Effective Theories of High Density QCD

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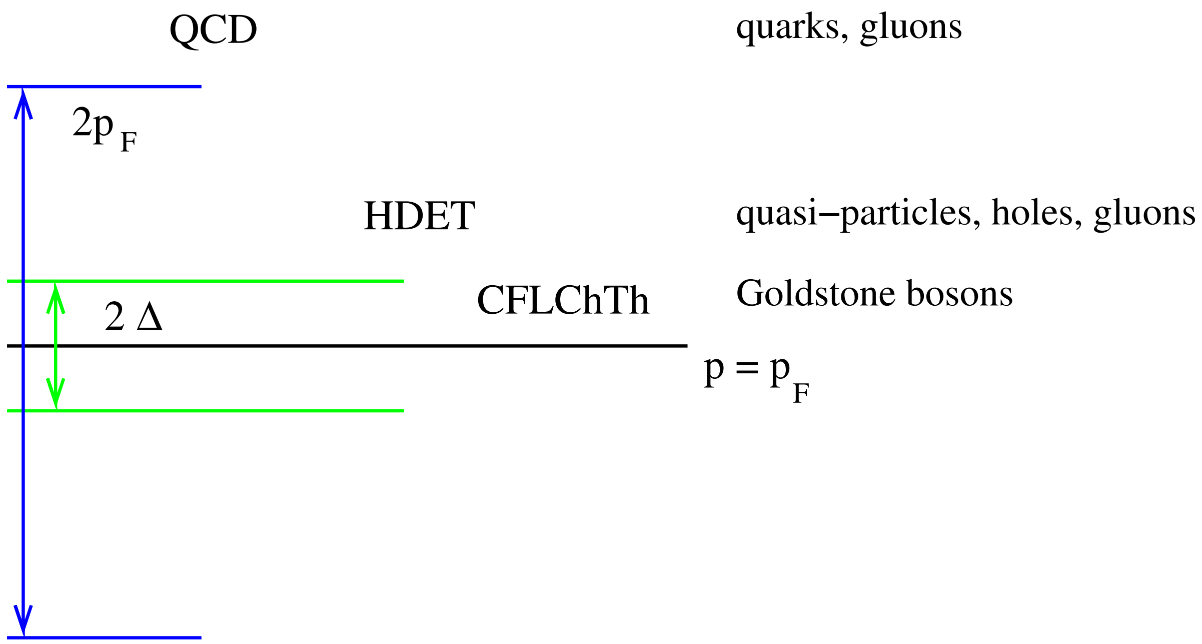


Why Effective Field Theories?

- weak coupling expansion $\neq$ loop expansion

  organize perturbation theory
  resum logarithms, etc.

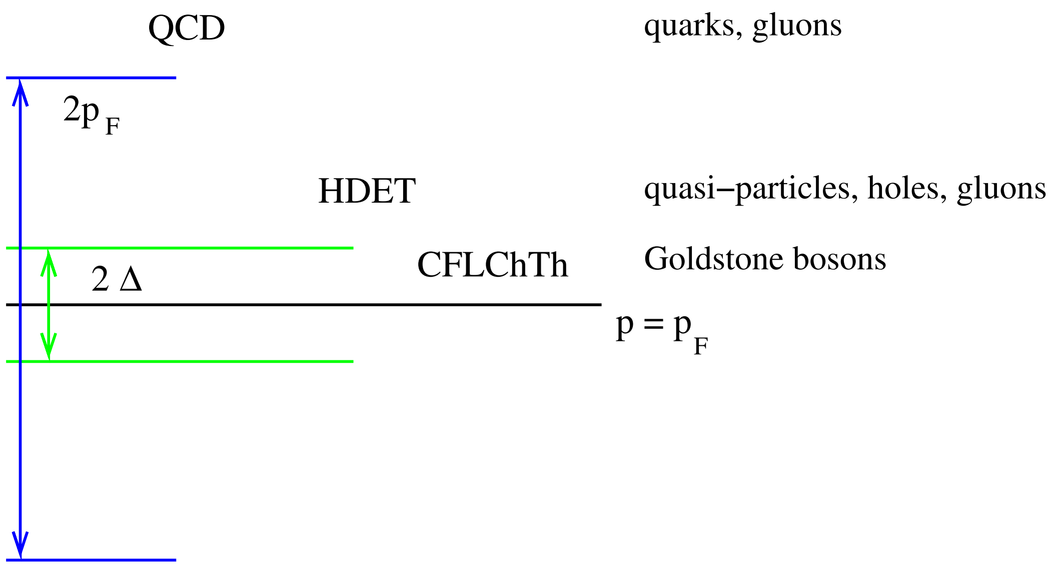
- studying and extending the reach of weak coupling methods

  improved dimensional analysis
  match to $N_c = 2$ lattice

- effects of perturbations, external fields

  quark masses
  electron chemical potentials, etc.

Effective Field Theories

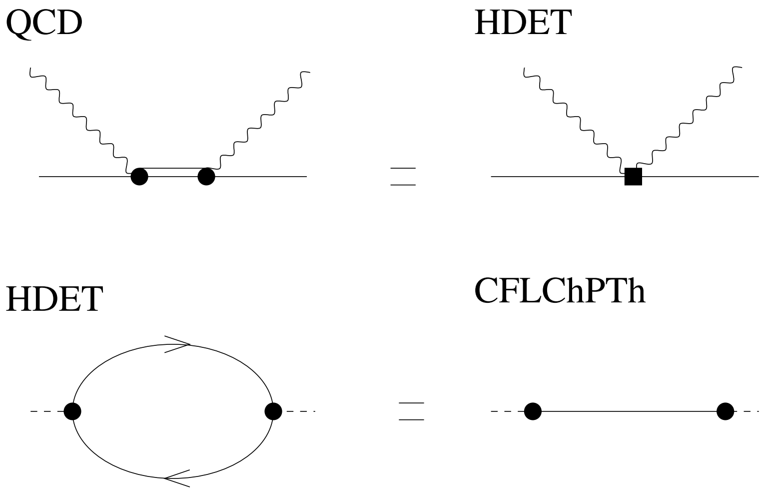


QCD $\mathcal{L} = \bar{\psi}(i\not{D} + \mu\gamma_0)\psi - \frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu}$

HDET $\mathcal{L} = \psi_v^\dagger(iv\cdot D)\psi_v - \frac{\Delta}{2}\psi^T_{-v}C\psi_v - \frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} + \ldots$

