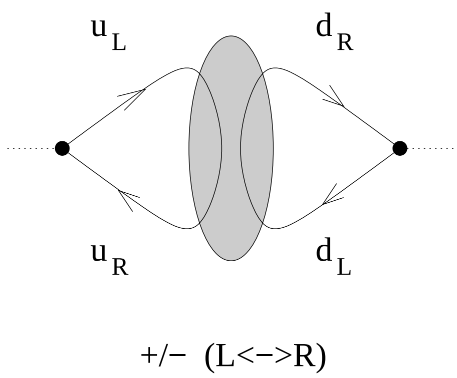


scalar channels: π and δ



OZI violation: $\eta' - \pi$, $\sigma - \delta$, $\omega - \rho$, $a_1 - b_1$





Summary

- No interaction in $(\bar{L}L \pm \bar{R}R)^2$.
- Sign changes for $\bar{L}R \leftrightarrow \bar{R}L$.
- Sign changes for $(\bar{u}d)(\bar{u}d) \leftrightarrow (\bar{u}u)(\bar{d}d)$.

$$\boxed{\mathcal{L} = G \det_f(\bar{\psi}_L \psi_R) + (L \leftrightarrow R)}$$

Chiral symmetry breaking

- Study motion of quarks in given A_μ, then average over A_μ
  $\Rightarrow \chi SB$ related to spectral density of $\not{D}$
- Diagonalize Dirac operator

$$i\not{D}\psi_\lambda = \lambda\psi_\lambda,$$

- quark propagator $S(x,y) = -\langle x|i\not{D}^{-1}|y\rangle$ is given by

$$S(x,y) = -\sum_\lambda \frac{\psi_\lambda(x)\psi_\lambda^\dagger(y)}{\lambda + im}$$

  quark condensate

$$\langle\bar{q}q\rangle = i\int d^4x\,\mathrm{tr}\,(S(x,x)) = -i\sum_\lambda \frac{1}{\lambda + im}.$$

- Note: $\quad i\not{D}\psi_\lambda = \lambda\psi_\lambda \quad \Rightarrow \quad i\not{D}(\gamma_5\psi_\lambda) = -\lambda(\gamma_5)\psi_\lambda$
- non-zero eigenvalues are paired

$$i\int d^4x\,\mathrm{tr}\,(S(x,x)) = -\sum_{\lambda\geq 0} \frac{2m}{\lambda^2 + m^2}.$$

- spectral density $\rho(\nu) = \langle\sum_\lambda \delta(\nu - \lambda)\rangle$ $(\langle\ldots\rangle = \int \mathcal{D}A\ldots)$.

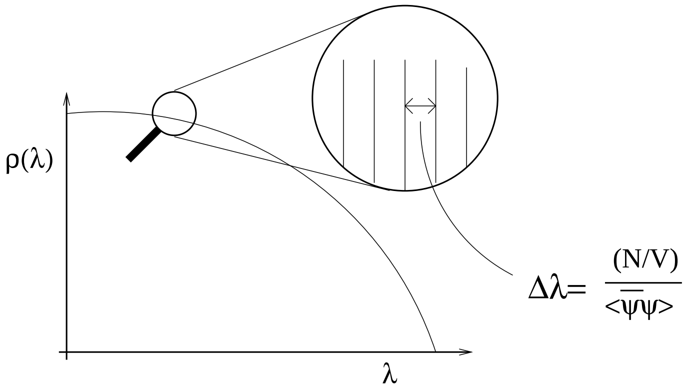
$$\langle\bar{q}q\rangle = -\int_0^\infty d\lambda\,\rho(\lambda)\frac{2m}{\lambda^2 + m^2}.$$

- Take $V \to \infty$ first, then $m \to 0$

$$\boxed{\langle\bar{q}q\rangle = -\pi\rho(\lambda = 0) \quad (Banks - Casher)}$$

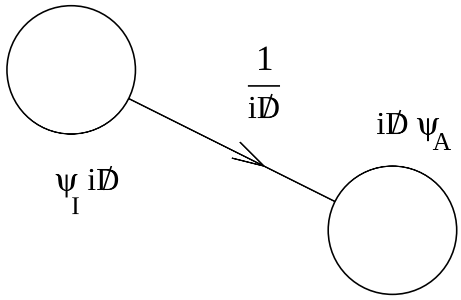
  $\Rightarrow \chi SB$ connected with "almost" zero modes

- zero mode basis



$$i\not{D} = \begin{pmatrix} 0 & T_{IA} \\ T_{AI} & 0 \end{pmatrix},$$

- overlap matrix elements T_{IA}



$$\begin{aligned} T_{IA} &= \int d^4x\, \psi_I^\dagger(x - z_I) i\not{D} \psi_A(x - z_A) \\ &\sim i(u \cdot \hat{R}) \frac{(\rho_I \rho_A)^3}{R^4}. \end{aligned}$$

