Emergence of Collectivity in Small Systems

Part 3: Heavy Ion-Collisions

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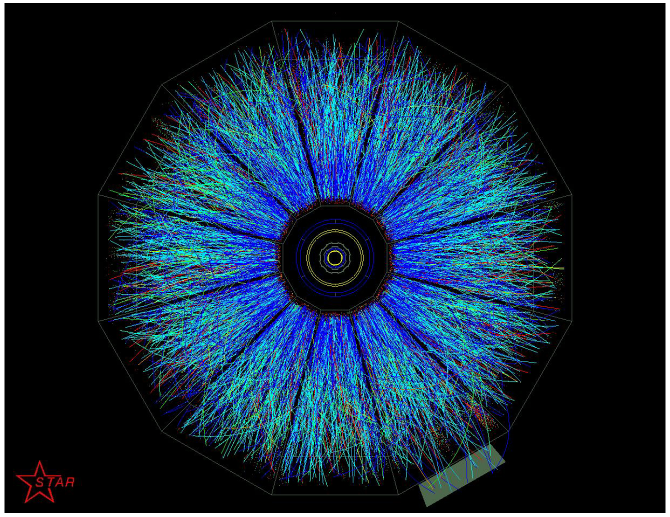
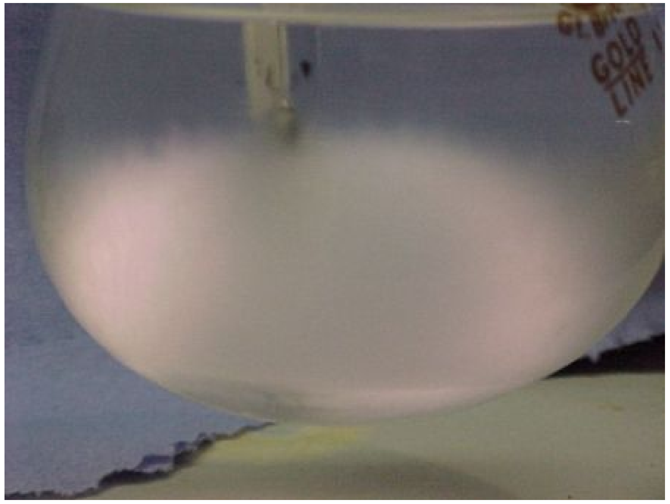
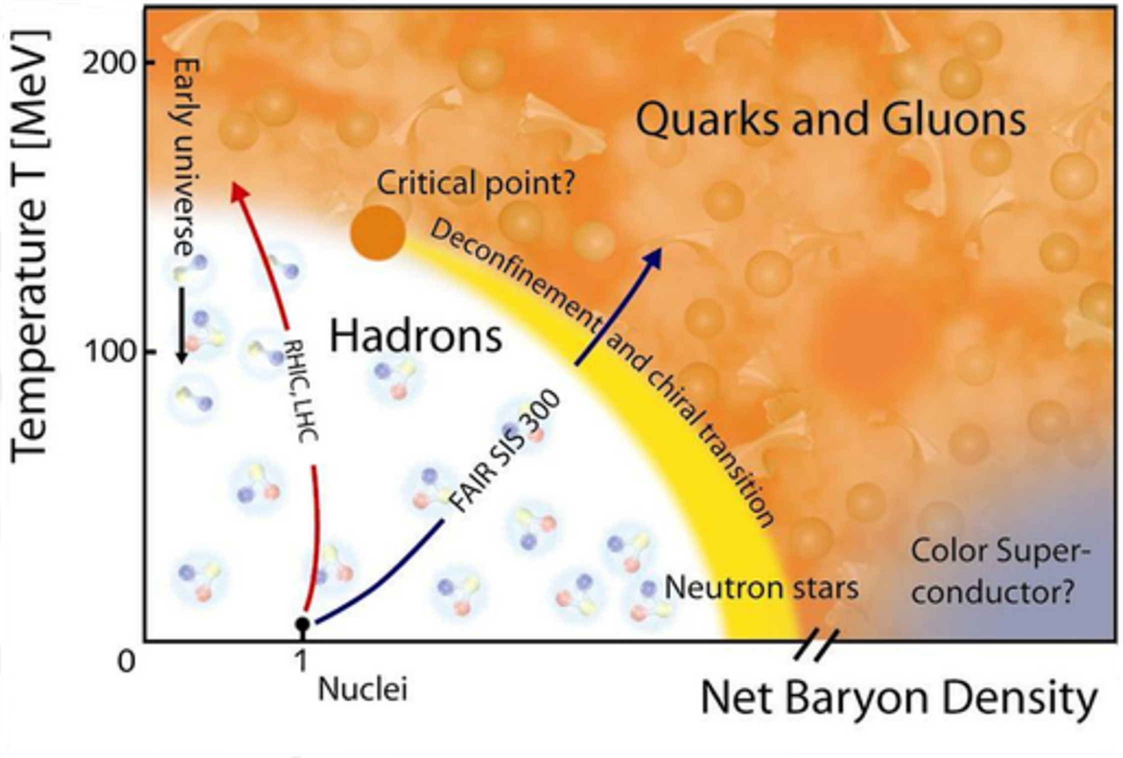


