Instantons in QCD

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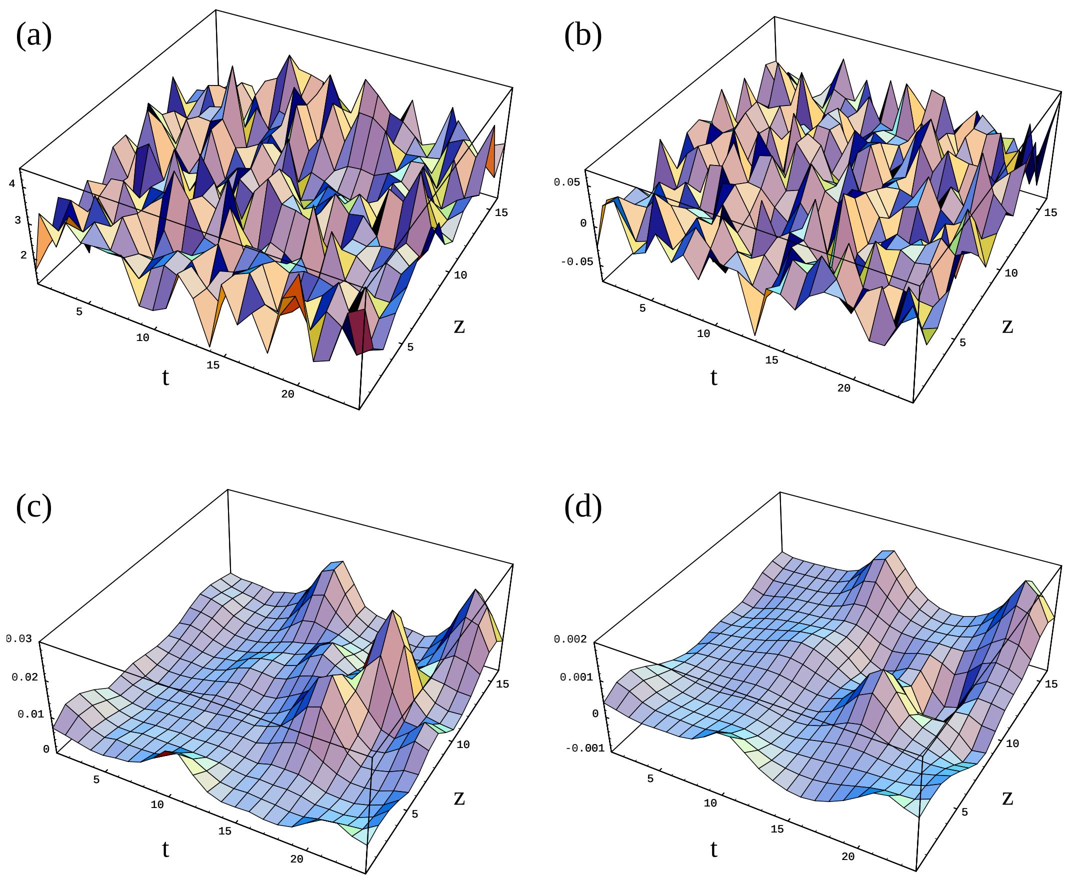
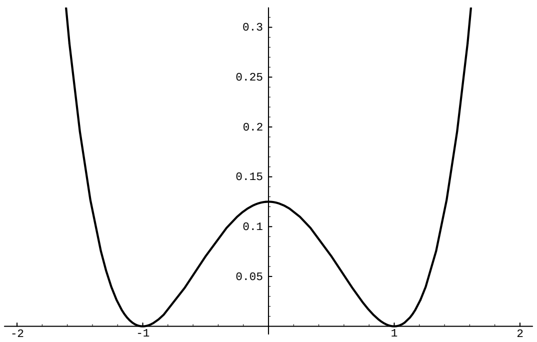


Instantons in Quantum Mechanics

- Consider particle moving in a potential



$$H = T + V$$

$$V = \lambda(x^2 - \eta^2)^2$$

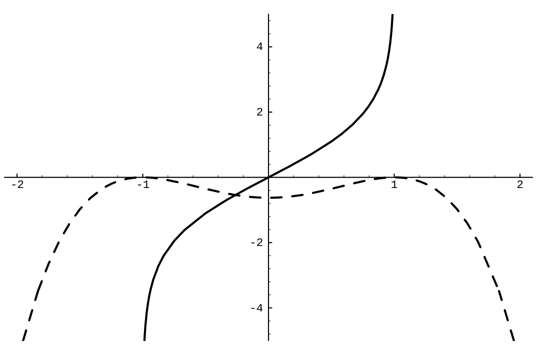
- Physics: degenerate states are mixed by tunneling, $E_{1,2} \simeq E_0 \pm \Delta E$.
- Not a classical process, but there are classical contributions to the euclidean ($\tau = it$, $e^{iS} \to e^{-S_E}$) partition function

$$Z = \mathrm{tr}(e^{-\beta H}) = \int Dx(\tau)\, e^{-S_E} \qquad S_E = \int_0^\beta d\tau \left(\frac{1}{2}m\dot{x}^2 + V(x)\right)$$

- saddle point approximation

$$\langle x|e^{-H\tau}|y\rangle = e^{-S_0} \int dx(\tau)\, e^{-\frac{1}{2}xS''x+\ldots}$$

- saddle points $\frac{\delta S}{\delta x(\tau)} = 0 \Rightarrow$ equation of motion $m\ddot{x} = V'$



$$\frac{1}{2}m\dot{x}^2 - V(x) = 0$$

$$x_{cl}(\tau) = \eta \tanh\left[\frac{\omega}{2}(\tau - \tau_0)\right]$$

- This solution has euclidean action

$$\boxed{S = \frac{\omega^3}{12\lambda}}$$

- $S \gg 1$ ($S \gg \hbar$): semi-classical expansion

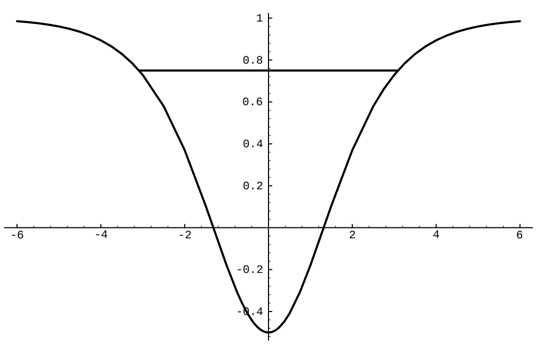
$$\langle -\eta | e^{-H\tau} | \eta \rangle = e^{-S_0} \int Dx(\tau)\, \exp\left(-\frac{1}{2} \delta x \left. \frac{\delta^2 S}{\delta x^2} \right|_{x_{cl}} \delta x + \ldots \right)$$

- pre-exponent = fluctuations around the classical path

$$P \sim \det(O)^{-1/2} \exp(-S)$$

$$O = -\frac{1}{2} \frac{d^2}{d\tau^2} + \left. \frac{d^2 V}{dx^2} \right|_{x = x_{cl}}$$

- Have $\det(O) = \Pi_n \, \epsilon_n$ with



$$\left(-\frac{d^2}{d\tau^2} + \omega^2 \left[1 - \frac{3}{2\cosh^2(\omega\tau/2)} \right] \right) x_n(\tau) = \epsilon_n x_n(\tau)$$

- Problem: there is a zero mode $\Rightarrow$ tunneling rate $= \infty$!

$$x_0(\tau) = \sqrt{\frac{3\omega}{8}} \frac{1}{\cosh^2(\omega\tau/2)}$$

- Zero mode is due to translational invariance. Write

$$x_0(\tau) = -S_0^{-1/2} \frac{d}{d\tau_0} x_{cl}(\tau - \tau_0)$$

- Integrate over τ_0 exactly. Write $x(\tau) = \Sigma_n c_n x_n(\tau)$

$$\begin{aligned} dx &= x_0 dc_0 + \ldots \\ dx &= \frac{dx_{cl}}{d\tau_0} d\tau_0 = -\sqrt{S_0} x_0(\tau) d\tau_0 \end{aligned}$$

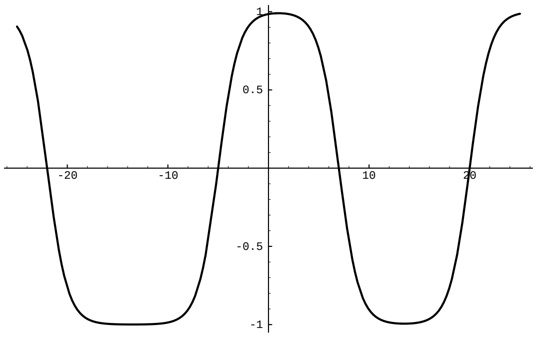
- This implies $dc_0 = \sqrt{S_0} d\tau_0$ and

$$\det(O)^{-1/2} = \left[\prod_{n>0} \left(\frac{2\pi}{\epsilon_n}\right)\right]^{1/2} \sqrt{S_0} \int d\tau_0,$$

- non-zero mode determinant (hard work)

$$\langle -\eta | e^{-H\tau_m} | \eta \rangle = \left[\sqrt{\frac{\omega}{\pi}} \exp\left(-\frac{\omega\tau_m}{2}\right)\right] \left[\sqrt{\frac{6S_0}{\pi}} \exp\left(-S_0\right)\right] \left(\omega\tau_m\right),$$

- valid for $\omega^{-1} \ll \tau \ll \Delta E^{-1}$. For level spacing, have to go to large time $\Rightarrow$ multi-instantons