- width of the zero mode zone $\sim$ average matrix element

$$\langle |T_{IA}|^2 \rangle = \frac{2\pi^2}{3N_c} \frac{N\rho^2}{V}.$$

- Gaussian ensemble $\Rightarrow$ spectral density is a semi-circle

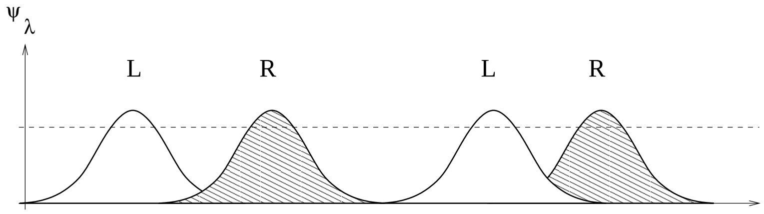
$$\rho(\lambda) = \frac{N}{\pi\sigma V} \left(1 - \frac{\lambda^2}{4\sigma^2}\right)^{1/2}.$$

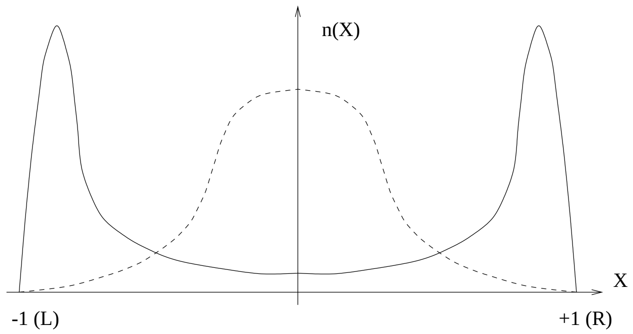
- Casher-Banks gives

$$\langle \bar{q}q \rangle = -\frac{1}{\pi\rho} \left(\frac{3N_c}{2} \frac{N}{V} \right)^{1/2} \simeq -(240\,\mathrm{MeV})^3,$$