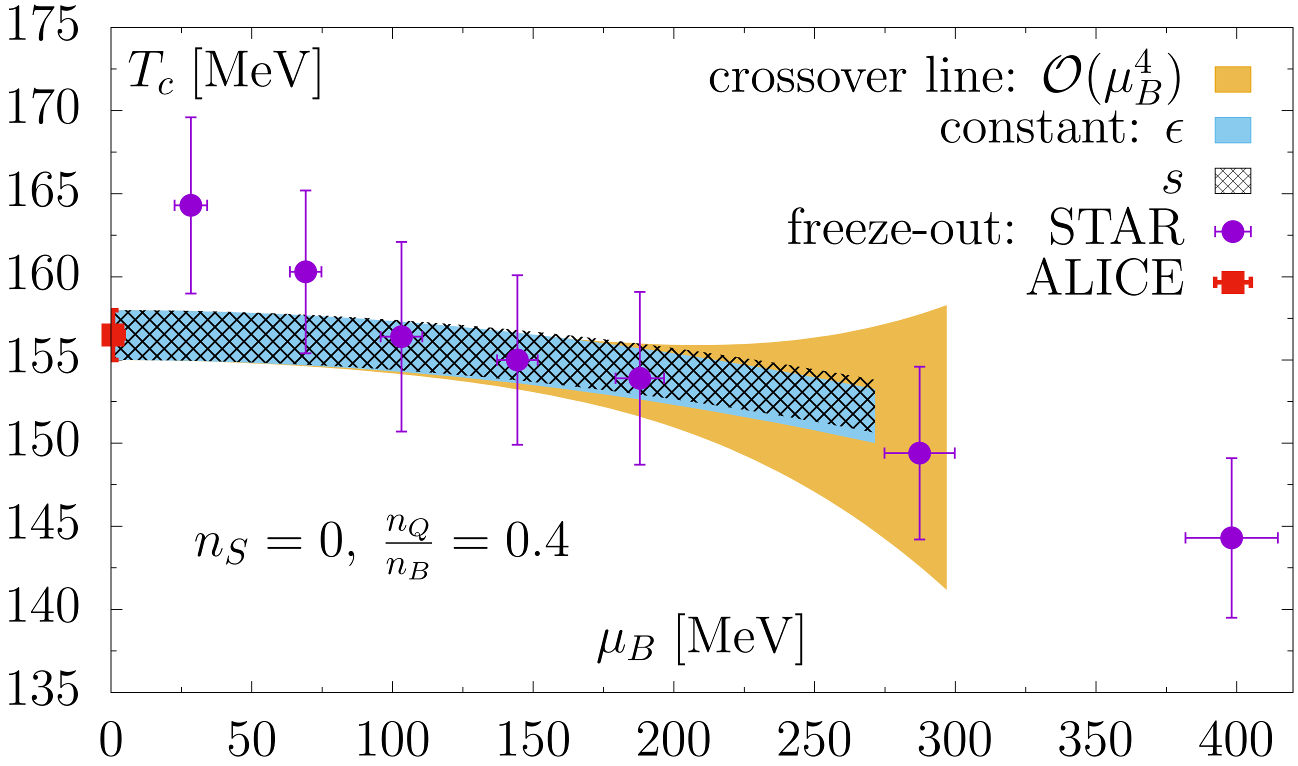


The phase diagram of QCD

$$\boxed{\mathcal{L} = \bar{q}_f(i\not{D} - m_f)q_f - \frac{1}{4g^2}G^a_{\mu\nu}G^a_{\mu\nu}}$$



Lattice results: Crossover transition



Curvature of crossover transition small, no hint of sharpening.

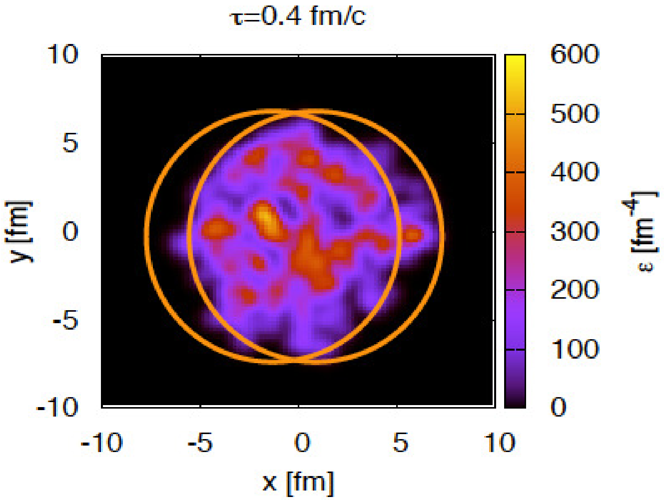
Large μ regime inaccessible (sign problem).