CFLChPTh $\mathcal{L} = \frac{f_\pi^2}{4}\mathrm{Tr}\left[\nabla_0\Sigma\nabla_0\Sigma^\dagger - v^2\vec{\nabla}\Sigma\vec{\nabla}\Sigma^\dagger\right] + \ldots$

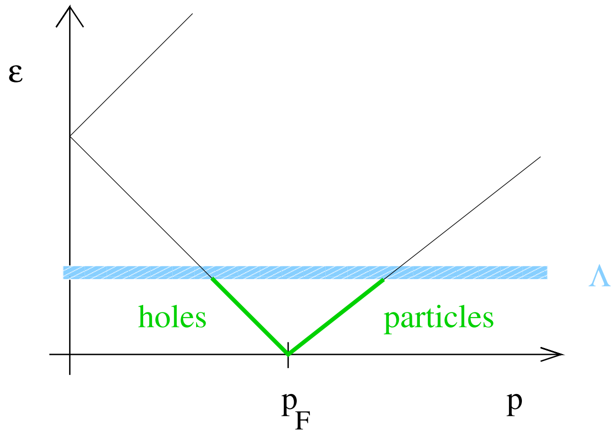
- coefficients determined by matching Green functions



High density effective theory

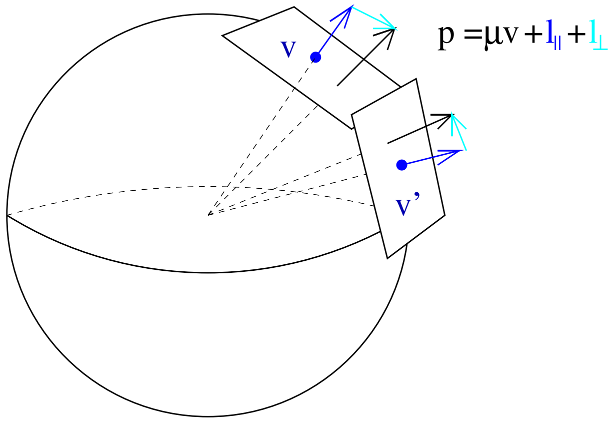
- quasi-particles (holes)

$$E_\pm = -\mu + \sqrt{\vec{p}^2 + m^2}$$
$$\simeq -\mu \pm |\vec{p}|$$



- effective field theory on v-patches

$$\psi_{v\pm} = e^{-i\mu v\cdot x}\left(\frac{1 \pm \vec{\alpha}\cdot\vec{v}}{2}\right)\psi$$



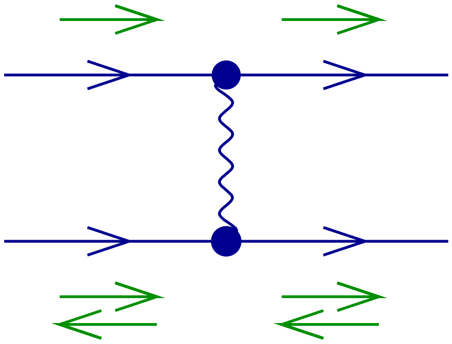
- effective lagrangian for ψ_{v+}

$$\mathcal{L} = \sum_v \psi_v^\dagger (iv\cdot D)\psi_v - \frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} + O(1/\mu) \quad [v^2 = 0]$$

D.K. Hong (2000)

Symmetries

- approximate spin symmetry



$$\psi_v \to (1 + i\vec{\theta} \cdot \vec{\sigma})\psi_v$$

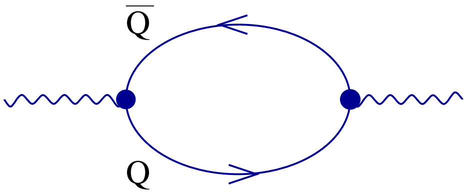
$$\log\left[\Delta(\uparrow\downarrow)\right] \sim \log\left[\Delta(\uparrow\uparrow)\right]$$

- reparameterization invariance: $p_\alpha = \mu v_\alpha + l_\alpha$

$$\begin{aligned} \vec{v} &\to \vec{v} + \vec{\epsilon}/\mu \\ \vec{l} &\to \vec{l} - \vec{\epsilon} \\ \psi_v &\to \psi_v + \delta\psi_v \end{aligned}$$

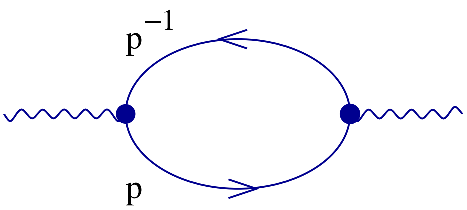
fixes certain higher order terms

- particle number conservation (as in HQET)?



$= 0$

particle-anti-particle pairs



$\neq 0$

particle-hole pairs

Formal reason: boundary conditions ($i\epsilon$ prescription)

Higher Order Terms

- power corrections: $1/\mu$ terms

$$\mathcal{L} = -\frac{1}{2\mu}\sum_v \psi_v^\dagger \left\{ D_\perp^2 - \frac{ag}{2}\sigma^{\mu\nu}G_{\mu\nu}^\perp - MM^\dagger \right\} \psi_v$$

higher order terms: $\frac{c_n}{\mu^n}\psi_v^\dagger D_\perp^{2n_1}(\bar{v}\cdot D)^{n_2}\psi_v, \ \ldots$

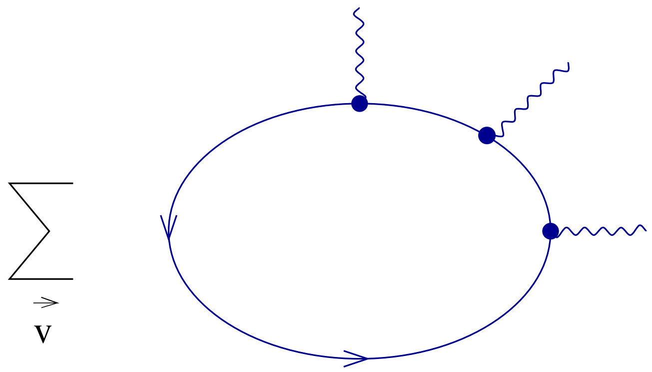
four fermion operators: $\frac{1}{\mu^2}c^{\Gamma\Gamma'}(\vec{v}_i)(\psi_{v_1}\Gamma\psi_{v_2})(\psi_{v_3}^\dagger\Gamma'\psi_{v_4}^\dagger)\delta_{\sum v_i}$