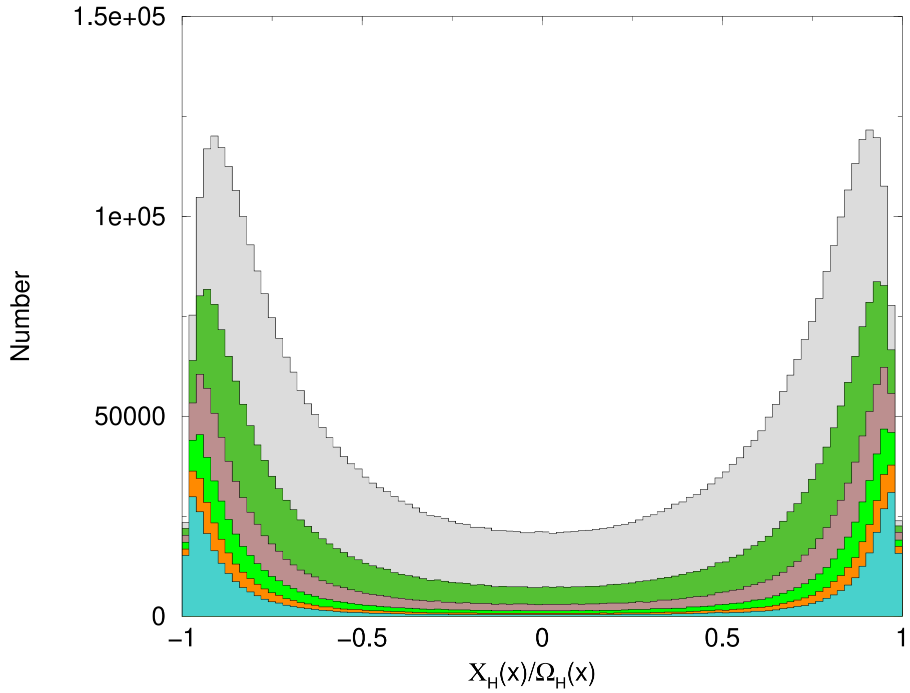
Chiral Symmetry Breaking on the Lattice

- Chirality distribution function



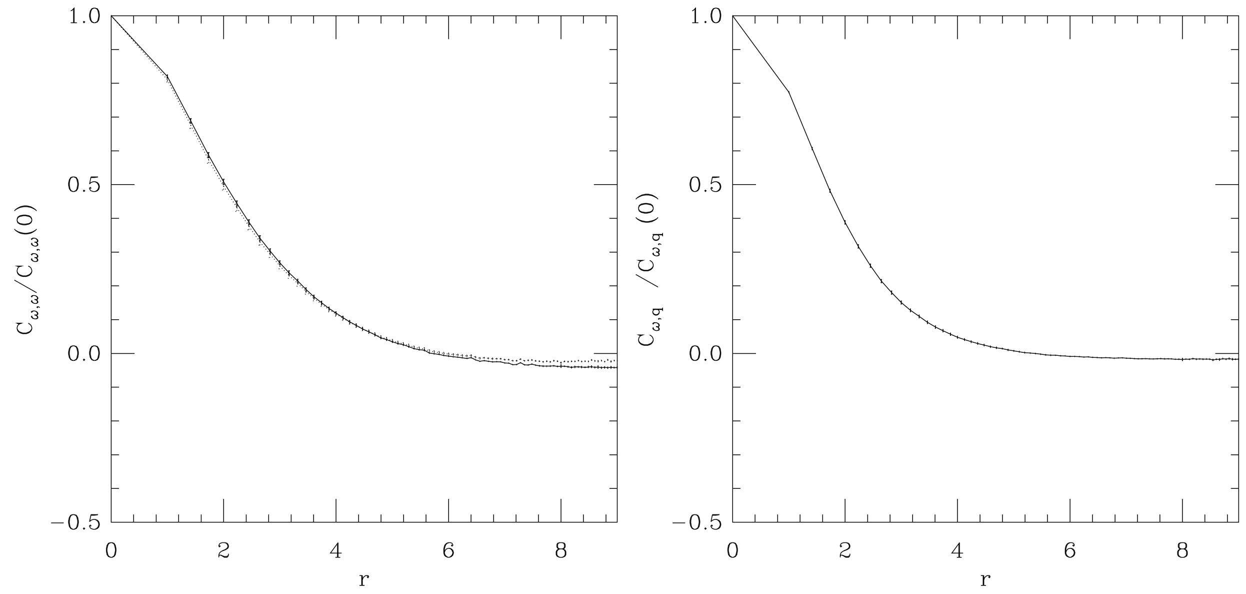


- Chirality distribution measured using overlap fermions



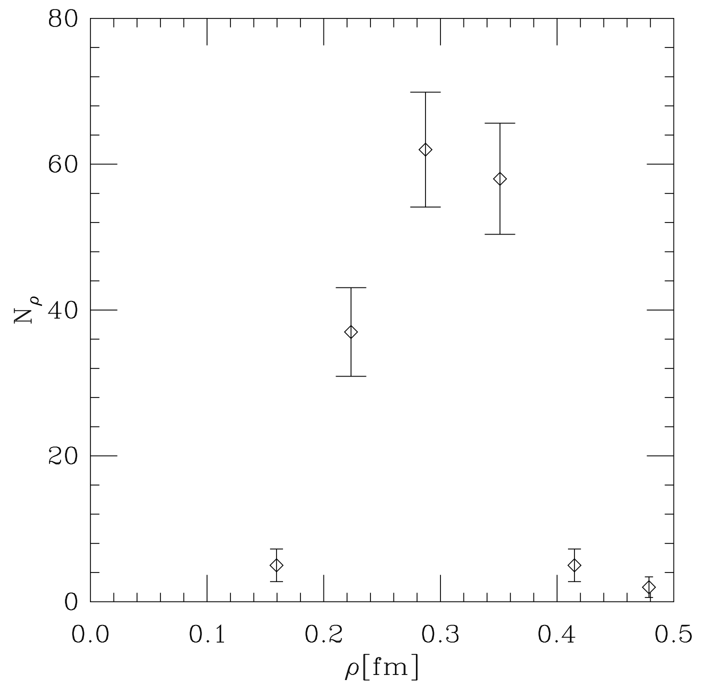
T. Blum et al., [hep-lat/0105006]

• chirality-chirality and chirality-topology correlation function



T. DeGrand, A. Hasenfratz, Phys.Rev.D64:034512,2001, [hep-lat/0012021]

• size distribution



QCD with two colors ($N_c = N_f = 2$)

- $N_c = 2$ QCD has an extra (Pauli-Gursey) symmetry

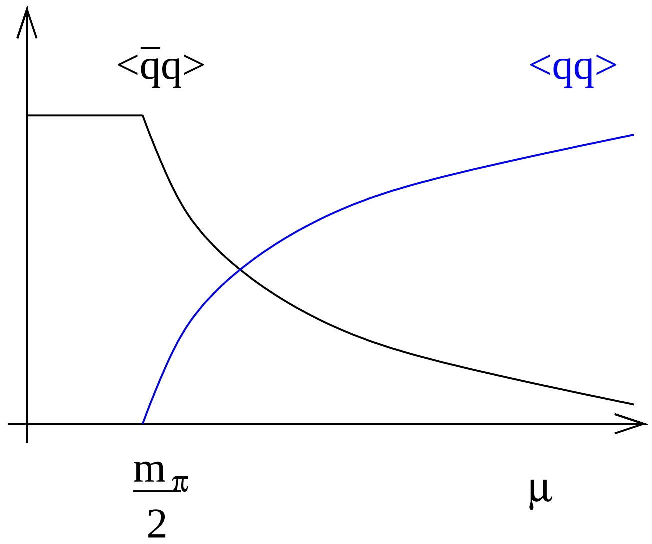
$$\langle \bar{q}q \rangle \leftrightarrow \langle q^T C\gamma_5 \lambda_2 \tau_2 q \rangle$$

- enlarged chiral symmetry group

$$SU(4) \to Sp(4) \qquad \text{not} : SU(2)_L \times SU(2)_R \to SU(2)_V$$

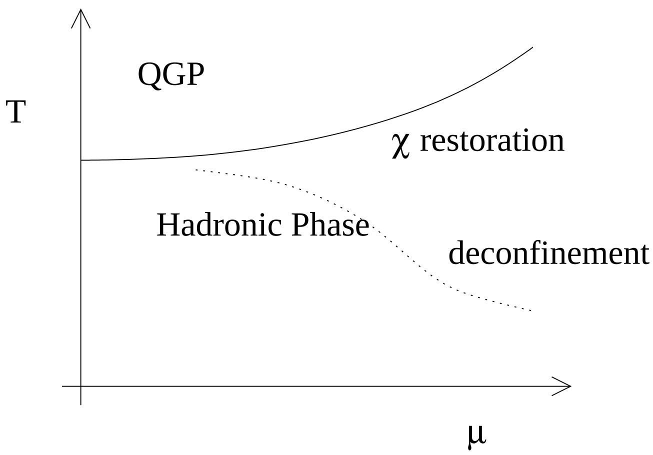
have 5 Goldstone bosons

- groundstate depends on external field: m, Δ, μ



Phase	GB′s	Higgs
$\langle \bar{q}q \rangle \neq 0$	$(\vec{\pi}, S, \bar{S})$	σ
$\langle qq \rangle \neq 0$	$(\vec{\pi}, S, \sigma)$	$\bar{S}$