Bazavov et al. [1812.08235], Borsanyi et al. [2002.02821].

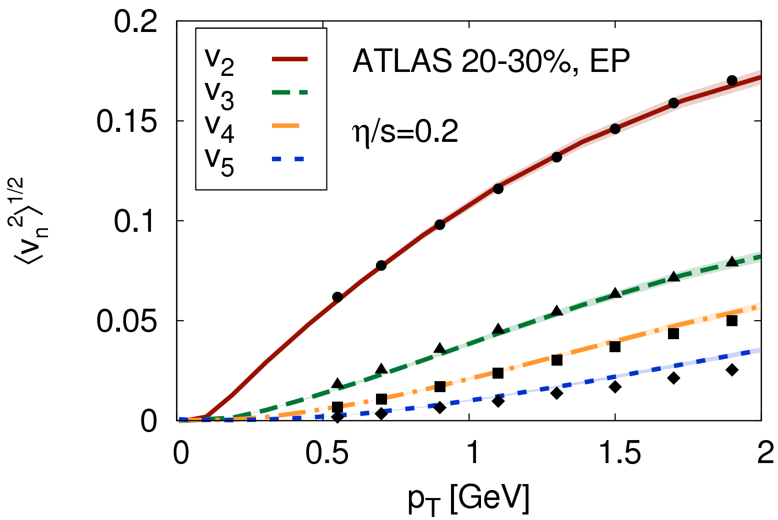
Central Experimental Result: Hydrodynamic Flow

Heavy ion collisions at RHIC are described by a very simple theory:

$$\pi\alpha\nu\tau\alpha\ \rho\varepsilon\iota \quad (\textit{everything flows})$$



B. Schenke

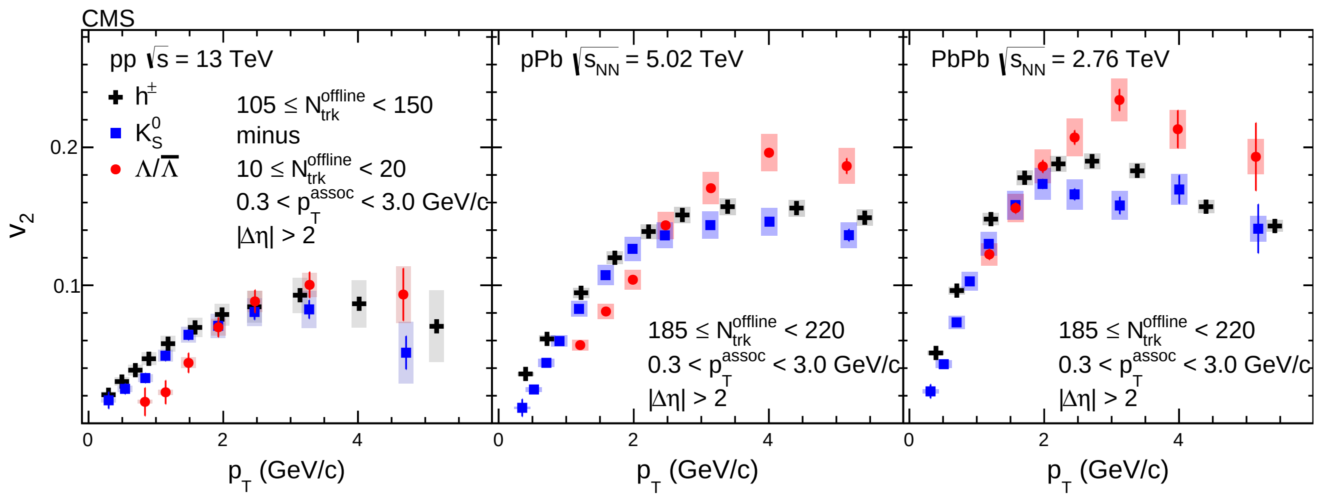


C. Gale et al.

Hydro converts initial state geometry, including fluctuations, to flow. Attenuation coefficient is small, $\eta/s \simeq 0.08\hbar/k_B$, indicating that the plasma is strongly coupled.

LHC: Flow in Small Systems

Even the smallest droplets of QGP fluid produced in (high multiplicity) pp and pA collisions exhibit collective flow.



Small viscosity $\eta/s \simeq 0.08\hbar/k_B$ implies short mean free path and rapid hydrodynamization.