- power counting trivial? amplitude $\sim l^\delta$

$$\boxed{\delta = 4 + \sum_i N_i(d_i - 4)}$$

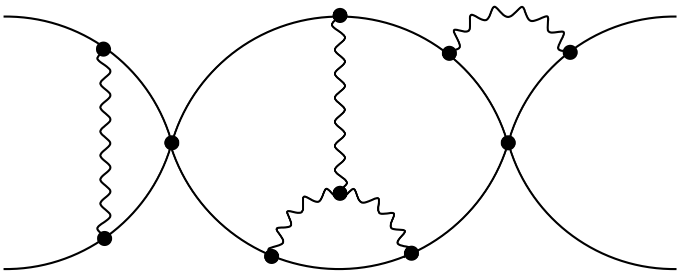
graphs with higher order vertices suppressed

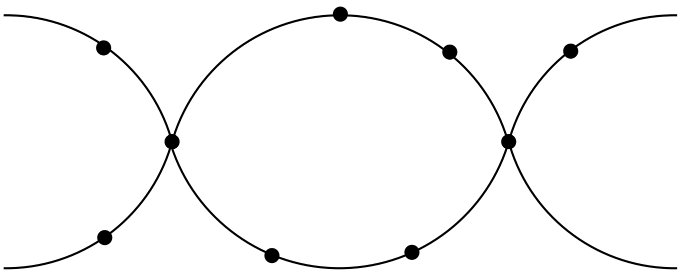
- problem: “hard” loops



$$\frac{1}{2\pi}\sum_{\vec{v}}\int\frac{d^2 l_\perp}{(2\pi)^2} = \frac{\mu^2}{2\pi^2}\int\frac{d\Omega}{4\pi}.$$

Modified Power Counting





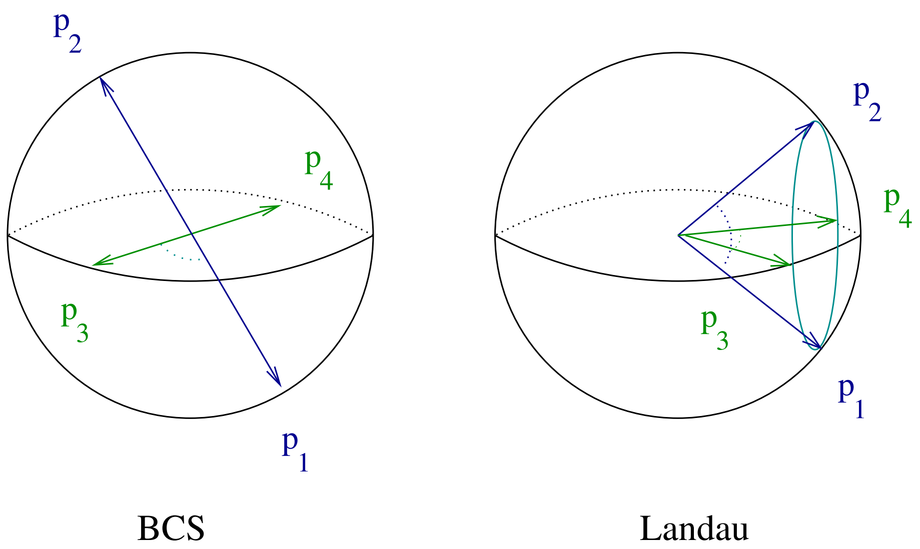
$$\boxed{\delta = \sum_i \left[(k-4)V_k^S + (k-2-f_k)V_k^H \right] + E_Q + 4 - 2N_c}$$

V_k^S	soft vertices of $O(l^k)$
V_k^H	hard vertices of $O(l^k)$
f_k	number of fermion fields in V_k^H
E_Q	external quark lines
N_c	connected graphs in hard graph

- quark loops in gluon n-pt fcts blow up at $l \sim g\mu$
- four quark operators are leading order
- six quark operators (etc) are suppressed

Four Quark Operators

- quark-quark scattering: $(v_1, v_2) \to (v_3, v_4)$



$$\mathcal{L}_{BCS} = \frac{1}{\mu^2} \sum_{v,v'} \sum_{\Gamma,\Gamma'} V_l^{\Gamma\Gamma'} R_l^{\Gamma\Gamma'}(\vec{v}\cdot\vec{v}')(\psi_v \Gamma \psi_{-v})(\psi^\dagger_{v'} \Gamma' \psi^\dagger_{-v'}),$$

$$\mathcal{L}_{FL} = \frac{1}{\mu^2} \sum_{v,v',\phi} \sum_{\Gamma,\Gamma'} F_l^{\Gamma\Gamma'}(\phi) R_l^{\Gamma\Gamma'}(\vec{v}\cdot\vec{v}')(\psi_v \Gamma \psi_{v'})(\psi^\dagger_{\tilde{v}} \Gamma' \psi^\dagger_{\tilde{v}'})$$

- spin structure: helicity amplitudes

$$(++) \to (++) \qquad (+-) \to (+-)$$

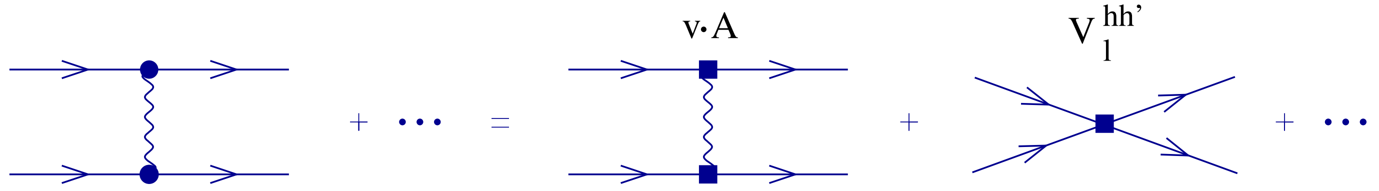
- BCS channel

$$\mathcal{L} = \frac{1}{\mu^2} \sum_{v,v'} V_l^{++} P_l(\vec{v}\cdot\vec{v}')(\psi_v \sigma_2 H_+ \psi_{-v})(\psi^\dagger_{v'} \sigma_2 H_+ \psi^\dagger_{-v'})$$

$$+ \frac{1}{\mu^2} \sum_{v,v'} V_l^{+-} d_{11}^{(l)}(\vec{v}\cdot\vec{v}')(\psi_v \sigma_2 H_- \vec{\sigma} H_+ \psi_{-v})(\psi^\dagger_{v'} \sigma_2 H_- \vec{\sigma} H_+ \psi^\dagger_{-v'})$$