- For a dilute system of instantons

$$\begin{aligned}\langle -\eta | e^{-H\tau_m} | \eta \rangle &= \sqrt{\frac{\omega}{\pi}} e^{-\omega\tau_m/2} \sum_{n\,\mathrm{odd}} \int_{-\tau_m/2<\tau_1<\ldots<\tau_m/2} \left[\prod_{i=1}^{n} \omega d\tau_i\right] \left(\sqrt{\frac{6S_0}{\pi}} \exp(-S_0)\right)^n \\ &= \sqrt{\frac{\omega}{\pi}} e^{-\omega\tau_m/2} \sum_{n\,\mathrm{odd}} \frac{(\omega\tau_m d)^n}{n!} = \sqrt{\frac{\omega}{\pi}} e^{-\frac{\omega\tau_m}{2}} \sinh(\omega\tau_m d)\,,\end{aligned}$$

- Compare to

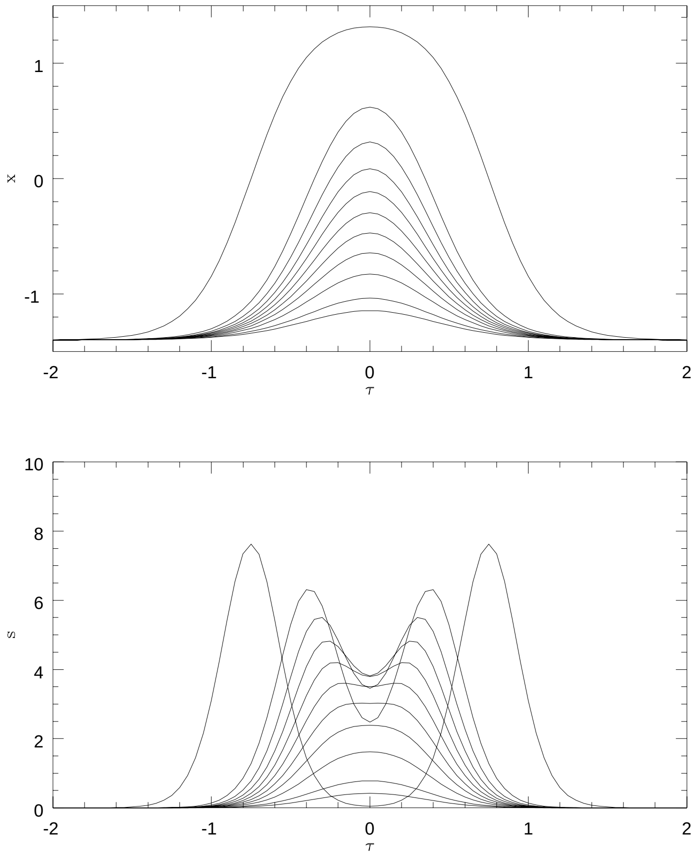
$$\begin{aligned}\langle -\eta | e^{-H\tau_m} | \eta \rangle &= \psi_0^\dagger(-\eta)\psi_0(\eta) e^{-\tau_m E_0} + \psi_1^\dagger(-\eta)\psi_1(\eta) e^{-\tau_m E_1} + \ldots \\ &= 2\psi_0^\dagger(-\eta)\psi_0(\eta) e^{-\tau_m \bar{E}} \sinh(\tau_m \Delta E)\end{aligned}$$

from which we finally read off the level spacing

$$\Delta E = \sqrt{\frac{6S_0}{\pi}} \omega \exp(-S_0)$$

- If the tunneling rate is small, instantons are well separated, and everything is under control. But if $\Delta E \sim \omega$, instantons start to overlap

- Approximate instanton-anti-instanton solution: "streamline" or "valley" method



$$f(\lambda)\frac{dx_\lambda(\tau)}{d\lambda} = \frac{\delta S}{\delta x_\lambda(\tau)}$$

- IA-contribution related to large-order perturbation theory: $\bar{E} = \sum_k \bar{E}_k g^{2k}$ $(g^2 = 2\lambda)$

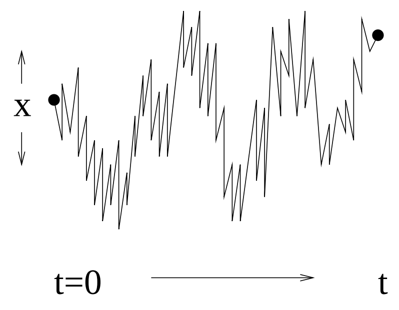
$$\bar{E}_k = \frac{1}{\pi}\int_0^\infty dg^2\, \frac{\mathrm{Im}E_k(g^2)}{g^{2k+2}}$$

$$\bar{E}_k = \frac{3^{k+1}k!}{\pi}\left(1 - \frac{53}{18k} + \dots\right)$$

Correlation Functions

- consider coordinate correlation function

$$\Pi(\tau) = \langle x(\tau)x(0)\rangle$$



- spectral representation

$$\Pi(\tau) = \sum_n |\langle 0|x|n\rangle|^2 e^{-\Delta E_n \tau}$$

dipole matrix element $\times$ exponential decay

- energy gap

$$\Delta E_1 = -\frac{d}{d\tau}\log(\Pi(\tau))$$

Lattice Quantum Mechanics

- Partition Function (euclidean space)

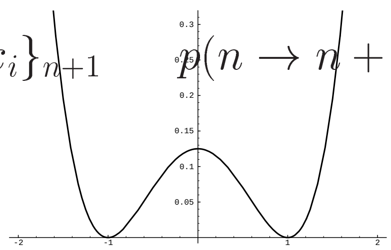
$$Z = \mathrm{tr}(e^{-\beta H}) = \int Dx(\tau)\, e^{-S} \qquad S = \int d\tau\, \left(\frac{1}{2}m\dot{x}^2 + V(x)\right)$$

- Lattice version: discretize time $x(\tau = na) = x_n$

$$S = \sum_n \left[\frac{m}{2a}(x_n - x_{n-1})^2 + aV(x_n)\right]$$

- Consider double well potential

$$\{x_i\}_n \to \{x_i\}_{n+1} \qquad p(n \to n+1) = \exp(-\Delta S)$$



- Typical lattice configuration

- Can be extracted using cooling (or more sophisticated techniques)

$$\{x_i\}_n \to \{x_i\}_{n+1} \qquad S_n > S_{n+1} \ \ (\hbar \to 0)$$

- Effective description: Tunneling events (instantons)

$$\omega = -\frac{d}{d\tau}\log(\Pi(\tau))$$

Quantum Chromo Dyanmics

- lagrangian

$$L = \frac{1}{4g^2} G^a_{\mu\nu} G^a_{\mu\nu} + \bar{\psi}(i\not{D} + im)\psi$$

- field strength tensor and covariant derivative

$$\begin{aligned} G_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \\ D_\mu \psi &= (\partial_\mu + iA_\mu)\psi, \qquad A_\mu = A^a_\mu \frac{\lambda^a}{2} \end{aligned}$$

- gauge transformations

$$\begin{aligned} A_\mu &\to U^\dagger A_\mu U + iU^\dagger \partial_\mu U \\ G_{\mu\nu} &\to U^\dagger G_{\mu\nu} U \\ \psi &\to U\psi \end{aligned}$$

- electric and magnetic fields

$$E_i = G_{0i}, \qquad B_i = \frac{1}{2}\epsilon_{ijk} G_{jk}$$

- dual field strength tensor

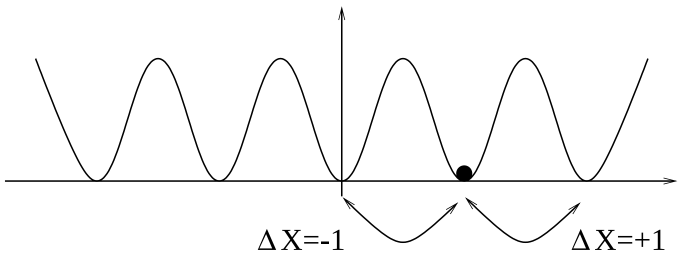
$$\tilde{G}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} G_{\alpha\beta} \qquad E_i \leftrightarrow B_i$$

- action, topological charge density

$$\begin{aligned} \frac{1}{4} G_{\mu\nu} G_{\mu\nu} &= \frac{1}{2}\left(E^2 + B^2\right) \\ \frac{1}{4} G_{\mu\nu} \tilde{G}_{\mu\nu} &= E \cdot B \end{aligned}$$

Tunneling in QCD

- Classical potential is periodic in variable X



$$X = \int d^3x \, K_0(x,t)$$

$$\partial^\mu K_\mu = \frac{1}{32\pi^2} G^a_{\mu\nu} \tilde{G}^a_{\mu\nu}$$

- classical minima correspond to pure gauge configurations ($A_0 = 0$)

$$A_i(x) = iU^\dagger(x) \partial_i U(x)$$

- Tunneling processes change X by $\Delta X = \pm 1$

$$\Delta X = X(t = +\infty) - X(t = -\infty) = \frac{1}{32\pi^2} \int d^4x \, G^a_{\mu\nu} \tilde{G}^a_{\mu\nu} = Q_{top}$$

Instanton solution in QCD

- Use

$$S = \frac{1}{4g^2} \int d^4x \, G^a_{\mu\nu} G^a_{\mu\nu} \; = \; \int d^4x \left[\pm \frac{1}{4g^2} G^a_{\mu\nu} \tilde{G}^a_{\mu\nu} + \frac{1}{8g^2} \left(G^a_{\mu\nu} \mp \tilde{G}^a_{\mu\nu} \right)^2 \right],$$

where $\tilde{G}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} G_{\rho\sigma}$ is the dual field strength tensor.

- last term is always positive, first one is a topological invariant $\Rightarrow$ action is minimal if the field is (anti) self-dual

$$\boxed{G^a_{\mu\nu} = \pm \tilde{G}^a_{\mu\nu},}$$

Can show $G^a_{\mu\nu} = \pm \tilde{G}^a_{\mu\nu} \Rightarrow D_\mu G_{\mu\nu} = 0$.