Outline

I. Relativistic Fluid Dynamics

II. Small Systems

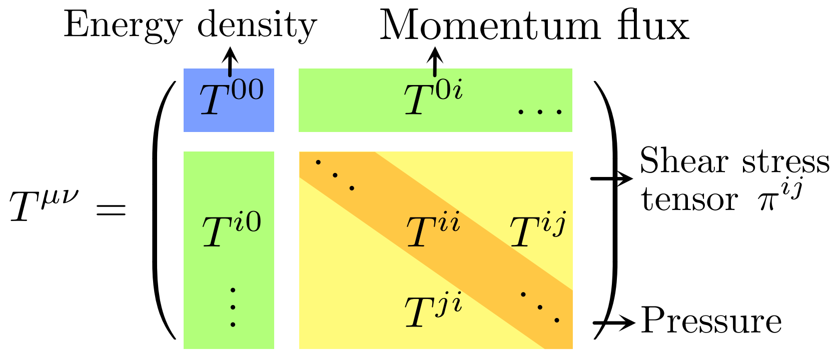
III. Fluctuations

I. Relativistic Fluid Dynamics

Conservation of energy, momentun, and baryon number (extend to BSQ?)

$$\partial_\mu T^{\mu\nu} = 0$$

$$\partial_\mu j^\mu = 0$$



Constitutive relations: $T^{\mu\nu} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)} + \ldots$

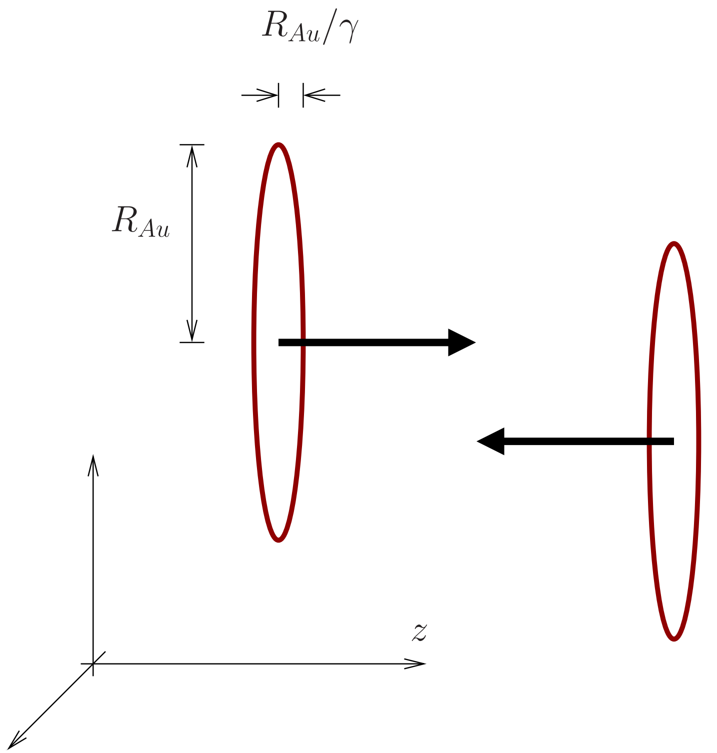
$$T^{\mu\nu}_{(0)} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu}$$

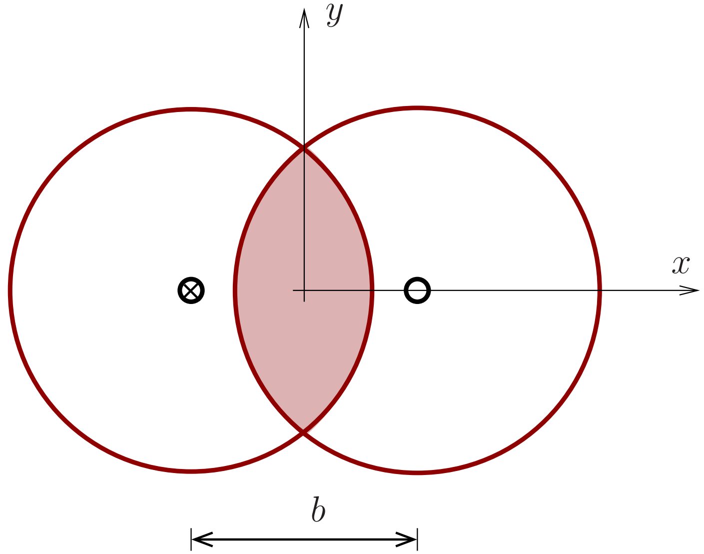
$$T^{\mu\nu}_{(1)} = -\eta \Delta^{\mu\alpha}\Delta^{\nu\alpha}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3} g_{\alpha\beta}\partial\cdot u\right) - \zeta\Delta^{\mu\nu}\partial\cdot u$$

Equation of state: $P = P(\epsilon, n)$

Many technical details: Stability, causality, initial conditions, freezeout

Heavy ion collision: Geometry





rapidity : $y = \dfrac{1}{2}\log\left(\dfrac{E+p_z}{E-p_z}\right)$

transverse momentum : $p_T^2 = p_x^2 + p_y^2$

Bjorken expansion

Experimental observation: At high energy ($\Delta y \to \infty$) rapidity distributions of produced particles (in both pp and AA) are "flat"