Four Fermion Operators: Matching

- match moments of scattering amplitudes on Fermi surface



- leading order result (color $[\bar{3}]$ channel)

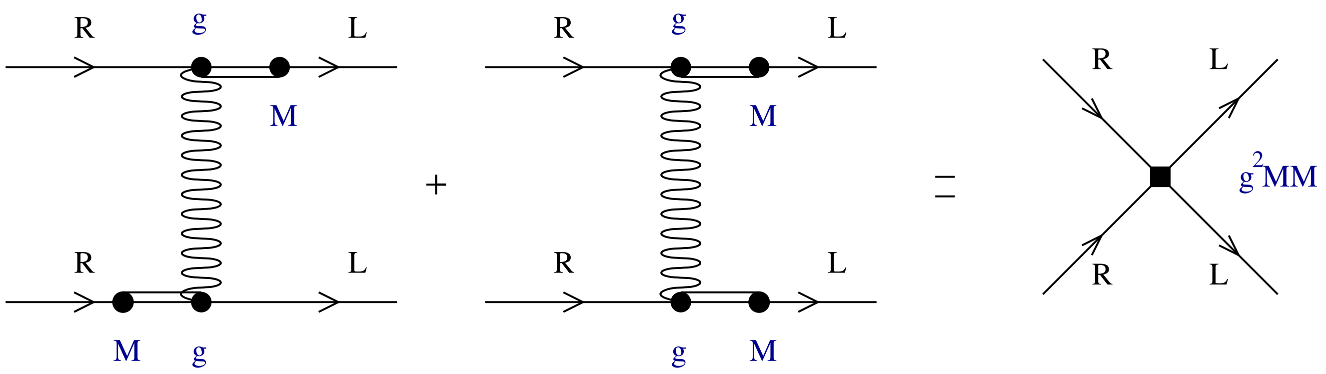
$$\begin{aligned} V_0^{++}(\Lambda_\perp^2 = 2\mu^2) &= 0 \\ V_1^{+-}(\Lambda_\perp^2 = 2\mu^2) &= -\tfrac{9}{2}\tfrac{2g^2}{3} \\ &\vdots \end{aligned}$$

scale dependence

$$\Lambda_\perp^2 \frac{d}{d\Lambda_\perp^2} V_0^{++}(\Lambda_\perp^2) = \frac{g^2}{3}$$

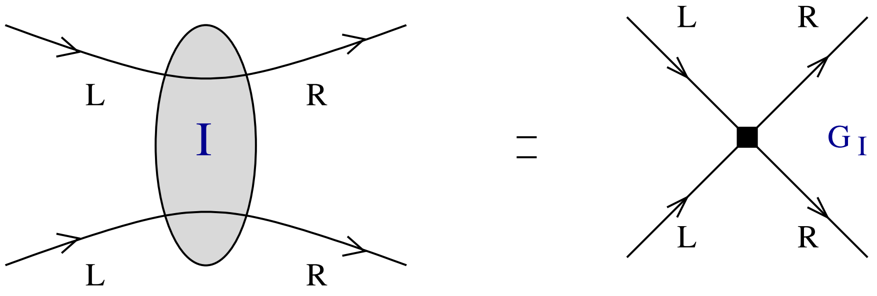
Helicity Flip Operators

- mass terms: $(++) \to (--)$



$$V_0^{++;--} = \frac{2g^2}{3}(M)_{ik}(M)_{il}$$

- instanton effects

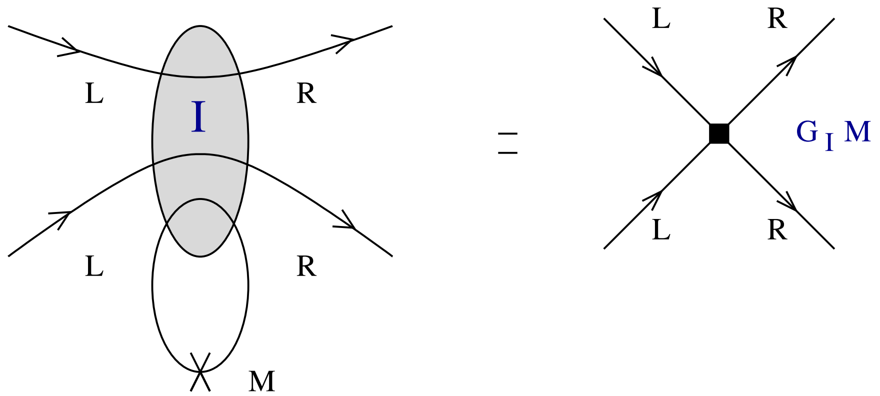


$$V_0^{++;--} = C_{3,2}\left[\log\left(\frac{\mu}{\Lambda}\right)\right]^6\left(\frac{\Lambda}{\mu}\right)^{35/3}\Lambda^{-2}$$

higher partial waves vanish: $\quad V_{l>0}^{++;--} = 0$

scale independent: $\quad \Lambda_\perp^2 \frac{d}{d\Lambda_\perp^2} V_0^{++;--} = 0$

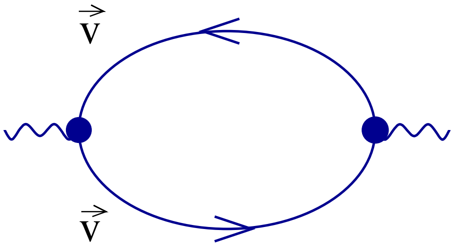
• instanton effects ($N_f = 3$)



$$V_0^{++;--} = C_{3,3}\mathrm{Tr}[M]\left[\log\left(\frac{\mu}{\Lambda}\right)\right]^6\left(\frac{\Lambda}{\mu}\right)^{12}\Lambda^{-2}$$

Hard Loops: Gluon Self Energy

- gluon self energy: sum over patches



$$\sum_{\vec{v}} \int \frac{d^4k}{(2\pi)^4} \frac{v_\mu v_\nu}{(k_0 - l_k)(k_0 + p_0 - l_{k+p})}$$

- use

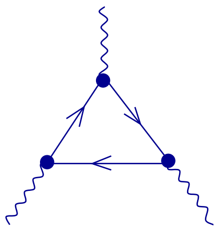
$$\frac{1}{2\pi} \sum_{\vec{v}} \int \frac{d^2 l_\perp}{(2\pi)^2} = \frac{\mu^2}{2\pi^2} \int \frac{d\Omega}{4\pi}.$$