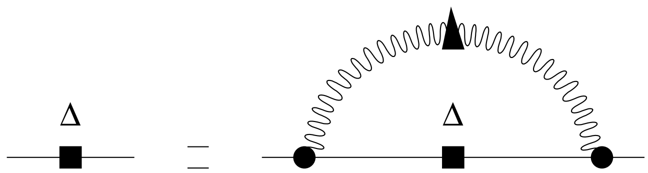
- diquark phase ($\Delta > m$): no phase transition



Note: $N_c = 2$ QCD (N_f even) can be simulated.

η' mass at large baryon density $(N_c = N_f = 2)$

- $\mu \gg \Lambda_{QCD}$: superconductivity driven by perturbative gluon exchange



$$\Delta = 512\pi^4 b_0' \mu g^{-5} \exp\left(-\frac{2\pi^2}{g}\right)$$

$$\epsilon^{ab}\epsilon_{ij}\langle q_i^a C q_j^b\rangle = \frac{4\pi}{g}\Delta\left(\frac{\mu^2}{2\pi^2}\right)$$

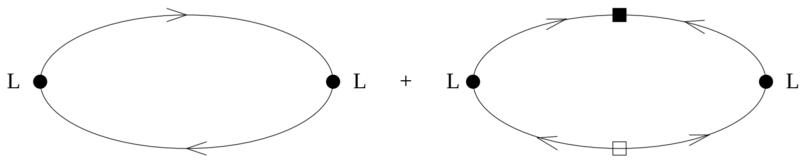
- diquark condensate breaks $U(1)_B$ and $U(1)_A$

$$\langle q_L q_L\rangle = \Phi\, e^{i(\chi+\phi)/2}, \qquad \langle q_R q_R\rangle = \Phi\, e^{i(\chi-\phi)/2}$$

- effective lagrangian for $U(1)_A$ Goldstone boson

$$\mathcal{L} = \frac{f^2}{2}\left[(\partial_0\phi)^2 - v^2(\partial_i\phi)^2\right] - V(\phi+\theta)$$

- decay constant calculable in perturbation theory



$$f^2 = \left(\frac{\mu^2}{4\pi^2}\right)$$

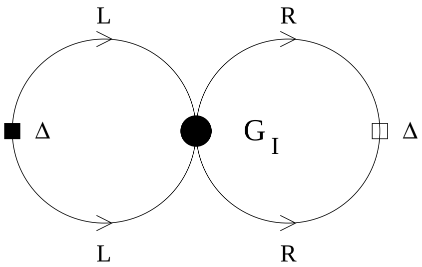
... but $V(\phi)$ vanishes in perturbative QCD

- instanton induced effective interaction for quarks with $p \sim p_F$

$$\mathcal{L} = \int n(\rho,\mu) d\rho\, \rho^6 \epsilon_{f_1 f_2}\epsilon_{g_1 g_2} \left\{ \frac{2N_c-1}{2N_c} (\bar\psi_{R,f_1}\psi_{L,g_1})(\bar\psi_{R,f_2}\psi_{L,g_2}) - \frac{1}{8N_c} (\bar\psi_{R,f_1}\sigma_{\mu\nu}\psi_{L,g_1})(\bar\psi_{R,f_2}\sigma_{\mu\nu}\psi_{L,g_2}) \right\}$$

$$n(\rho,\mu) = n(\rho,0)\exp\left[-N_f\rho^2\mu^2\right] \quad \Rightarrow \quad \boxed{\rho \sim \mu^{-1} \ll \Lambda_{QCD}^{-1}}$$

- instanton contribution to vacuum energy



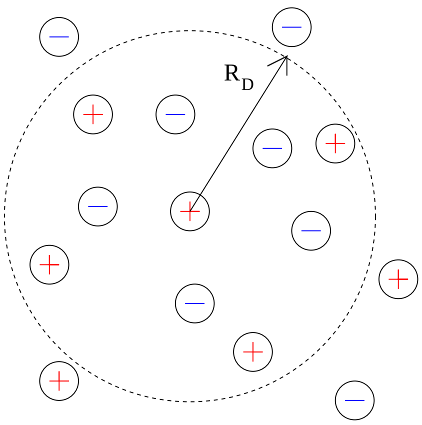
$$\langle \mathcal{L} \rangle = A\cos(\phi+\theta)$$

$$A = C_{N_c}^{N_f}\Phi^2 \left[\log\left(\frac{\mu}{\Lambda}\right)\right]^4 \left(\frac{\Lambda}{\mu}\right)^8 \Lambda^{-2}.$$