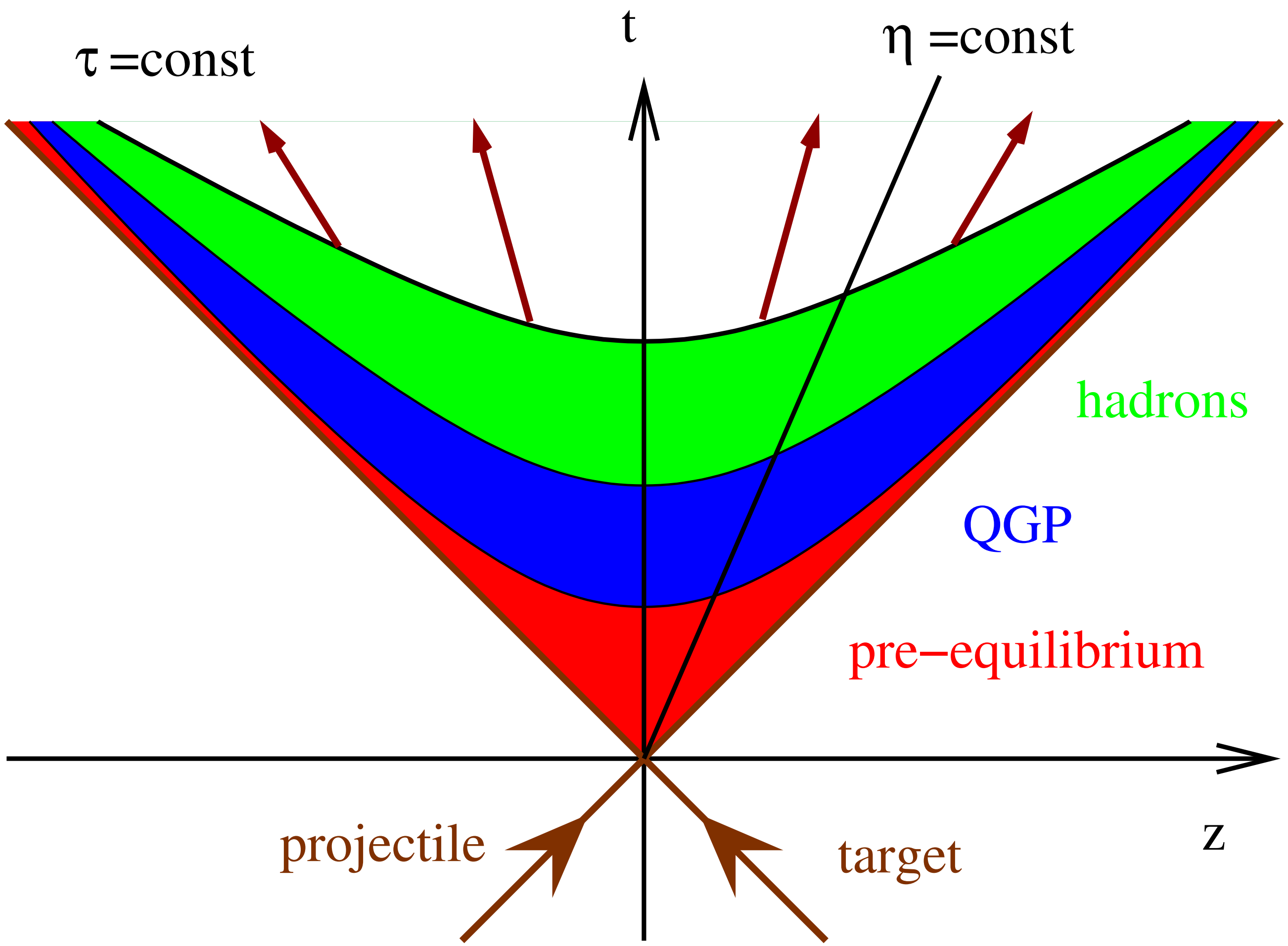
$$\frac{dN}{dy} \simeq const$$

Physics depends on proper time $\tau = \sqrt{t^2 - z^2}$, not on y

All comoving ($v = z/t$) observers are equivalent

Analogous to Hubble expansion

Bjorken expansion



$$\tau = \sqrt{t^2 - z^2} \qquad\qquad \eta = \frac{1}{2}\log\left(\frac{t+z}{t-z}\right)$$

Bjorken expansion: Hydrodynamics

Boost invariant expansion

$$u^\mu = \gamma(1, 0, 0, v_z) = (t/\tau, 0, 0, z/\tau)$$

solves Euler equation (no longitudinal acceleration)

$$\partial^\mu (s u_\mu) = 0 \qquad \Rightarrow \qquad \frac{d}{d\tau}\left[\tau s(\tau)\right] = 0$$

Solution for ideal Bj hydrodynamics

$$\boxed{s(\tau) = \frac{s_0 \tau_0}{\tau}} \qquad\qquad T = \frac{const}{\tau^{1/3}}$$

Exact boost invariance, no transverse expansion, no dissipation, . . .