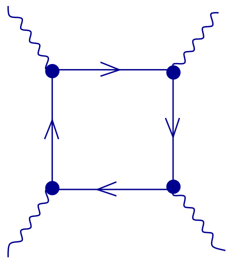
- HDL polarization function

$$\Pi^{ab}_{\mu\nu}(p) = 2m^2 \delta^{ab} \int \frac{d\Omega}{4\pi} v_\mu v_\nu \left\{ 1 - \frac{p_0}{p_0 - l_p} \right\}$$

- need counter term $m^2 A_\perp^2$
- hard loops modify power counting
- have to resum for $E \sim g\mu$

leading order
hard loop diagrams

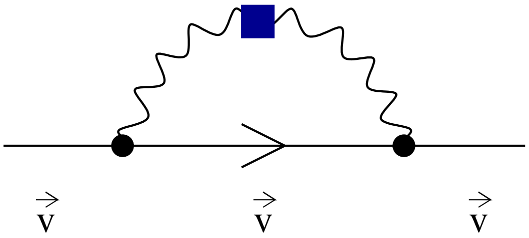




$\ldots$

Soft Loops: Quark Self Energy

- quark self energy: no sum



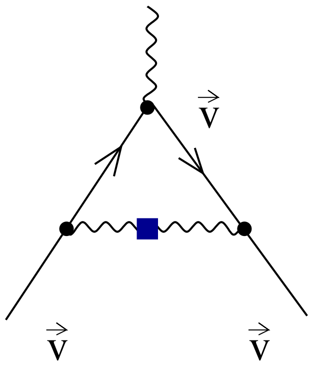
$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{p_0 + k_0 - l_{p+k}} v_\mu v_\nu D_{\mu\nu}(k)$$

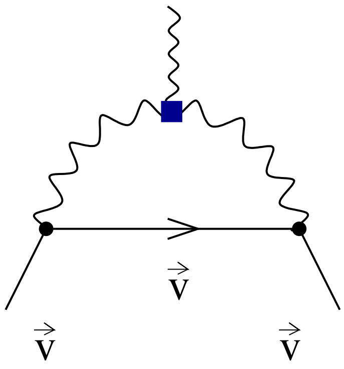
- logarithmic IR divergence

$$\Sigma(p_4) \simeq \frac{g^2 C_F}{12\pi^2} p_4 \log\left(\frac{\Lambda}{p_4}\right)$$

Note: Λ dependence cancels against FL operators

- similar structure in vertex



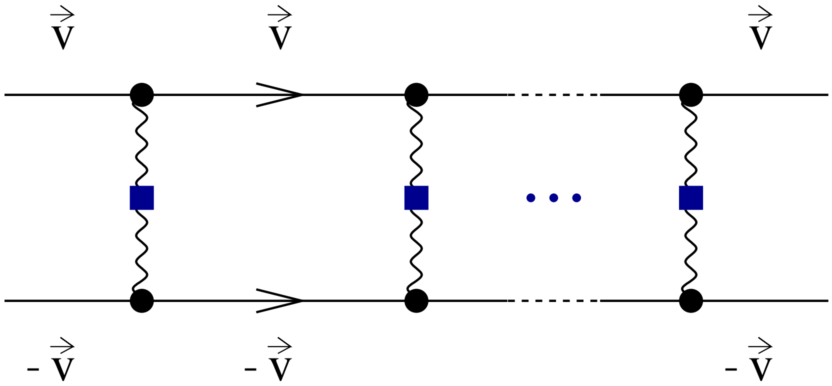


$$\lim_{(p_1)_4 \to (p_2)_4} \lim_{l_{p_1} \to l_{p_2}} \Gamma_\alpha(p_1, p_2) = \frac{g^3 C_F v_\alpha}{12\pi^2} \log\left(\frac{\Lambda}{p_4}\right)$$

natural size. But: have to resum for $E \sim \mu \exp(-1/g^2)$

Superconductivity

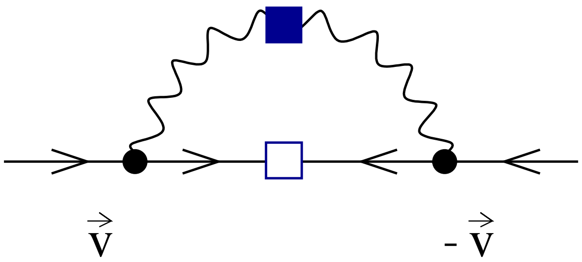
- $E \sim \mu \exp(-1/g)$: BCS ladder diagrams IR divergent



introduce gap term

$$\mathcal{L} = \Delta\, R_l^\Gamma (\vec{v}\cdot\hat{\Delta})\psi_{-v}\sigma_2\Gamma\psi_v + h.c.$$

- gap equation: only two patches (v,-v)



$$\Delta(p_4) = \frac{g^2}{18\pi^2}\int_0^{\Lambda_\|} dq_4 \frac{\Delta(q_4)}{\sqrt{q_4^2+\Delta(q_4)^2}} \log\left(\frac{\Lambda_\perp}{|p_4^2-q_4^2|^{1/2}}\right).$$

- solution (can set $\Lambda_\| = \Lambda_\perp$ at this order)

$$\Delta_0 = 2\Lambda \exp(-3\pi^2/(\sqrt{2}g))$$

Higher Order Corrections

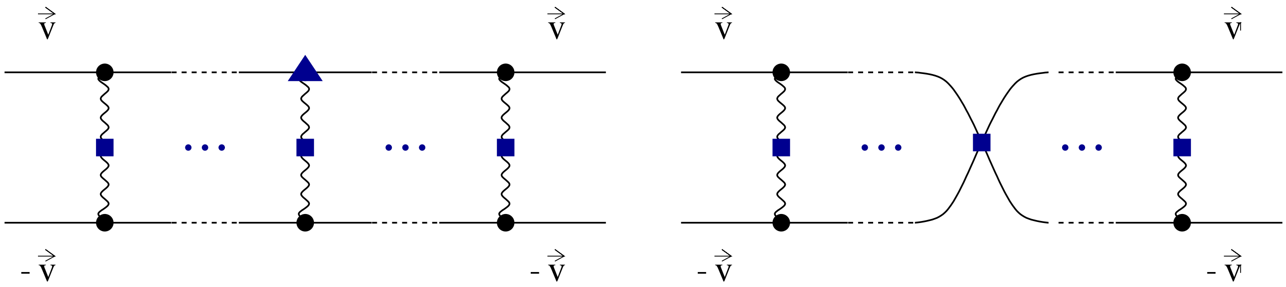
- write gap equation as an eigenvalue problem

$$\Delta(\bar{y}) = \bar{x}_0^2 \int_0^1 d\bar{x}\, \Delta(\bar{x}) K(\bar{x}, \bar{y})$$

with $x_0 = \log(2\Lambda/\Delta_0)$. Get othogonal set of solutions

$$\Delta_n(\bar{x}) = \Delta_{n,0} \sin\left(\bar{x}_{0,n}\bar{x}\right), \qquad x_{0,n} = (2n+1)\frac{\pi}{2}$$