- The action of a solution with topological charge Q is

$$S = \frac{8\pi^2}{g^2}|Q|$$

- $x \to \infty$: winding number 1 configuration. Use hedgehog ansatz

$$A_\mu = iU\partial_\mu U^\dagger, \qquad U = i\hat{x}_\mu \tau_\mu^+, \qquad \tau_\mu^\pm = (\vec{\tau}, \mp i).$$

The corresponding gauge potential is

$$A_\mu^a = 2\frac{\eta_{a\mu\nu}x_\nu}{x^2}, \qquad \eta_{a\mu\nu} = \begin{cases} \epsilon_{a\mu\nu} & \mu,\, \nu = 1,\, 2,\, 3, \\ \delta_{a\mu} & \nu = 4, \\ -\delta_{a\nu} & \mu = 4. \end{cases}$$

- ansatz $A_\mu^a = 2\eta_{a\mu\nu}x_\nu f(x^2)/x^2$. Self-duality requires

$$f(1-f) - x^2 f' = 0$$

$$\Rightarrow f = \frac{x^2}{(x^2+\rho^2)}$$

- gauge potential and field strength

$$A_\mu^a(x) = 2\frac{\eta_{a\mu\nu}x_\nu}{x^2+\rho^2}, \qquad (G_{\mu\nu}^a)^2 = \frac{192\rho^4}{(x^2+\rho^2)^4}.$$

Collective coordinates

- Coordinates: position z, size ρ, and gauge rotations $A^a_\mu \to R^{ab}A^b_\mu$.

$$A^a_\mu(x) = 2R^{ab}\frac{\eta_{b\mu\nu}(x-z)_\nu}{(x-z)^2+\rho^2}$$

  Instantons live in $SU(2)$ subgroup $\Rightarrow$ not all gauge transformations rotate the instanton. (#coordinates) $= 4N_c$.

- ordinary rotations $\equiv$ gauge rotations
- Coordinate inversion: Not a new solution, but singular gauge transform of the regular solution

$$A^a_\mu(x) = 2\frac{x_\nu}{x^2}\frac{\overline{\eta}_{a\mu\nu}\rho^2}{x^2+\rho^2}.$$

  solution is well behaved at infinity, but has a singularity at the origin.

Theta vacua

- Instantons connect the topological vacua $|n\rangle$. Correct groundstate is a linear superposition
$$|\theta\rangle = \sum_n e^{in\theta}|n\rangle$$
- For a dilute system of instantons
$$E(\theta) = -2K\cos(\theta)\exp(-S)$$
- The state $|\theta = 0\rangle$ has the lowest energy, but in general θ can take any value
$$\mathcal{L} = \frac{i\theta}{32\pi^2} G^a_{\mu\nu}\tilde{G}^a_{\mu\nu}$$
- $\theta \neq 0$ leads to CP violations, but in nature, $\theta < 10^{-9}$. Exactly why, nobody knows (The strong CP problem).

Tunelling rate

- semi-classical approximation

$$P_{tunneling} \sim \exp(-8\pi^2/g^2)(\det O)^{-1/2}(\det \not{D}\,)^{N_f}$$

- pre-exponent = fluctuations around the instanton. There are $4N_c$ zero modes $\Rightarrow (\sqrt{S})^{4N_c} d\rho\, dz\, dU$

$$dn_I \sim \left(\frac{8\pi^2}{g^2}\right)^{2N_c} \exp\left(-\frac{8\pi^2}{g^2}\right) \rho^{-5} d\rho\, dz\, dU,$$

- Pauli-Vilars regulator $\det(O)/\det(O+M^2)$. Zero modes give M^{4Nc}. Non-zero modes give $\sim \log(M\rho)$

$$dn_I \sim \exp\left(-\frac{8\pi^2}{g^2} + 4N_c \log(M\rho) - \frac{N_c}{3}\log(M\rho)\right) \rho^{-5} d\rho dz_\mu \;\equiv\; \exp\left(-\frac{8\pi^2}{g^2(\rho)}\right) \rho^{-5} d\rho dz_\mu,$$

where we have introduced the running coupling

$$\frac{8\pi^2}{g^2(\rho)} = \frac{8\pi^2}{g^2} - \frac{11N_c}{3}\log(M\rho)$$

- remaining constants (hard work! $\Rightarrow$ 't Hooft)

$$dn_I = \frac{0.466\exp(-1.679N_c)}{(N_c-1)!(N_c-2)!} \left(\frac{8\pi^2}{g^2}\right)^{2N_c} \exp\left(-\frac{8\pi^2}{g^2(\rho)}\right) \frac{d^4z d\rho}{\rho^5}.$$

- one-loop beta function

$$\frac{dn_I}{d^4z} \sim \frac{d\rho}{\rho^5}(\rho\Lambda)^b.$$

b is large $\Rightarrow$ small size instantons are strongly suppressed. But: rate appears to diverge at large ρ

Phenomenology

- tunneling lowers the ground state energy

$$\epsilon = -\frac{b}{4}\left(\frac{N}{V}\right), \qquad b = \frac{11}{3}N_c - \frac{2}{3}N_f$$

- QCD vacuum energy $\epsilon \simeq -0.5\mathrm{GeV}/\mathrm{fm}^3$. Expect

$$\left(\frac{N}{V}\right) \simeq 1\,\mathrm{fm}^{-4}$$

- estimate typical ρ from size distribution

$$\left(\frac{N}{V}\right) = \int d\rho\, n(\rho) \qquad \Rightarrow \rho \simeq 0.3\,\mathrm{fm}$$

- instanton liquid is dilute

$$\left(\frac{N}{V}\right)\rho^4 = 0.01$$

$\Rightarrow$ inhomogenous vacuum!

- topological susceptibility

$$\chi_{top} = \frac{\langle Q_{top}^2\rangle}{V} = \frac{f_\pi^2}{2N_f}(m_{\eta'}^2 + m_\eta^2 - 2m_K^2) \simeq (180\,\mathrm{MeV})^2 \quad (\mathrm{Witten-Veneziano})$$

- dilute instanton gas

$$\chi_{top} = \frac{\langle (N_I - N_A)^2\rangle}{V} \simeq \left(\frac{N}{V}\right) \qquad \Rightarrow \left(\frac{N}{V}\right) \simeq 1\,\mathrm{fm}^{-4}$$

Mean field approximation

- Interacting instanton ensemble (pure gauge theory)

$$Z = \frac{1}{N_+!N_-!}\prod_i^{N_++N_-}\int [d\Omega_i\, n(\rho_i)]\, \exp(-S_{int}).$$

$\Omega_i = (z_i, \rho_i, U_i)$: collective coordinates

- MFA: Use trial function given by the product of single instanton distributions $\mu(\rho)$

$$Z_1 = \frac{1}{N_+!N_-!}\prod_i^{N_++N_-}\int d\Omega_i\, \mu(\rho_i) = \frac{1}{N_+!N_-!}(V\mu_0)^{N_++N_-}$$

- Variational principle gurantees that we get an upper bound on the vacuum energy

$$Z = Z_1\langle \exp(-(S-S_1))\rangle \;\geq\; Z_1 \exp(-\langle S-S_1\rangle),$$

- Need the average interaction. In the sum ansatz

$$\langle S_{int}\rangle = \gamma^2\rho_1^2\rho_2^2, \qquad \gamma^2 = \frac{27}{4}\frac{N_c}{N_c^2-1}\pi^2$$

- Variational principle gives

$$\mu(\rho) = n(\rho)\exp\left[-\frac{\beta\gamma^2}{V}N\overline{\rho^2}\rho^2\right],$$

$\Rightarrow$ size integral regularized by repulsive interaction.

- Final result

$$\left(\frac{N}{V}\right) = \Lambda_{PV}^4 \left[C_{N_c}\beta^{2N_c}\Gamma(\nu)(\beta\nu\gamma^2)^{-\nu/2}\right]^{\frac{2}{2+\nu}},$$

Note that Λ_{PV} is the only dimensionful parameter.

$$\chi_{top} \simeq \left(\frac{N}{V}\right) = (0.65\Lambda_{PV})^4, \quad (\overline{\rho^2})^{1/2} = 0.47\Lambda_{PV}^{-1},$$

- Free energy

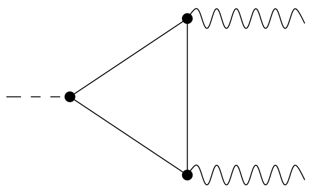
$$F = \frac{N}{V}\left\{\log\left(\frac{N}{2V\mu_0}\right) - \left(1+\frac{\nu}{2}\right)\right\}.$$

(N/V) determined by $\partial F/(\partial(N/V)) = 0$. Vacuum energy $\epsilon = F_0$

$$\epsilon = -\frac{b}{4}\left(\frac{N}{V}\right)$$

Instantons and Light Quarks

- Fermions are intimately related with tunneling: Different topological sectors have different axial charge!
- ABJ discovered the famous anomaly in the axial current



$$\partial_\mu j^5_\mu = \frac{N_F}{16\pi^2} G^a_{\mu\nu}\tilde{G}^a_{\mu\nu}$$

- Integrating this equation, we find

$$\Delta Q_5 = Q_5(t=+\infty) - Q_5(t=-\infty) \;=\; \int d^4x\, \partial_\mu j^5_\mu \;=\; 2N_f Q_{top}$$

- Calculate ΔQ_5 from mode expansion of quark propagator

$$S(x,y) = \sum_\lambda \frac{\psi_\lambda(x)\psi^\dagger_\lambda(y)}{\lambda}$$

Get

$$\begin{aligned}\Delta Q_5 &= \int d^4x\, N_f\; \partial_\mu \left[\mathrm{tr}\left(S(x,x)\gamma_\mu\gamma_5\right)\right] \\ &= N_f \int d^4x\, \mathrm{tr}\left(\sum_\lambda \frac{\psi_\lambda(x)\psi^\dagger_\lambda(x)}{\lambda} 2\lambda\gamma_5\right) \\ &= 2N_f(n_L - n_R)\end{aligned}$$

$\Rightarrow \Delta Q_5 =$ (# left handed) $-$ (# right handed) zero modes

- (Anti)Instantons: Dirac operator has a left (right) handed zero mode.