Viscous corrections

Longitudinal expansion: Bj expansion solves Navier-Stokes equation

entropy equation
$$\frac{1}{s}\frac{ds}{d\tau} = -\frac{1}{\tau}\left(1 - \frac{\frac{4}{3}\eta + \zeta}{sT\tau}\right)$$

Viscous corrections small if $\quad \dfrac{4}{3}\dfrac{\eta}{s} + \dfrac{\zeta}{s} \ll (T\tau)$

early	$T\tau \sim \tau^{2/3}$	$\eta/s \sim const$	$\eta/s < \tau_0 T_0$
late	$T\tau \sim const$	$\eta \sim T/\sigma$	$\tau^2/\sigma < 1$

Hydro valid for $\tau \in [\tau_0, \tau_{fr}]$

Viscous corrections to T_{ij} (radial expansion)

$$T_{zz} = P - \frac{4}{3}\frac{\eta}{\tau} \qquad T_{xx} = T_{yy} = P + \frac{2}{3}\frac{\eta}{\tau}$$

increases radial flow (central collision)

decreases elliptic flow (peripheral collision)

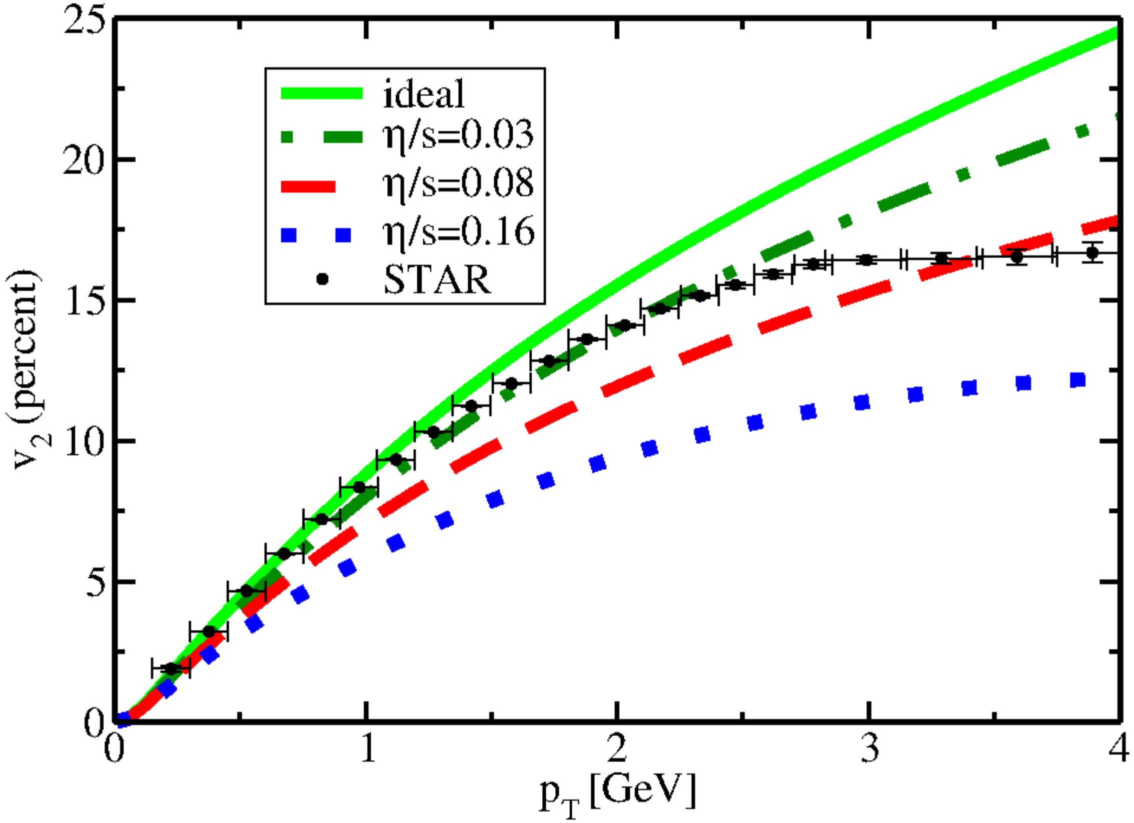
Modification of distribution function ($\Gamma_s = (\frac{4}{3}\eta + \zeta)/(sT)$)

$$\delta f = \frac{3}{8}\frac{\Gamma_s}{T^2} f_0(1+f_0) p_\alpha p_\beta \nabla^{\langle\alpha} u^{\beta\rangle}$$

Correction to spectrum grows with $p_\perp^2$

$$\frac{\delta(dN)}{dN_0} = \frac{\Gamma_s}{4\tau_f}\left(\frac{p_\perp}{T}\right)^2$$