- perturbation $K(x,y) = K_0(x,y) + \delta K(x,y)$



$$\begin{aligned} \Delta(\bar{x}) &= \Delta^{(0)}(\bar{x}) + \Delta^{(1)}(\bar{x}) + \Delta^{(2)}(\bar{x}) + \ldots, \\ \bar{x}_0 &= \bar{x}_0^{(0)} + \bar{x}_0^{(1)} + \bar{x}_0^{(2)} + \ldots \end{aligned}$$

- "Rayleigh-Schroedinger" perturbation theory

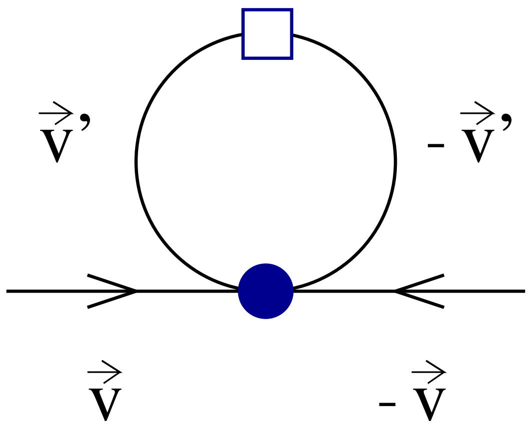
$$\bar{x}_0^{(1)} = -\frac{1}{2}\left(\bar{x}_0^{(0)}\right)^2 \int d\bar{x} \int d\bar{y}\, \Delta_0^{(0)}(\bar{x}) \delta\bar{K}(x_0\bar{x}, x_0\bar{y}) \Delta_0^{(0)}(\bar{y})$$

$$c_k^{(1)} = \frac{1}{1-\left(\frac{1}{2k+1}\right)^2} \bar{x}_0^{(0)} \int d\bar{x} \int d\bar{y}\, \Delta_0^{(0)}(\bar{x}) \delta\bar{K}(x_0\bar{x}, x_0\bar{y}) \Delta_k^{(0)}(\bar{y})$$

with $\Delta^{(1)}(x) = \Sigma\, c_k^{(1)} \Delta_k^{(0)}(x)$

Contact Terms

- corrections to kernel: four-fermion operators



sum over patches
$\Rightarrow$ no power suppression

- kernel

$$K(p_4,q_4) = \log\left(\frac{\Lambda_\perp}{|p_4^2-q_4^2|^{1/2}}\right) + \log\left(\frac{512\pi^4\mu}{g^5\Lambda_\perp}\left(\frac{2}{N_f}\right)^{\frac{5}{2}}\right)$$

cutoff dependence cancels

kernel is gauge independent

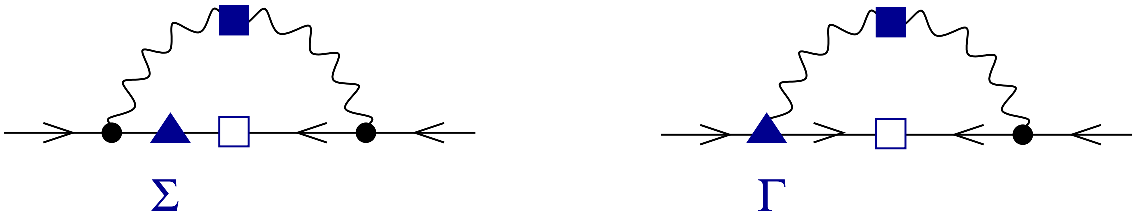
- note: particle-hole pairing (a.k.a "chiral density wave", "Overhauser pairing", "DGR state")

$$\mathcal{L} = \Sigma_0 e^{2ip_f\cdot x}\psi^\dagger_{-v}\psi_v$$

no hard loop enhancement

Non-Fermi Liquid Effects

- effect of soft loops in the kernel (a.k.a non-Fermi liquid effects)



$$\delta\bar{K}(x_0\bar{x}, x_0\bar{y}) = -\frac{g^2}{9\pi^2}(x_0\bar{x})\,\bar{K}_0(x_0\bar{x}, x_0\bar{y})$$

- correction to eigenvalue

$$\bar{x}_0^{(1)} = \frac{1}{2}\left(\bar{x}_0^{(0)}\right)^2 \langle 0|\delta\bar{K}|0\rangle = \frac{4+\pi^2}{8}\frac{g}{3\sqrt{2}\pi}$$

- perturbative expansion

$$\bar{x}_0 \sim g\log(\Delta) = O(g^0) + O(g\log(g)) + O(g) + \ldots$$

- result to $O(g)$

$$\boxed{\Delta = 512\pi^4\mu g^{-5} e^{-\frac{4+\pi^2}{8}} e^{-\frac{3\pi^2}{\sqrt{2}g}}}$$

W. E. Brown, J. T. Liu and H. c. Ren (2000)

Effective Chiral Theory

- oscillations of the order parameter

$$X_i^a = \epsilon_{ijk}\epsilon^{abc}(\psi_L)_j^b(\psi_L)_k^c, \qquad \langle X_i^a\rangle \sim \delta_i^a$$

$$Y_i^a = \epsilon_{ijk}\epsilon^{abc}(\psi_R)_j^b(\psi_R)_k^c, \qquad \langle Y_i^a\rangle \sim \delta_i^a$$

- low energy degrees of freedom

$$\Sigma = XY^\dagger = \exp\left(\frac{i\phi^a\lambda^a}{f_\pi}\right) \qquad \phi^a = (\pi, K, \eta, \eta')$$

$$\text{e.g. } K^0 \sim \epsilon^{abc}\epsilon_{ade}(\bar{u}_R^b C\bar{s}_R^c)(d_L^d C u_L^e)$$