- To show this, note that

$$(i\not{D})^2 = \left(-D^2 + \frac{1}{2}\sigma_{\mu\nu}G_{\mu\nu}\right)$$

$$\sigma_{\mu\nu}G^{(\pm)}_{\mu\nu} = \mp\gamma_5\sigma_{\mu\nu}G^{(\pm)}_{\mu\nu}$$

- Dirac equation in a self dual field

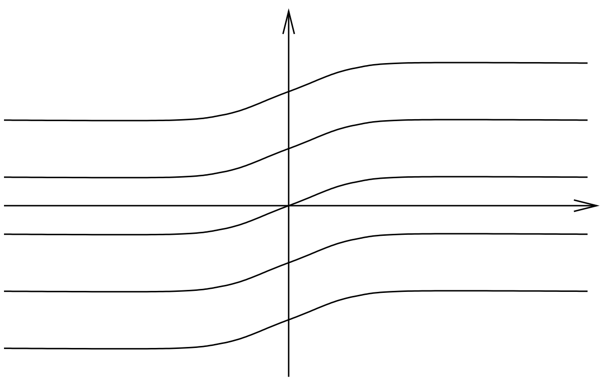
$$\left(-D^2 + \frac{1}{2}\sigma_{\mu\nu}G^{(+)}_{\mu\nu}\right)\chi_L = 0, \qquad -D^2\chi_R = 0,$$

- $\Rightarrow$ zero mode is left-handed. More work:

$$\psi_0(x) = \frac{\rho}{\pi}\frac{1}{(x^2+\rho^2)^{3/2}}\frac{\gamma\cdot x}{\sqrt{x^2}}\frac{1+\gamma_5}{2}\phi$$

with $\phi_{ia} = \epsilon_{ia}$

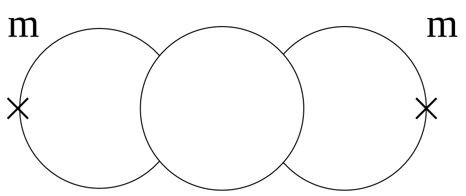
- Note: Axial charge violation = level crossing:



$$\psi_i(\vec{x},t) = \psi_i(\vec{x},t=-\infty)\exp\left(-\int_{-\infty}^{t} dt'\,\epsilon_i(t')\right).$$

Tunneling amplitude

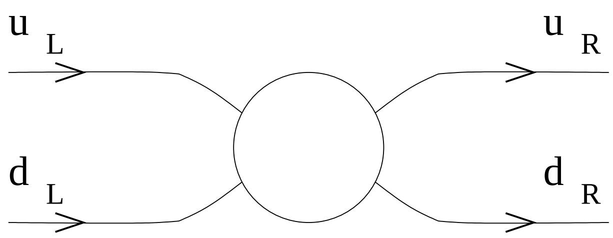
- tunneling amplitude (with fermions)



$$\det\left[\not{D}+m\right] = m^{N_f}\prod_{\lambda>0}(\lambda^2+m^2)^{N_f}$$

zero in the chiral limit!

- No surprise: The different vacua have different axial charge!
- Consider a $2N_f$ fermion correlation function



$$\langle \prod_f \bar{\psi}_f(x_f)\Gamma\psi_f(y_f)\rangle \sim (tr(S(x,y)\Gamma))^{N_f}$$

this amplitude is $m^{N_f}\frac{1}{m^{N_f}} \sim O(1)$.

- represent as effective $2N_f$ quark interaction ('t Hooft)

$$\begin{aligned}\mathcal{L}_{N_f=2} \;=\; & \int d\rho\, n_0(\rho)\left[\prod_f \left(m\rho - \frac{4}{3}\pi^2\rho^3 \bar{q}_{f,R}q_{f,L}\right)\right.\\ & \left. + \frac{3}{32}\left(\frac{4}{3}\pi^2\rho^3\right)^2 \left(\bar{u}_R\lambda^a u_L \bar{d}_R\lambda^a d_L - \frac{3}{4}\bar{u}_R\sigma_{\mu\nu}\lambda^a u_L \bar{d}_R\sigma_{\mu\nu}\lambda^a d_L\right)\right],\end{aligned}$$

- Chiral and flavor structure

$$\boxed{\mathcal{L} \sim \det_{fg}(\bar{\psi}^f_L\psi^g_R) + (L \leftrightarrow R)}$$

- Implies

$\sigma, \vec{\pi}$	attractive
$\eta', \vec{a}_0$	repulsive
$\vec{\rho}, \vec{a}_1$	no interaction

Chiral symmetry breaking I

- Quark condensate given by

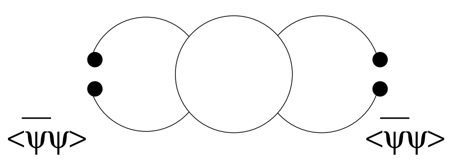
$$\langle \bar{q}q \rangle = i \int d^4x \, \mathrm{tr} \left(S(x,x) \right) .$$

- Consider $N_f = 1$: Zero mode contribution is $\langle \bar{q}q \rangle = -m^{-1}$. THen

$$\langle \bar{q}q \rangle = -\frac{1}{m} \left(\frac{N}{V} \right)$$

  Since $(N/V) \sim (\rho m)$, $\langle \bar{q}q \rangle \simeq (N/V)_0$.

- $N_f > 1$: $\langle \bar{q}q \rangle$ is still $(N/V)/m$, but $(N/V) \sim m^{N_f}$.
- Consider effective lagrangian (for $N_f = 2$)



$$\begin{aligned} \mathcal{L} &= \int d\rho \, n_0(\rho) \left[\prod_f \left(m\rho - \frac{4}{3}\pi^2\rho^3 \bar{q}_{f,R} q_{f,L} \right) \right. \\ &+ \left. \frac{3}{32} \left(\frac{4}{3}\pi^2\rho^3 \right)^2 \left(\bar{u}_R \lambda^a u_L \bar{d}_R \lambda^a d_L - \ldots \right) \right] , \end{aligned}$$

  $\Rightarrow$ if chiral symmetry is broken, $(N/V) \sim \langle \bar{q}q \rangle^2$

- Replace $m^{N_f} \to (m^*)^{N_f}$ with the effective quark mass

$$m_f^* = m_f - \frac{2}{3}\pi^2\rho^2 \langle \bar{q}_f q_f \rangle .$$

- Single instanton contribution $\langle \bar{q}q \rangle = -1/m^*$. Many instantons

$$\langle \bar{q}q \rangle = -\frac{1}{m^*} \left(\frac{N}{V} \right)$$

- Two equations for m^* and $\langle \bar{q}q \rangle \Rightarrow$ self-consistent result

$$\langle \bar{q}q \rangle = -\frac{1}{\pi\rho} \sqrt{\frac{3N}{2V}}$$

  Use $(N/V) = 1 \, \mathrm{fm}^{-4}$ and $\rho = 0.33 \, \mathrm{fm}$, $\Rightarrow \langle \bar{q}q \rangle \simeq -(215\mathrm{MeV})^3$,

Chiral symmetry breaking II

- Idea: Study the motion of quarks in given A_μ, then average over A_μ
  $\Rightarrow$ Chiral symmetry breaking is related to spectral density of $\not{D}$
- Diagonalize Dirac operator

$$i\not{D}\psi_\lambda = \lambda\psi_\lambda,$$

  where ψ_λ is an eigenstate with "virtuality" λ.

- quark propagator $S(x,y) = -\langle x|i\not{D}^{-1}|y\rangle$ is given by

$$S(x,y) = -\sum_\lambda \frac{\psi_\lambda(x)\psi_\lambda^\dagger(y)}{\lambda + im}$$

  quark condensate

$$i\int d^4x\, \mathrm{tr}\,(S(x,x)) = -i\sum_\lambda \frac{1}{\lambda + im}.$$

- Note: $\quad i\not{D}\psi_\lambda = \lambda\psi_\lambda \quad \Rightarrow \quad i\not{D}(\gamma_5\psi_\lambda) = -\lambda(\gamma_5)\psi_\lambda$
- non-zero eigenvalues are paired

$$i\int d^4x\, \mathrm{tr}\,(S(x,x)) = -\sum_{\lambda\geq 0} \frac{2m}{\lambda^2 + m^2}.$$

- Average over A_μ, introduce spectral density $\rho(\nu) = \langle \sum_\lambda \delta(\nu - \lambda)\rangle$.