- η' mass satisfies "Witten-Veneziano" relation

$$\boxed{f^2 m_\phi^2 = A}$$

- very dilute instanton gas



$$\rho \ll r_{IA} \ll R_D$$

$$r_{IA} = A^{1/4}, \quad R_D = m_\phi^{-1}$$

- A is the <u>local</u> topological susceptibility

$$A = \chi_{top}(V) = \frac{\langle Q_{top}^2 \rangle_V}{V} \qquad r_{IA}^4 \ll V \ll R_D^4$$

- <u>Global</u> topological susceptibility vanishes

$$\chi_{top} = \lim_{V \to \infty} \frac{\langle Q_{top}^2 \rangle_V}{V} = 0 \qquad (m = 0)$$

A. Polyakov (1977)
D. Son, M. Stephanov, A. Zhitnitsky, hep-ph/0103099.
E. Shuryak, J. Verbarrschot, hep-lat/9409020
R. Rapp, T. Schäfer, E. V. Shuryak, and M. Velkovsky, hep-ph/9904353.
T. Schäfer, hep-ph/0201189.

QCD at large N_c

- QCD ($m = 0$) is a parmater free theory

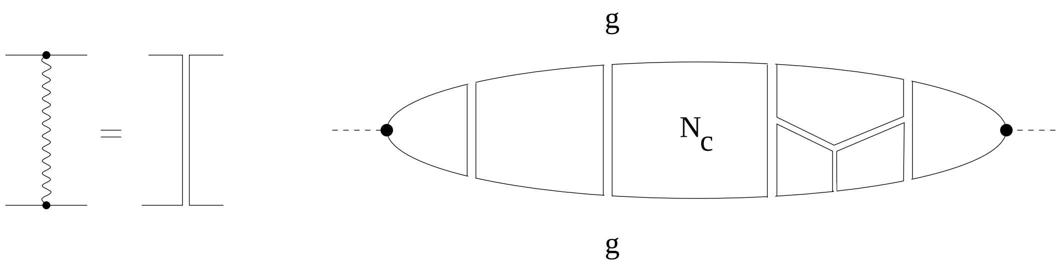
Very beautiful But: No expansion parameter

- 't Hooft: Consider $N_c \to \infty$ and use $1/N_c$ as a small parameter

$$N_c \to \infty \quad \Rightarrow \quad \text{classical master field}$$

- keep Λ_{QCD} fixed

$$\Rightarrow g^2 N_c = const$$



- Could the master field be a multi-instanton configuration?

$$\text{Witten (1979) : No} \qquad dn \sim \exp(-\frac{1}{g^2}) \sim \exp(-N_c)$$

$U(1)_A$ **anomaly at large** N_c

- consider θ term

$$\mathcal{L} = \frac{ig^2\theta}{32\pi^2} G^a_{\mu\nu}\tilde{G}^a_{\mu\nu},$$

- no θ dependence in perturbation theory.

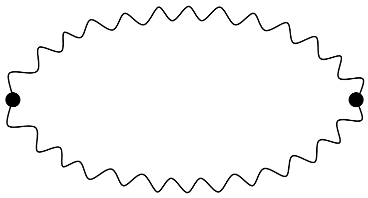
Witten: *non-perturbative θ dependence*

$$\chi_{top} = \left.\frac{d^2E}{d\theta^2}\right|_{\theta=0} \sim O(1)$$

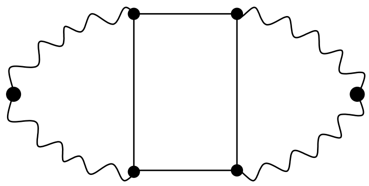
- massless quarks: toplogical charge screening

$$\lim_{m\to 0} \chi_{top} = 0$$

- How can that happen? Fermion loops are suppressed!



$g^4 N_c^2 \sim 1$



$g^8 N_c^3 \sim \frac{1}{N_c}$

Witten: *η' has to become light*

$$f_\pi^2 m_{\eta'}^2 = 2N_f \chi_{top} \quad \Rightarrow \quad m_{\eta'}^2 = O(1/N_c)$$

Digression: $\mathcal{N}=4$ SUSY Yang Mills

- string/field theory duality (Maldacena)

$$\begin{array}{ccc} \mathcal{N}=4 \text{ SUSY YM } (d=4) & \Leftrightarrow & \text{IIB strings on } AdS_5 \times S_5 \\ \lambda = g^2 N \to \infty & \Leftrightarrow & (l_s/R)^4 \to 0 \\ (g^2 \to 0) & & (g_s \to 0) \end{array}$$

- $\mathcal{N}=4$ conformal $\Rightarrow$ g does not run
- string theory contains D-instantons characterized by location on $AdS_5 \times S_5 \Leftrightarrow$ field theory instantons