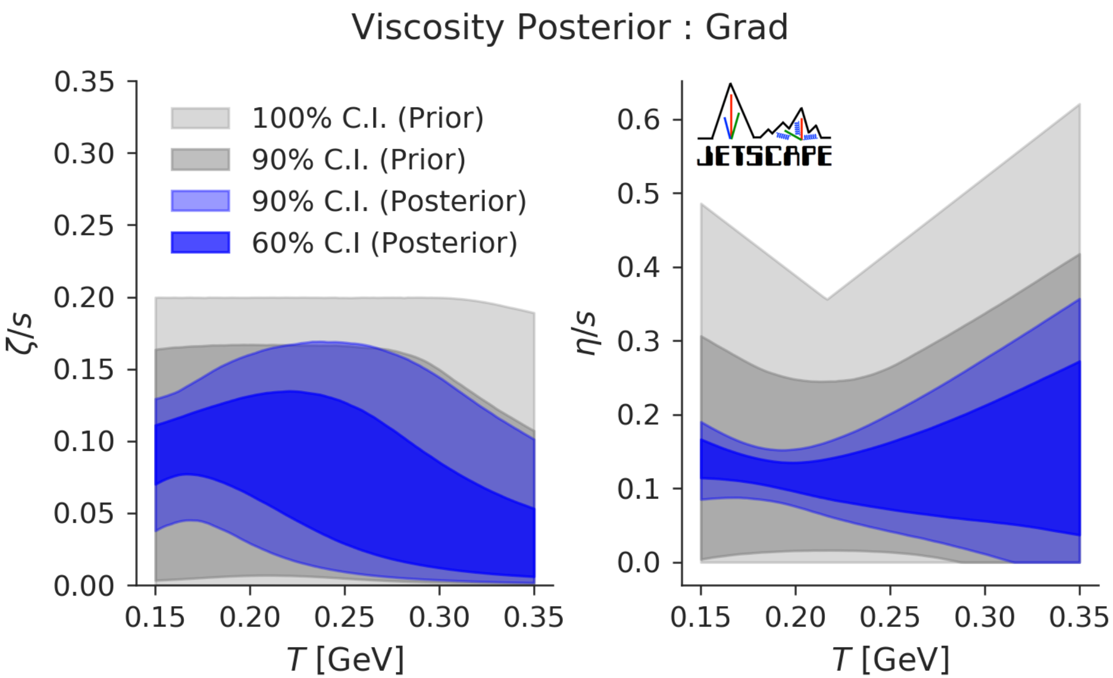
Viscous effects at RHIC: First Attempts

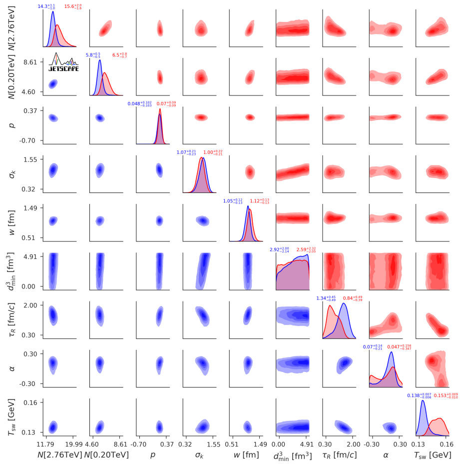


$$p_0 \left. \frac{dN}{d^3p} \right|_{p_z=0} = v_0(p_\perp)\left(1 + 2v_2(p_\perp)\cos(2\phi) + \ldots\right)$$

Romatschke (2007), Teaney (2003)