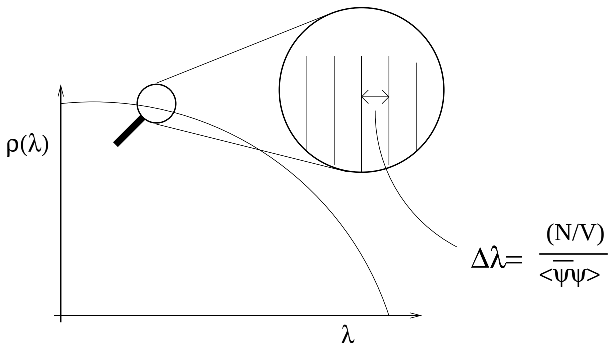
$$\langle\bar{q}q\rangle = -\int_0^\infty d\lambda\, \rho(\lambda)\frac{2m}{\lambda^2 + m^2}.$$

- Take $V \to \infty$ first, then $m \to 0$

$$\langle\bar{q}q\rangle = -\pi\rho(\lambda = 0), \quad (\mathrm{Banks-Casher})$$

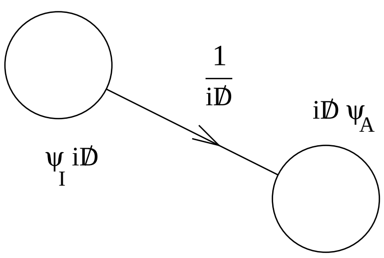
  $\Rightarrow$ chiral symmetry breaking connected with "almost" zero modes

• zero mode basis



$$i\not{D} = \begin{pmatrix} 0 & T_{IA} \\ T_{AI} & 0 \end{pmatrix},$$

• overlap matrix elements T_{IA}



$$\begin{aligned} T_{IA} &= \int d^4x\, \psi^\dagger_{0,I}(x-z_I) i\not{D} \psi_{0,A}(x-z_A) \\ &= i(u\cdot R) \frac{1}{\rho_I\rho_A} \frac{4.0}{(2.0 + R^2/(\rho_I\rho_A))^2}. \end{aligned}$$

• width of the zero mode zone ∼ average matrix element

$$\langle |T_{IA}|^2 \rangle = \frac{2\pi^2}{3N_c} \frac{N\rho^2}{V}.$$

• Gaussian ensemble ⇒ spectral density is a semi-circle

$$\rho(\lambda) = \frac{N}{\pi\sigma V} \left(1 - \frac{\lambda^2}{4\sigma^2} \right)^{1/2}.$$

• Casher-Banks gives

$$\langle \bar{q} q \rangle = -\frac{1}{\pi\rho} \left(\frac{3N_c}{2} \frac{N}{V} \right)^{1/2} \simeq -(240\,\mathrm{MeV})^3,$$

Chiral symmetry breaking III

- Opposite strategy: Average over the gauge fields first, study effective quark interaction
- Rewrite the partition function of the instanton liquid

$$Z = \frac{1}{N_+!N_-!}\prod_i^{N_++N_-}\int[d\Omega_i\, n(\rho_i)]\,\exp(-S_{int})\,\det(\not{D}+m)^{N_f}$$

  in terms of a fermionic effective action

$$\begin{aligned} Z &= \int d\psi d\psi^\dagger \exp\left(\int d^4x\, \psi^\dagger(i\not\partial + im)\psi\right)\left\langle \prod_{I,f}(\Theta_I - im_f)\prod_{A,f}(\Theta_A - im_f)\right\rangle, \\ \Theta_{I,A} &= \int d^4x\, \left(\psi^\dagger(x) i\not\partial \phi_{I,A}(x-z_{I,A})\right)\int d^4y\, \left(\phi^\dagger_{I,A}(y-z_{I,A}) i\not\partial \psi(y)\right), \end{aligned}$$

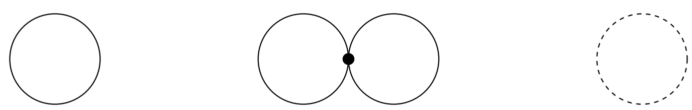
  which describes quarks interacting via the 't Hooft vertices $\Theta_{I,A}$.

- Vertices $\Theta_{I,A}$ can be exponentiated using standard tricks. For $N_f = 2$

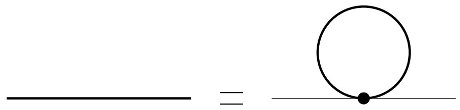
$$\mathcal{L} = \beta(2\pi\rho)^4 \frac{1}{4(N_c^2-1)}\left(\frac{2N_c-1}{2N_c}\left[(\psi^\dagger\tau_a^-\psi)^2 + (\psi^\dagger\gamma_5\tau_a^-\psi)^2\right] + \ldots\right).$$

- Free energy $F = -1/V \cdot \log Z$. In MFA $[M(k) = \beta\sigma k^2\varphi'^2(k)]$

$$F = 4N_c\int \frac{d^4k}{(2\pi)^4}\log\left(k^2 + \beta\sigma k^2\varphi'^2(k)\right) - 2\frac{2N_c(N_c^2-1)}{2N_c-1}\beta\sigma^2 - \frac{N}{V}\log\left(\frac{\beta V}{N}\right).$$



- Varying with respect to $\beta = \beta_\pm$ gives the gap equation



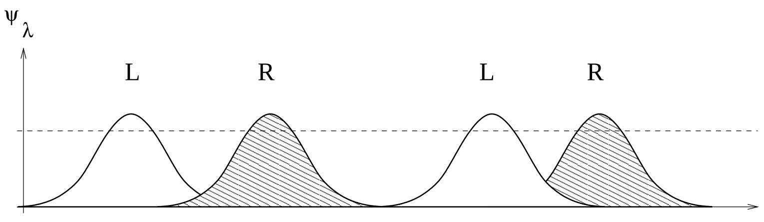
$$\boxed{\int \frac{d^4k}{(2\pi)^4} \frac{M^2(k)}{k^2 + M^2(k)} = \frac{N}{4N_c V},}$$

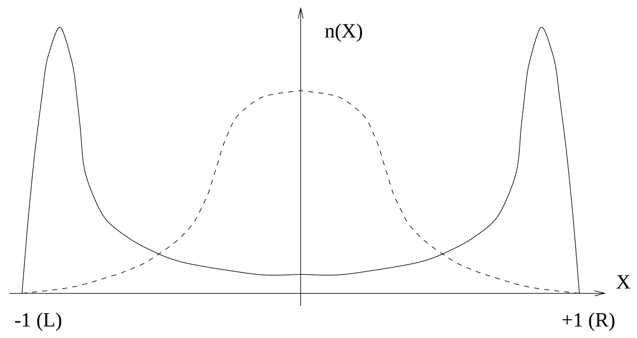
- Find $M \simeq 350$ MeV. For small N/V, have $M(0) \sim \rho(\frac{N}{2VN_c})^{1/2}$. Quark condensate

$$\langle \bar{q}q \rangle = -4N_c \int \frac{d^4k}{(2\pi)^4} \frac{M(k)}{M^2(k) + k^2} \; = \; -(255\,\mathrm{MeV})^3$$