T. Banks, M. Green

$$\int d^4x \int \frac{d\rho}{\rho^5} \int d\Lambda^{ab}$$

$$AdS_5 \times S_5$$

- k instanton amplitudes

M. Mattis et al.

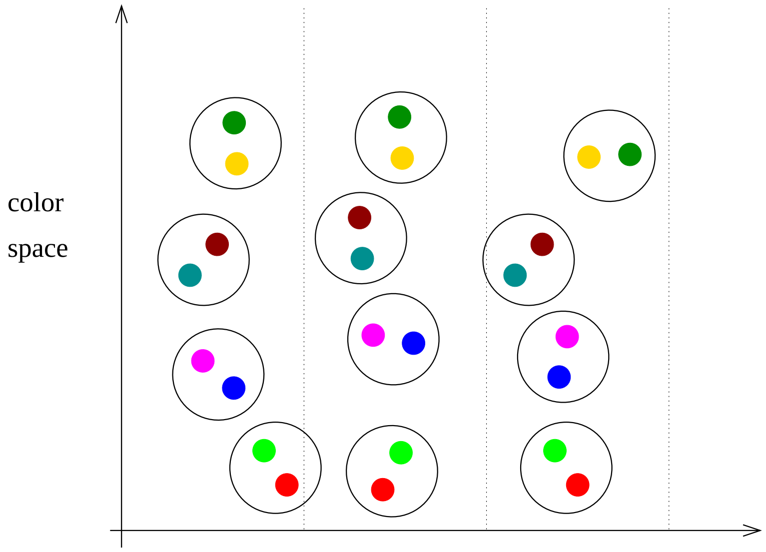
$$(AdS_5 \times S_5)^k \quad \to \quad (AdS_5 \times S_5)$$

k instantons in commuting $SU(2)'s$

(bound by fermions)

Instantons at large N_c

- semi-classical ensemble of instantons at large N_c



- instantons are $N_c = 2$ configurations

$$\left(\frac{N}{V}\right) = O(N_c) \quad \Rightarrow \epsilon_{vac} = O(bN_c) = O(N_c^2)$$

- instantons are semi-classical

$$\rho \simeq \rho^* = O(1) \quad S_{inst} = O(N_c)$$

- density $dn \sim \exp(-S_{inst}) = O(\exp(-N_c))$?

$$NO! \quad \textit{large entropy} \quad dn \sim \exp(+N_c)$$

- topological susceptibility $\chi_{top} \simeq (N/V) = O(N_c)$?

$$NO! \quad \textit{fluctuations suppressed} \quad \chi_{top} = O(1)$$

Instantons ensemble

- instanton ensemble

$$Z = \frac{1}{N_I! N_A!} \prod_I^{N_I+N_A} \int [d\Omega_I \, n(\rho_I)] \, \exp(-S_{int})$$

$$n(\rho) = C_{N_c} \left(\frac{8\pi^2}{g^2}\right)^{2N_c} \rho^{-5} \exp\left[-\frac{8\pi^2}{g(\rho)^2}\right]$$

$$C_{N_c} = \frac{0.47 \exp(-1.68 N_c)}{(N_c-1)!(N_c-2)!} \quad \frac{8\pi^2}{g^2(\rho)} = -b \log(\rho\Lambda), \quad b = \frac{11}{3} N_c$$

$$S_{int} = -\frac{32\pi^2}{g^2} |u|^2 \left\{ \frac{\rho_I^2 \rho_A^2}{R_{IA}^4} \left(1 - 4\cos^2\theta\right) + S_{core}\left(\frac{\rho_I^2 \rho_A^2}{R_{IA}^4}\right) \right\}$$

- complicated ensemble, size distribution

$$\rho^* \sim O(1) \qquad \begin{cases} \rho < \rho^* & n(\rho) \sim \exp(-N_c) \\ \rho \sim \rho^* & n(\rho) \sim const \end{cases}$$

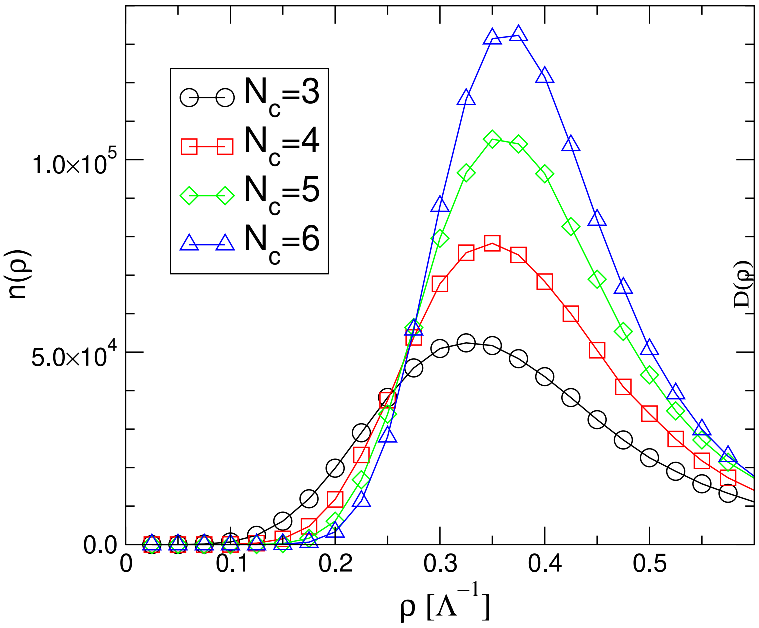
- total density determined by interactions