- effective theory

$$\begin{aligned}\mathcal{L} \;=\; & \tfrac{f_\pi^2}{4}\mathrm{Tr}\left(\nabla_0\Sigma\nabla_0\Sigma^\dagger - v^2\vec{\nabla}\Sigma\vec{\nabla}\Sigma^\dagger\right) \\ & + A\mathrm{Tr}(M\Sigma^\dagger)Ve^{i\Theta} \qquad \leftarrow \quad U_A(1) \text{ anomaly} \\ & + B_1\left[\mathrm{Tr}(M\Sigma^\dagger)\right]^2 + \ldots\end{aligned}$$

- expansion

$$\mathcal{L} \sim f_\pi^2\Delta^2\left(\frac{\vec{\partial}}{\Delta}\right)^{N_1}\left(\frac{\partial_0 + MM^\dagger/p_F}{\Delta}\right)^{N_2}\left(\frac{MM}{p_F^2}\right)^{N_3}(\Sigma)^{N_4}(\Sigma^\dagger)^{N_4}$$

Note:

$$\Lambda_{\chi SB} \sim \Delta \ll 4\pi f_\pi$$

$$\frac{M^2}{p_F^2} \ll \frac{M^2}{p_F\Delta}$$

Quark Mass expansion

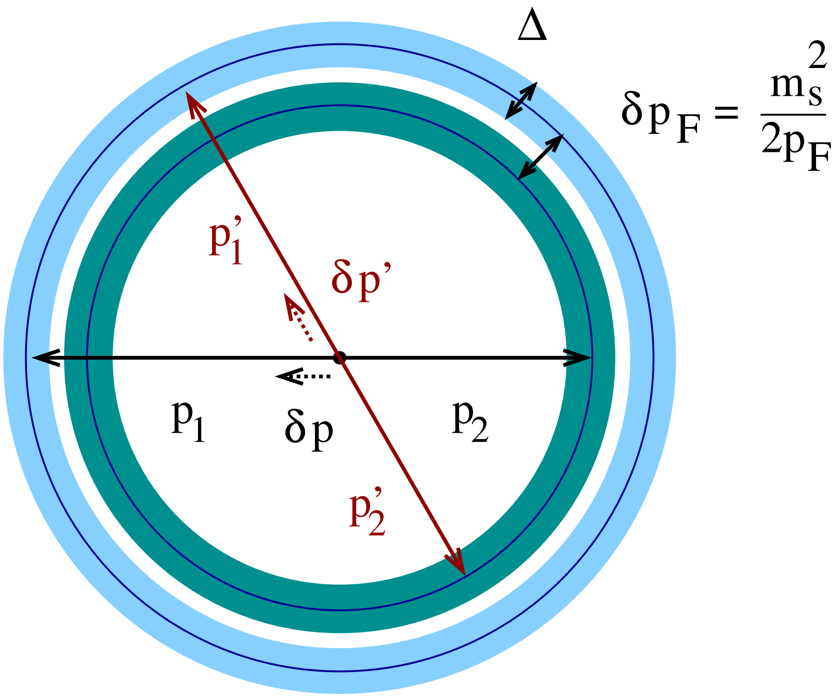
- expansion parameter

$$(m/p_F), (m/\Delta), (m/\Lambda), \ldots ?$$

- two parameters appear in matching procedure

$$\left(\frac{m^2}{p_F\Delta}\right), \quad \left(\frac{m^2}{p_F^2}\right) = \left(\frac{\Delta}{p_F}\right)\left(\frac{m^2}{p_F\Delta}\right)$$

- scale $m^2/(p_F\Delta)$ appears in BCS: consider pairing between two species with different Fermi momenta



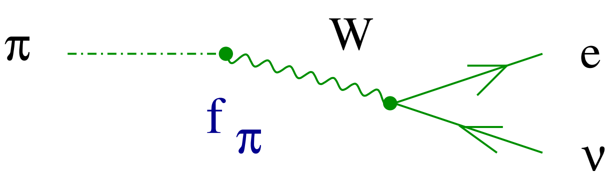
$$\delta p_f \sim \Delta \longrightarrow \text{unlocking transition}$$

$$\Rightarrow \text{mass expansion breaks down if } \frac{\delta m^2}{p_F\Delta} \sim 1$$

Matching, part I

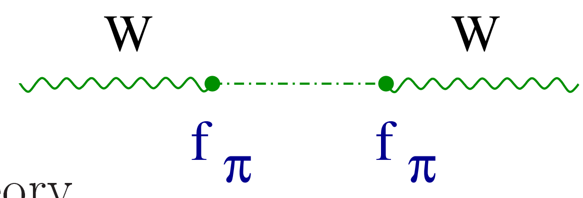
- Compute f_π: Gauge $SU(3)_L \times SU(3)_R$ flavor symmetry

$$\partial_\mu \Sigma \to \nabla_\mu \Sigma = \partial_\mu \Sigma - i W_\mu^L \Sigma + i \Sigma W_\mu^R$$

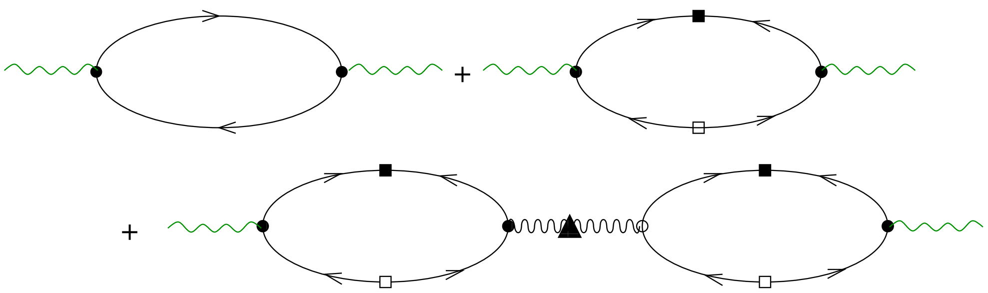


- Higgs phenomenon

$$\mathcal{L} = \frac{f_\pi^2}{4} \mathrm{Tr}\left[(W_0^L - W_0^R)^2 \right] + \ldots \qquad m_W^2 = f_\pi^2$$



- microscopic theory



$$\boxed{f_\pi^2 = \frac{21 - 8\log(2)}{18} \left(\frac{\mu^2}{2\pi^2} \right)}$$