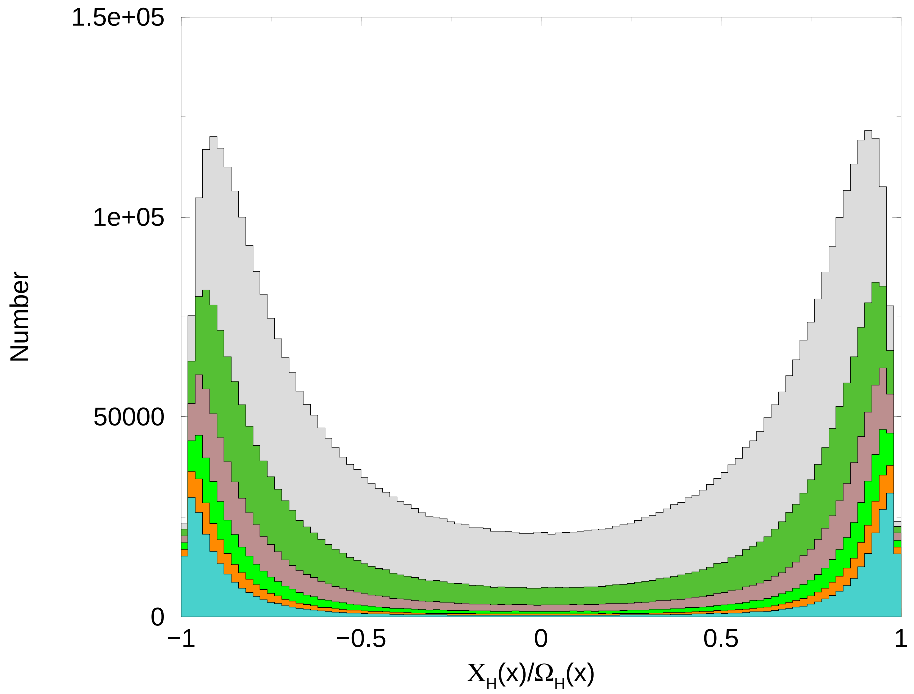
Chiral Symmetry Breaking on the Lattice

- Chirality distribution function



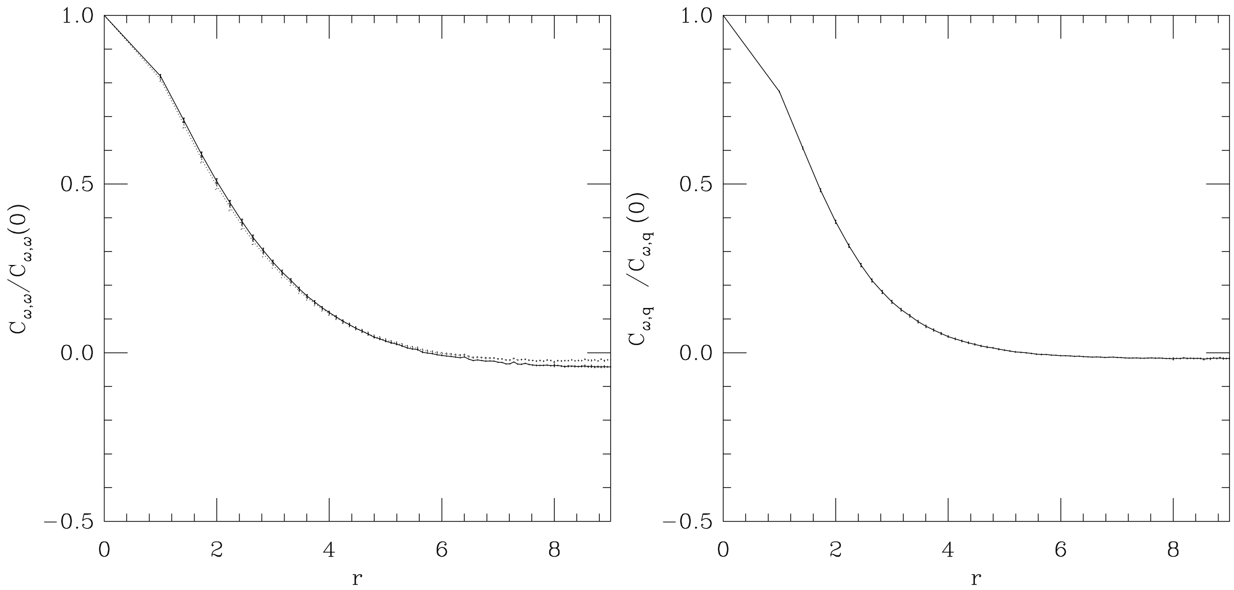


- Chirality distribution measured using overlap fermions



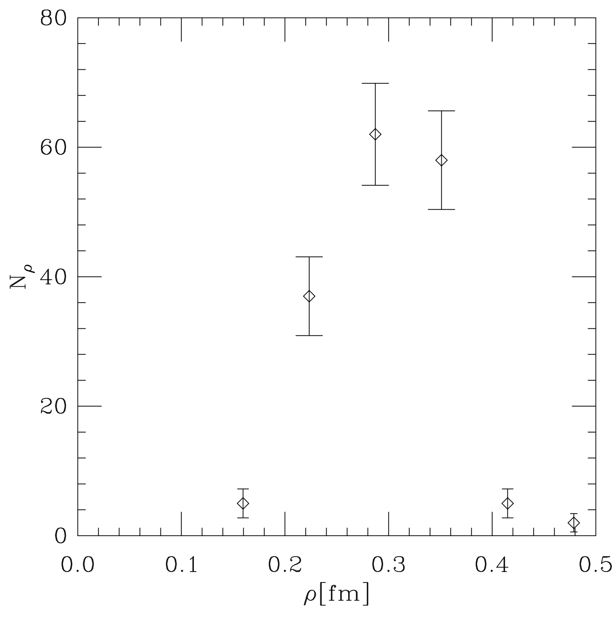
T. Blum et al., [hep-lat/0105006]

• chirality-chirality and chirality-topology correlation function



T. DeGrand, A. Hasenfratz, Phys.Rev.D64:034512,2001, [hep-lat/0012021]

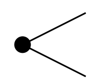
• size distribution



Correlation Functions in QCD

- pion source function

$$j(x) = \bar{u}(x)\gamma_5 d(x)$$

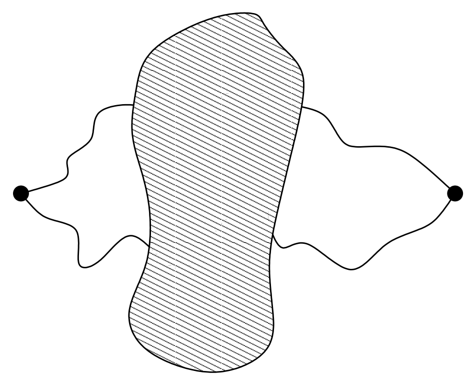


- correlation function

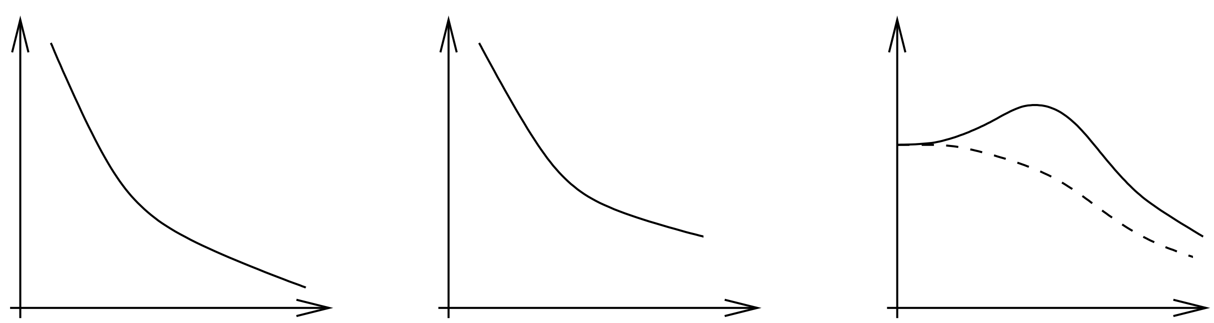
$$\Pi(x) = \langle j(0)j(x) \rangle$$

$$= P(\text{pion propagates from 0 to } x)$$

$$\sim \exp(-m_\pi x)$$



- normalize to non-interacting quarks



Correlation Functions

- hadronic currents

$$\begin{aligned} j_{mes}(x) &= \delta^{ab}\bar{\psi}^a(x)\Gamma\psi^b(x) \\ j_{had}(x) &= \epsilon^{abc}[\psi^a(x)C\Gamma\psi^b(x)]\Gamma'\psi^c(x) \\ j_{glue}(x) &= G^a_{\mu\nu}(x)G^a_{\mu\nu}(x) \end{aligned}$$

- Correlation functions

$$\Pi_{had}(x) = \langle 0|j_{had}(x)j_{had}(0)|0\rangle$$

- Spectral representation

$$\Pi_{had}(x) = \int ds\, \rho(s)D(\sqrt{s},x), \qquad D(m,x) = \frac{m}{4\pi^2 x}K_1(mx)$$

- flavor non-singlet mesons

$$\Pi_{NS}(x) = \langle \mathrm{Tr}\left[S^{ab}(0,x)\Gamma S^{ba}(x,0)\Gamma\right]\rangle$$

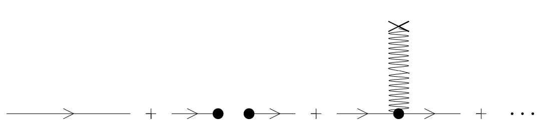
- flavor singlet mesons

$$\begin{aligned} \Pi_S(x) &= \langle \mathrm{Tr}\left[S^{ab}(0,x)\Gamma S^{ba}(x,0)\Gamma\right]\rangle \\ &\qquad - 2\langle \mathrm{Tr}\left[S^{aa}(0,0)\Gamma\right] \mathrm{Tr}\left[S^{bb}(x,x)\Gamma\right]\rangle \end{aligned}$$

- free propagator

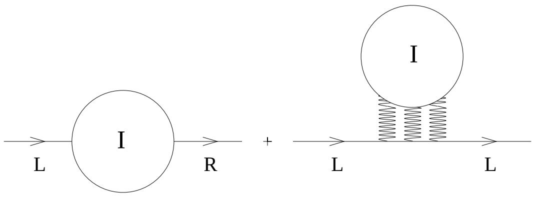
$$S_0(x) = \frac{i}{2\pi^2}\frac{\gamma\cdot x}{x^4}$$

- short distances: Operator Product Expansion (OPE)



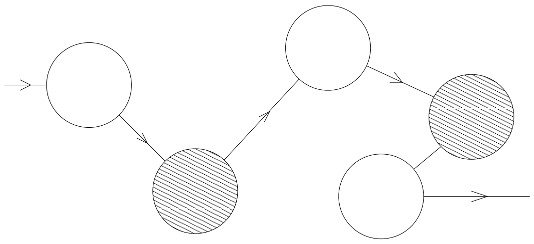
$$\begin{aligned} S(x) &= S_0(x) + q^a(0)\bar{q}^b(0) \\ &\qquad + \frac{\gamma\cdot x\sigma_{\alpha\beta} + \sigma_{\alpha\beta}\gamma\cdot x}{32\pi^2 x^2} i(\lambda^k)^{ab}G^k_{\alpha\beta} + \ldots \end{aligned}$$

- direct instanton contribution



$$S(x,y) = \frac{\psi_0(x)\psi_0^\dagger(y)}{m} + S_{NZM}(x,y)$$

- zero mode zone

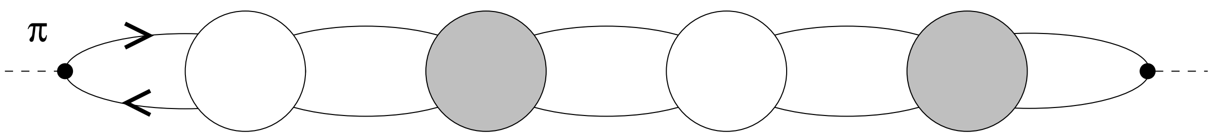


$$S(x,y) = \sum_{IJ} \psi_I(x) \left(\frac{1}{T+im}\right)_{IJ} \psi_J^\dagger(y)$$

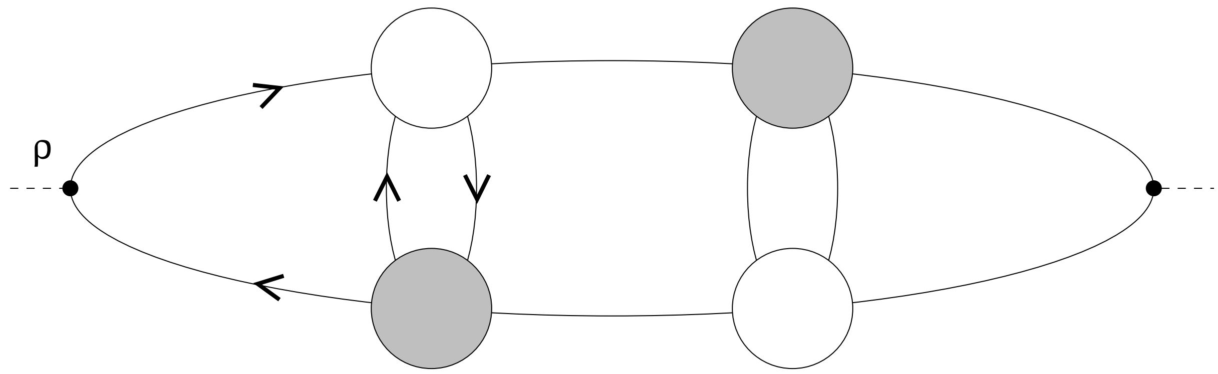
- Meson channels with direct instanton contribution: pion $\bar{u}_L d_R - \bar{u}_R d_L$
  Sum instanton chains using RPA

$$\Pi_\pi(x) = \int d^4x\, e^{iq\cdot x}\Gamma_5(q)\frac{1}{1-C_5(q)}\Gamma_5(q)$$

$$C_5(q) = 4N_c \left(\frac{N}{V}\right)^{-1} \int \frac{d^4p}{(2\pi)^4}\frac{M_1M_2(M_1M_2 - p_1\cdot p_2)}{(M_1^2+p_1^2)(M_2^2+p_2^2)}$$