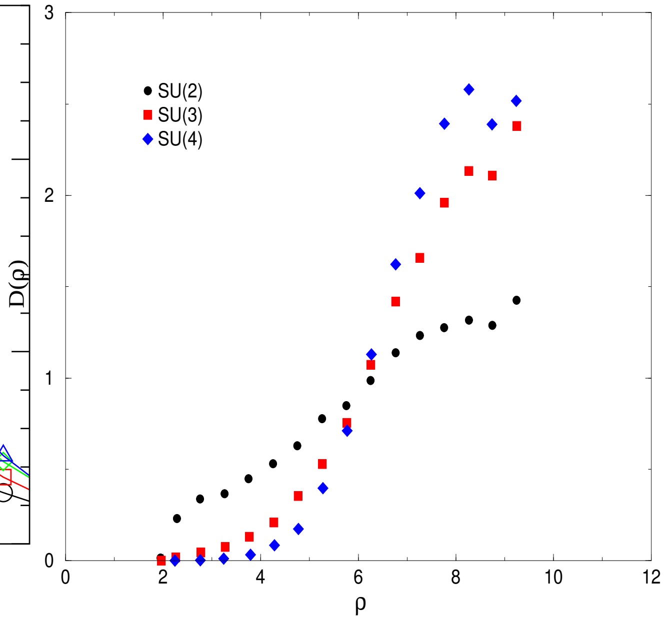
$$\begin{array}{ccc} S(1-body) & \sim & S(2-body) \\ N_c & \sim & N_c \quad \times \quad \frac{1}{N_c} \quad \times \quad \left(\frac{N}{V}\right) \\ classical & \sim & classical \times color\ overlap \times density \end{array}$$

- conclude

$$\boxed{\left(\frac{N}{V}\right) = O(N_c)}$$

- instanton size distribution





B. Lucini, M. Teper, hep-lat/0103027.

- fluctuations in N are $1/N_c$ suppressed

$$\langle N^2\rangle - \langle N\rangle^2 = \frac{4}{b}\langle N\rangle \qquad (\text{not } \langle N\rangle!)$$

low energy theorem

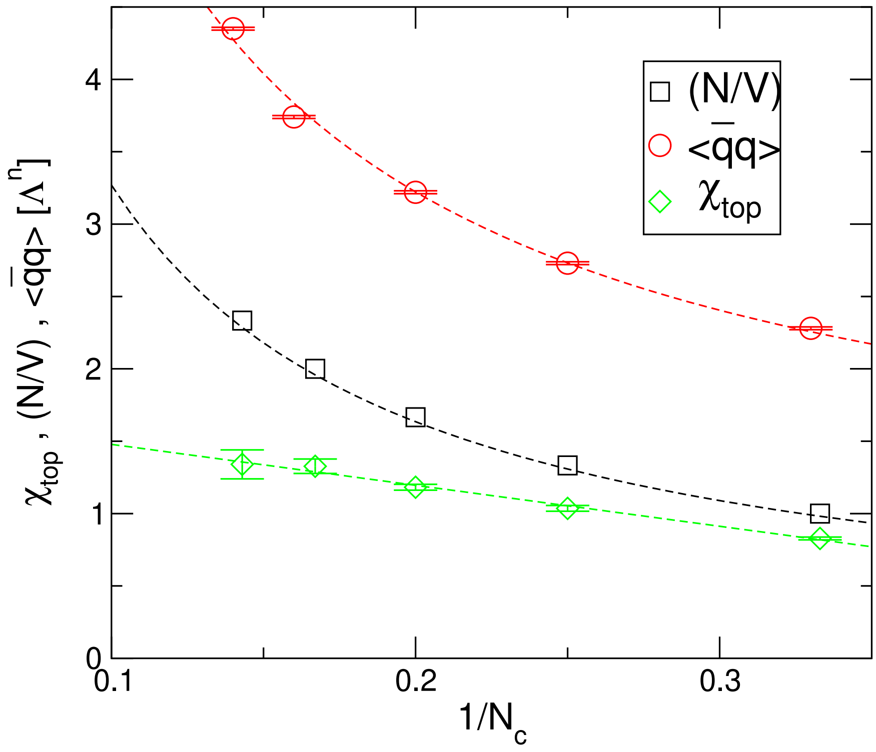
$$\frac{1}{(32\pi^2)^2}\int d^4x\ \left\{\langle g^2G^2(0)g^2G^2(x)\rangle - \langle g^2G^2(0)\rangle^2\right\} = \frac{4}{b}\frac{1}{32\pi^2}\langle g^2G^2\rangle$$

V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B191, 301 (1981).