- Note: $4\pi f_\pi \sim p_F \gg \Delta$

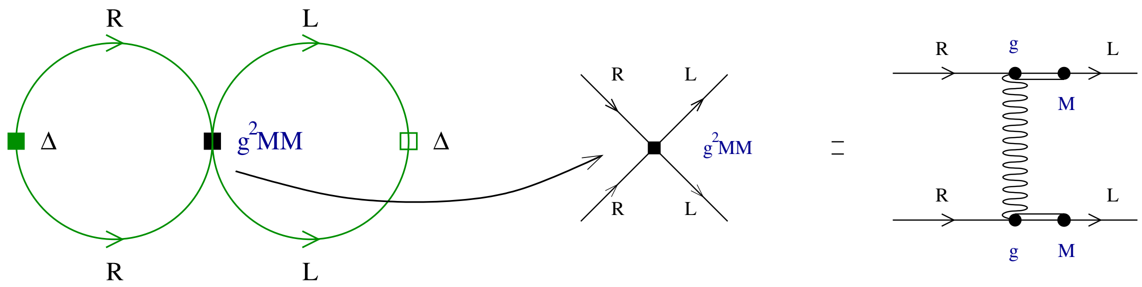
$$\rho \sim 5\rho_0 \;\Rightarrow\; p_F \sim 500\,\mathrm{MeV} \;\Rightarrow\; f_\pi \sim 100\,\mathrm{MeV}$$

Matching, part II

- Compute m dependence of vacuum energy

$$\Delta\mathcal{E} = -B_1 \left[\mathrm{Tr}(M)\right]^2 - B_2 \mathrm{Tr}(M^2) \qquad \Sigma = 1$$

- microscopic theory



$$\Delta\mathcal{E} \;\sim\; \left(\frac{g^2}{p_F^4}\right)\left[p_F^2\Delta\log(\Delta)\right]^2\left\{[\mathrm{Tr}(M)]^2 - \mathrm{Tr}(M^2)\right\}$$
$$\sim \Delta^2\left\{[\mathrm{Tr}(M)]^2 - \mathrm{Tr}(M^2)\right\}$$

$$B_1 = -B_2 = \frac{3\Delta^2}{4\pi^2} \qquad \left[\;\sim f_\pi^2\Delta^2\left(\frac{m^2}{p_F^2}\right)\;\right]$$

- meson masses

$$\boxed{\begin{aligned} m_\pi^2 &= \frac{3\Delta^2}{4f_\pi^2}(m_u+m_d)m_s \\ m_{K^\pm}^2 &= \frac{3\Delta^2}{4f_\pi^2}(m_u+m_s)m_d \end{aligned}}$$

- Note:

$$m_{GB} \sim 10 \text{ MeV} \qquad m_K < m_\pi$$

Matching, part III

- consider $1/p_F$ expansion

$$\mathcal{L} = \psi_L^\dagger \left(p_0 - \epsilon_p - \frac{MM^\dagger}{2p_F} \right) \psi_L + \frac{\Delta}{2} \psi_L C \psi_L$$

$$+ \, (L \leftrightarrow R, M \leftrightarrow M^\dagger) + O(1/p_F^2)$$

- $MM^\dagger$ and $M^\dagger M$ enter as gauge fields

$$W_L = \tfrac{MM^\dagger}{2p_F} \qquad \psi_L \to L\psi_L, \;\; \psi_R \to R\psi_R$$
$$W_R = \tfrac{M^\dagger M}{2p_F} \qquad W_L \to L W_L L^\dagger + i L \partial_0 L^\dagger, \ldots$$

- implement gauge symmetry in effective lagrangian

$$\mathcal{L} = \frac{f_\pi^2}{4} \mathrm{Tr} \left(\nabla_0 \Sigma \nabla_0 \Sigma^\dagger \right) + \ldots$$

$$\nabla_0 \Sigma = \partial_0 \Sigma + i \frac{MM^\dagger}{2p_F} \Sigma - i \Sigma \frac{M^\dagger M}{2p_F}$$

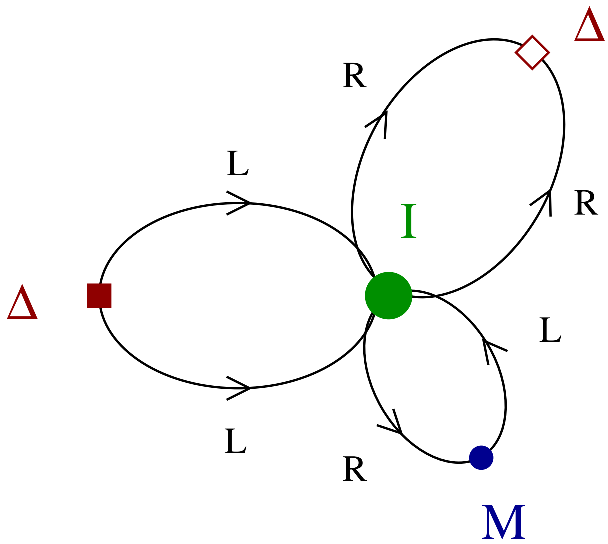
acts like an effective chemical potential $\hat{\mu} = MM^\dagger/(2p_F)$

- Note:

$$\frac{\nabla_0}{\Delta} \sim \left(\frac{\partial_0}{\Delta}, \frac{MM^\dagger}{p_F \Delta} \right)$$

Matching, anomalous part

- linear term $\mathrm{Tr}(M\Sigma)$ in vacuum energy related to instantons (violates $(Z_2)_A$ symmetry)



Instanton size

$$\rho \sim \mu^{-1} \ll \Lambda_{QCD}^{-1}$$

- instanton contribution to vacuum energy: $\mathcal{E} = A\mathrm{Tr}(M)$

$$A = C_{N_c}^{N_f} \langle \psi C \psi \rangle^2 \left(b \log \left(\frac{\mu}{\Lambda_{QCD}} \right) \right)^6 \left(\frac{\Lambda_{QCD}}{\mu} \right)^1 2\Lambda_{QCD}^{-3} e^{i\Theta}$$

$$\langle \psi C \psi \rangle = \frac{3\sqrt{2}\pi}{g} \Delta \left(\frac{\mu^2}{2\pi^2} \right)$$

- Note: $\langle \bar{\psi}\psi \rangle = -A$

$$\langle \bar{\psi}\psi \rangle \sim \left(\frac{\Lambda_{QCD}}{\mu} \right)^8 \Lambda_{QCD}^{-3} \ll \Lambda^{-3}$$

Summary: High Density Effective Theories

- Effective theories provide useful tools for systematic calculations in the regime $\Lambda < \Delta < \mu$.
- naive power counting modified by "hard loops"
- important questions: renormalization of four-fermion operators, counting and resumming logs