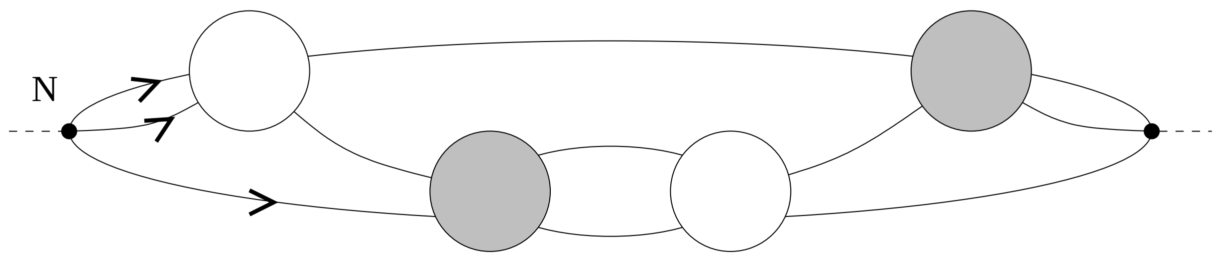


- Meson channels without direct instant contribution:



rho meson $\bar{u}_R \gamma_\mu d_R + \bar{u}_L \gamma_\mu d_L$

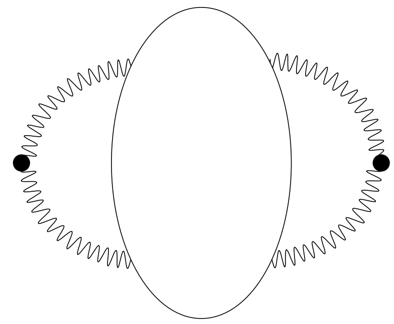
- Baryon channel with direct instanton contribution:



nucleon $(u_L C d_L) u_R$

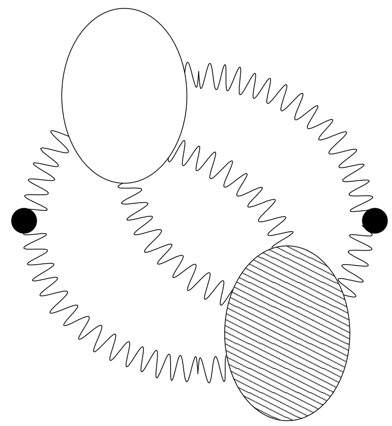
- Delta baryon: $(u_L C \gamma_\mu u_L) u_L$

- Glueball channel with direct instanton contribution:



scalar glueball $j_s = G^a_{\mu\nu} G^a_{\mu\nu}$

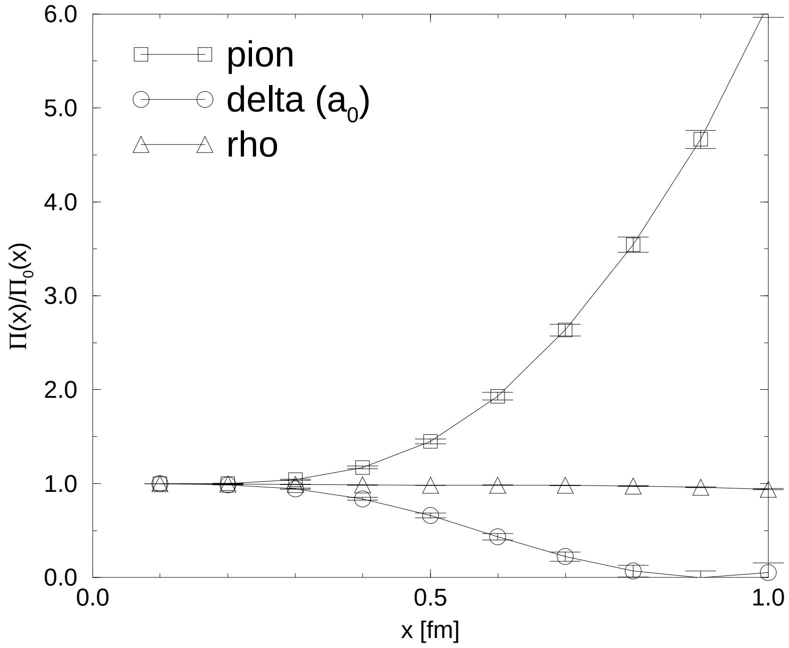
- Glueball channel without direct instanton contribution:



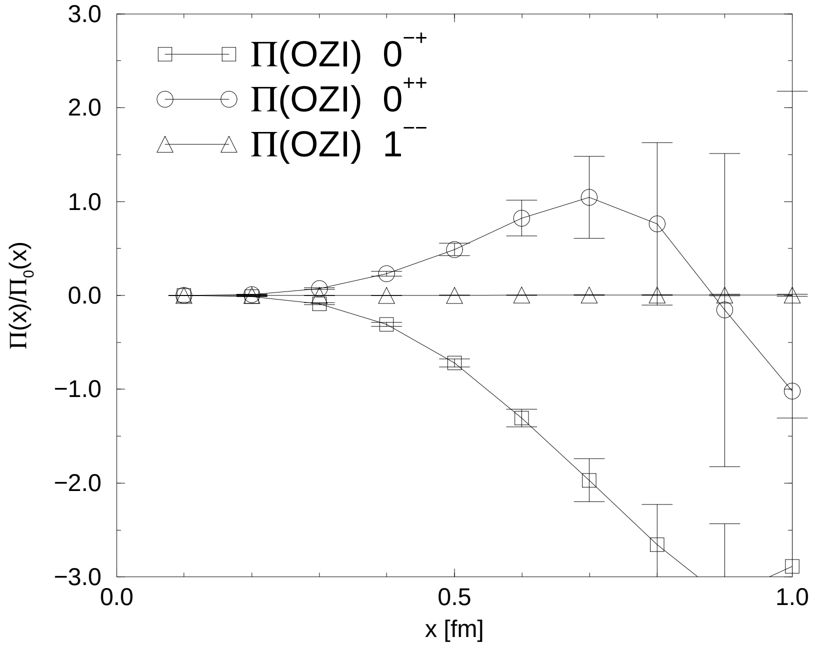
tensor glueball $j_T = T_{00} = (E^a)^2 - (B^a)^2$

- Note: $T_{\mu\nu} = 0$ for general self-dual field.
  But: Couples via trace anomaly: $\langle T_{\mu\mu} \rangle = \frac{b}{32\pi^2} \langle g^2 G^2 \rangle$

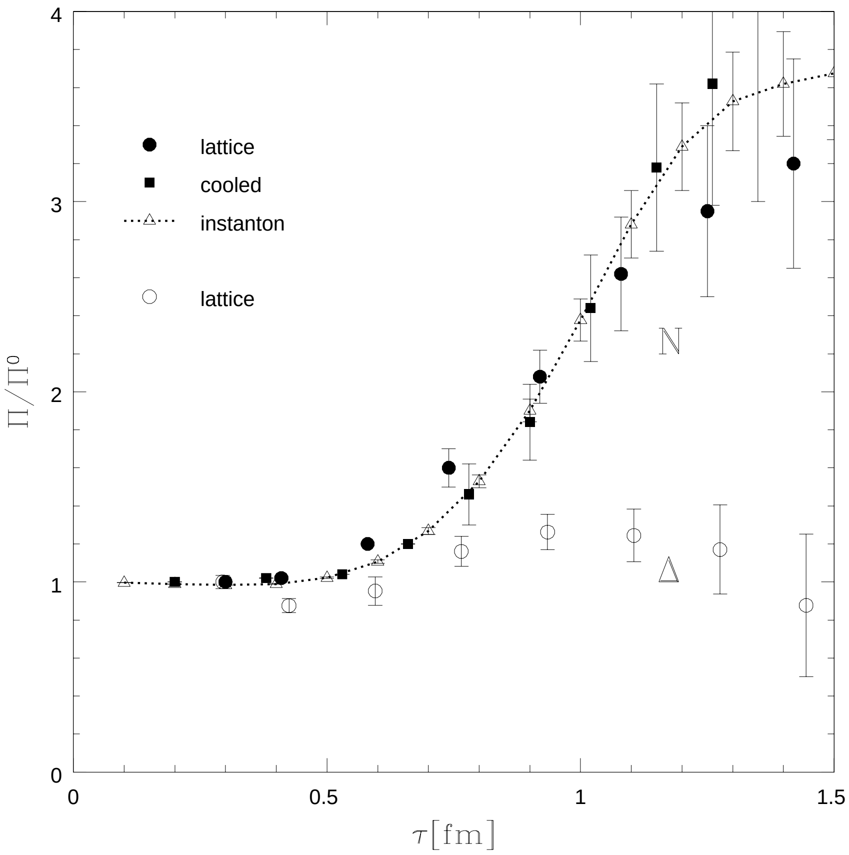
• flavor non-singlet mesons: π, a_0 and ρ