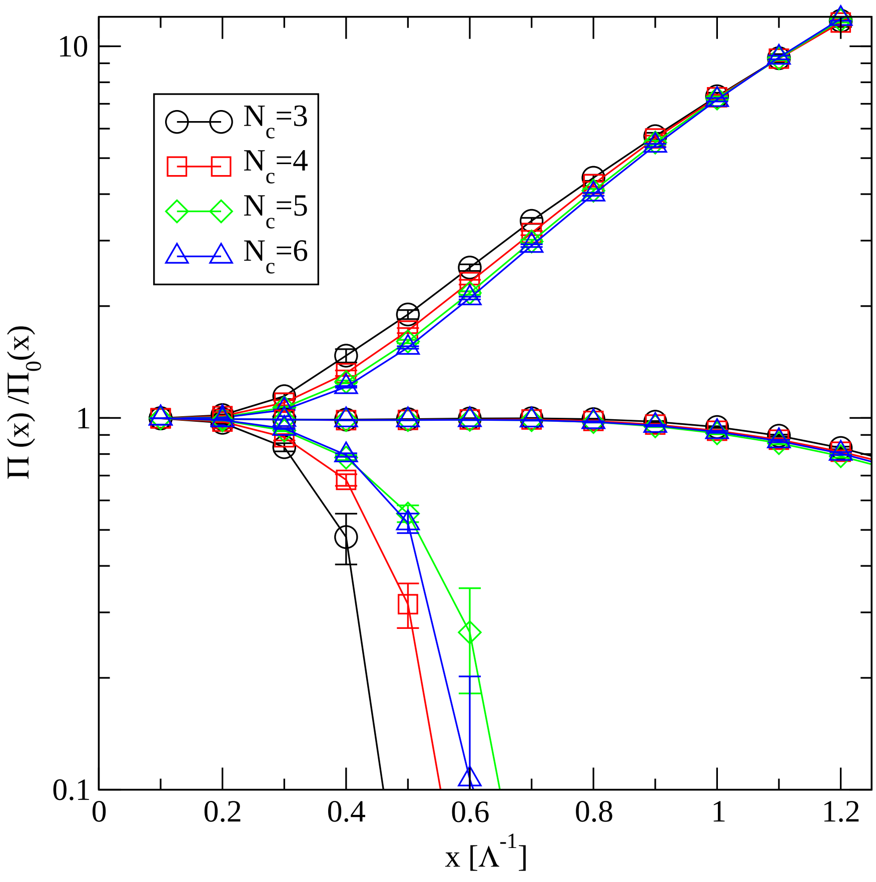
- also true for topological charge

$$\langle Q^2\rangle = \frac{4}{b - r(b-4)}\langle N\rangle$$

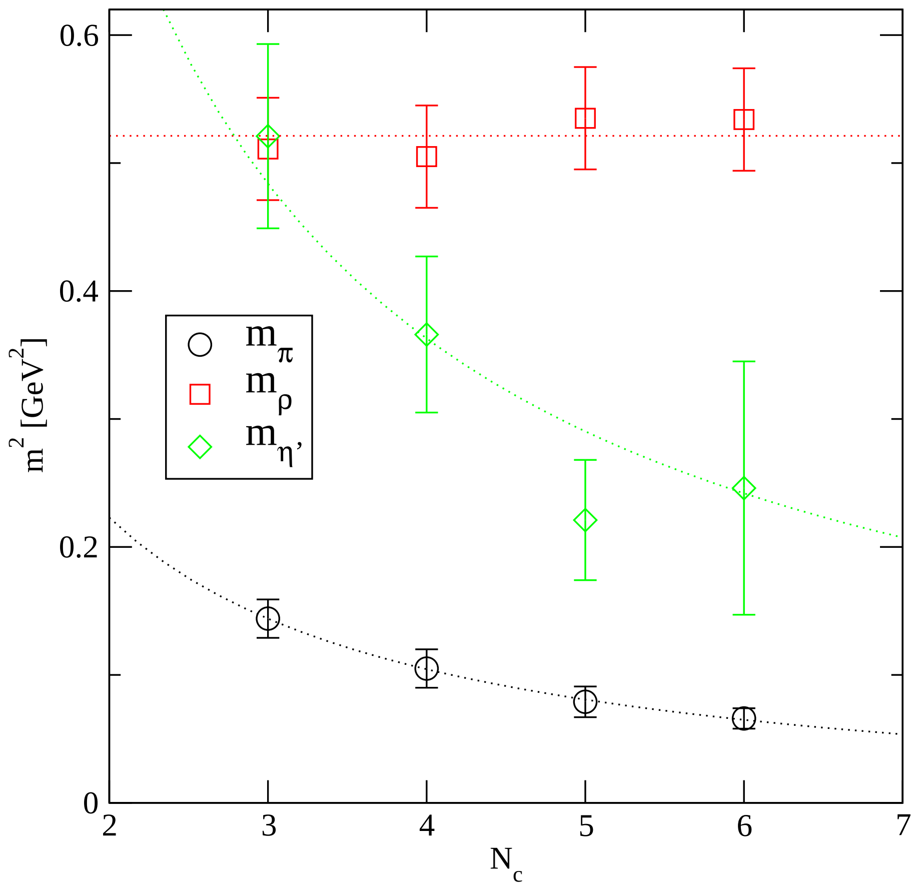
- instanton density, quark condensate, topological susceptibility



- meson correlation functions (π, ρ, η')



- meson masses: $m_\pi^2, m_\rho^2 \sim 1,\ m_{\eta'}^2 \sim 1/N_c$



Summary

- lattice evidence for role of instanton zero modes in chiral symmetry breaking
- nice example for semi-classical instanton liquid: QCD at large density
- instanton liquid can have a smooth large N_c limit

$$\left(\frac{N}{V}\right) = O(N_c), \qquad \chi_{top} = O(1), \qquad m^2_{\eta'} = O(1/N_c)$$

- how can we check this?