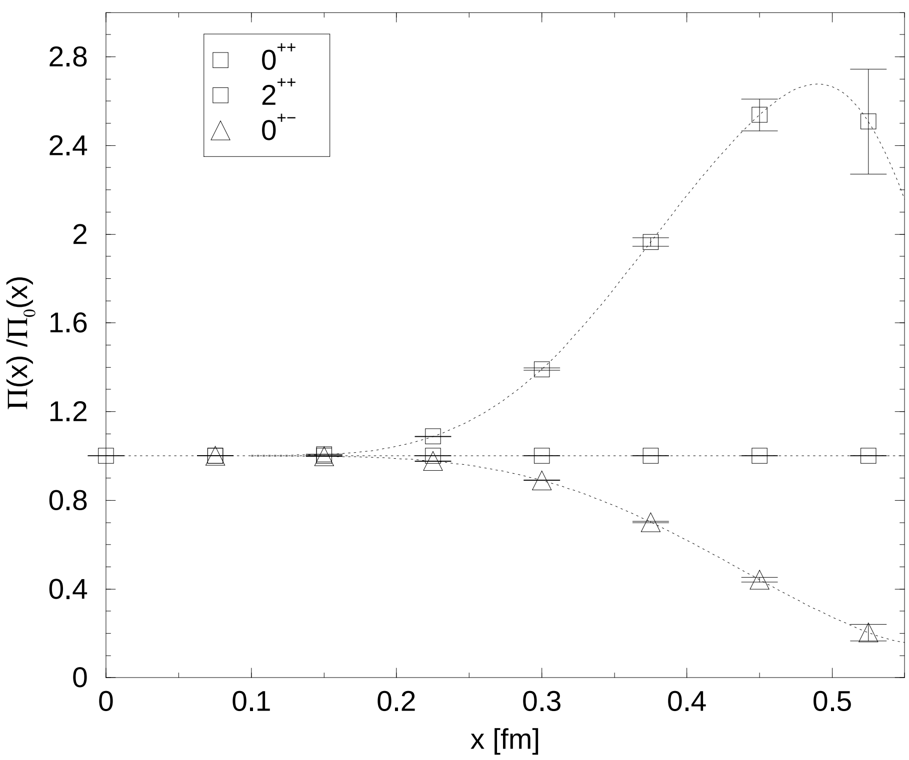
• disconnected, OZI violating, contribution: σ, η' and ω



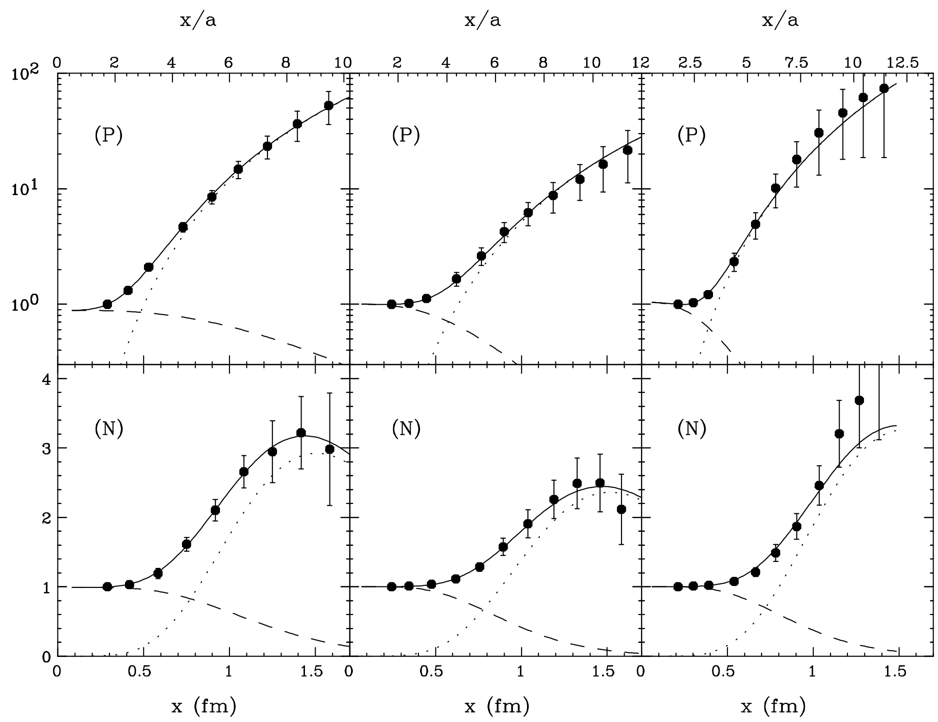
• baryons: nucleon and delta



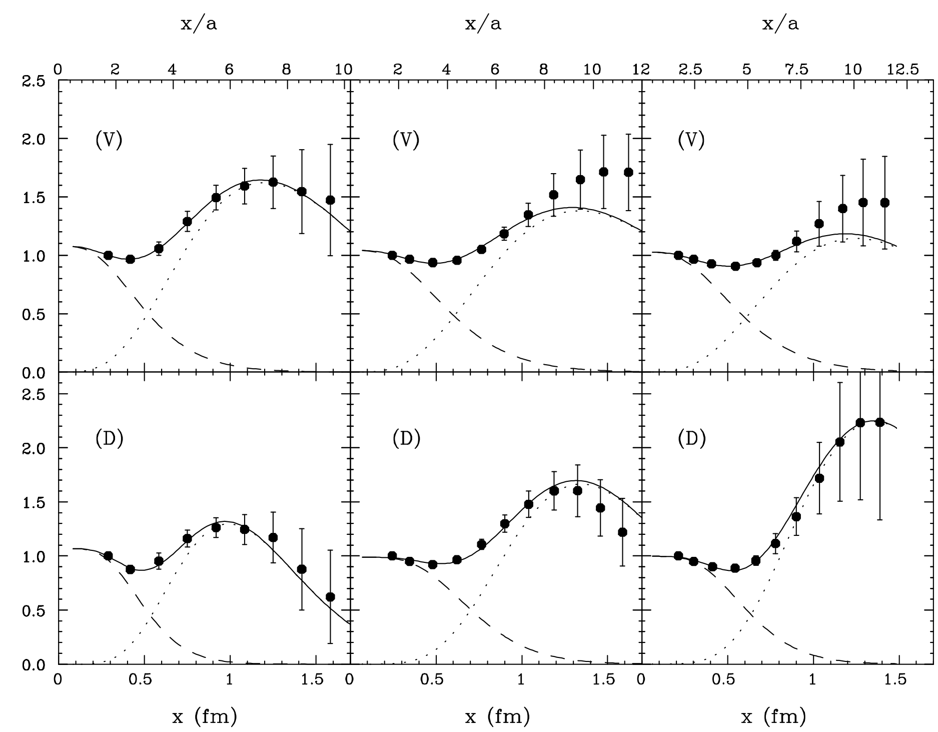
• glueballs: 0^{++}, 0^{+-} and 2^{++}



- cooled correlation functions: pion and proton



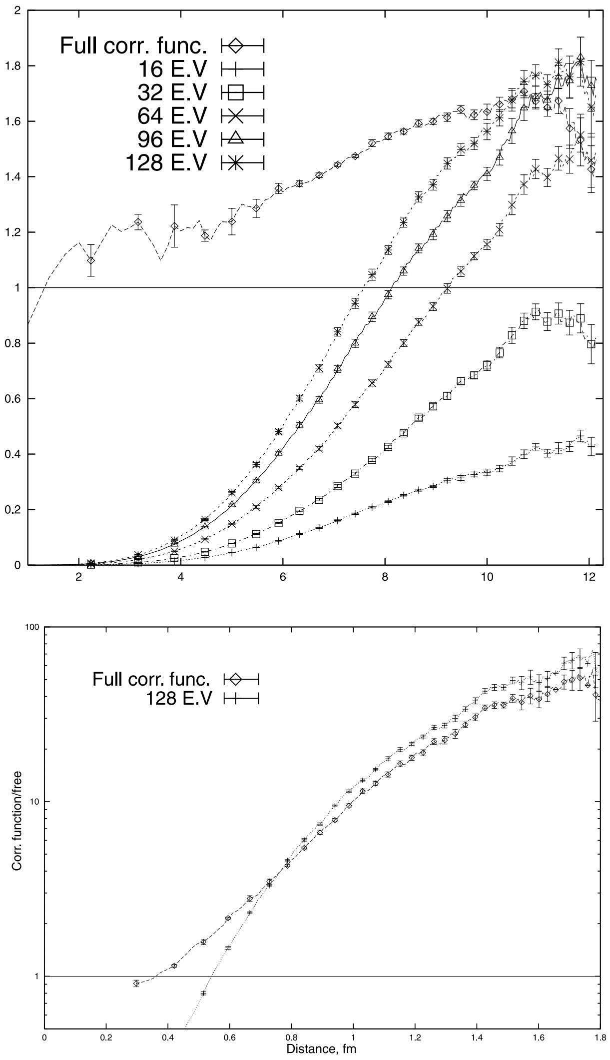
- cooled correlation functions: rho and delta



M.C. Chu et al., Phys.Rev.D49:6039-6050,1994, [hep-lat/9312071]

- mode expansion

T. Ivanenko, J. Negele, Nucl.Phys.Proc.Suppl.63:504,1998, [hep-lat/9709130]



$$\Pi(x) = \langle \mathrm{Tr}[S(x,y)\Gamma S(y,x)\Gamma] \rangle$$

$$S(x,y) = \sum_{\lambda=1}^{\nu_{max}} \frac{\phi_\lambda(x)\phi_\lambda^\dagger(y)}{\lambda + im}$$

- zero mode zone

$$\lambda < \sqrt{|T_{IA}|^2} = \rho\pi\sqrt{\frac{2}{3N_c}\frac{N}{V}} \simeq 100\,\mathrm{MeV}$$