Viscous effects: Bayesian analysis

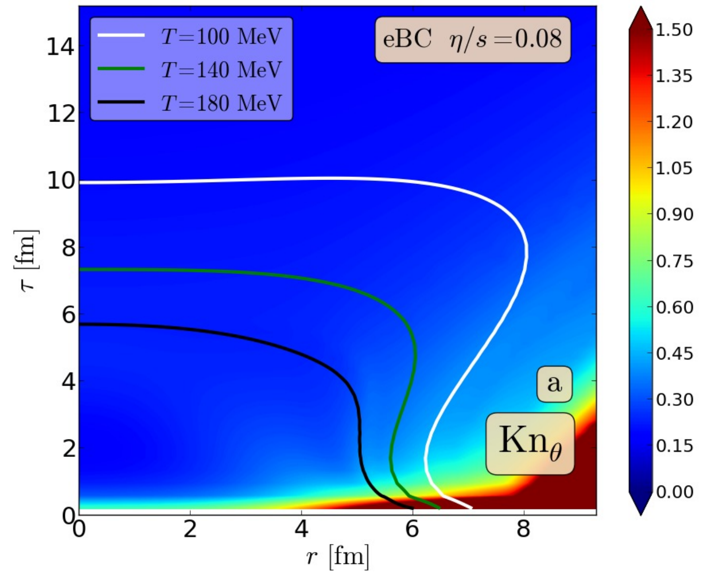




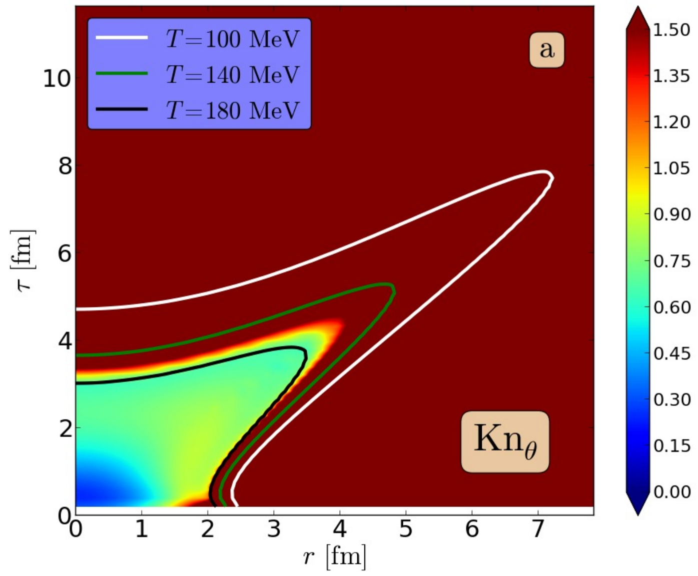
Combined analysis of LHC and RHIC data

JetScape collaboration arXiv:2011.01430.

II. Hydrodynamics in small systems



Pb+Pb LHC



p+Pb LHC

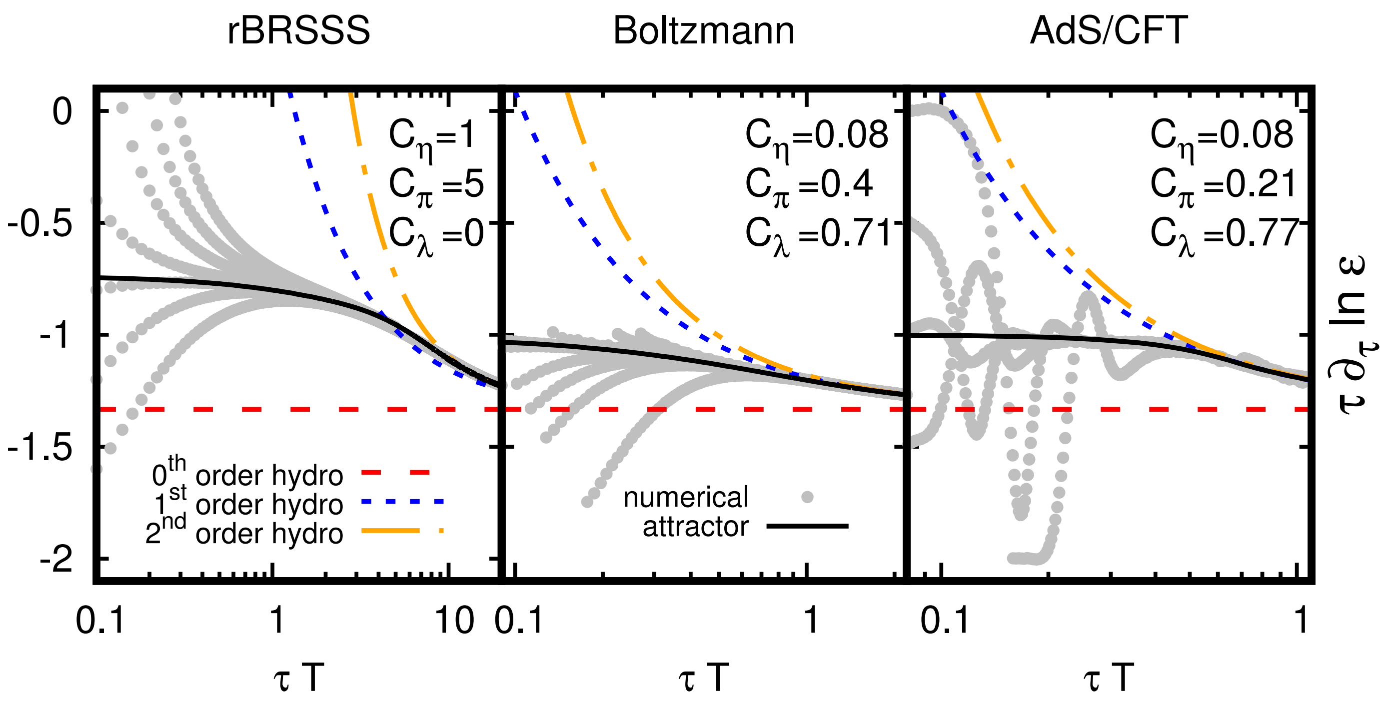
Ideas: 1) Attractors/Resurgence

2) Hydro+

3) Phenomenology

Plots from Niemi, Denicol arXiv:1404.7327.

1. Hydrodynamic Attractors



$$\tau\partial_\tau \log(\epsilon) = -(1+T_{zz}/\epsilon)$$

Attractor in a dynamical system: Asymptotic solution independent of initial conditions.

Romatschke arXiv:1704.08699.

How to characterize the attractor: Resurgence

Weak coupling expansion in QM or QFT

$$
\begin{aligned}
F(g^2) \;\; \sim \;\; & \Big(a_0^{(0)} + a_1^{(0)} g^2 + a_2^{(0)} g^4 + \ldots \Big) \\
& + \sigma e^{-S/g^2} \Big(a_0^{(1)} + a_1^{(1)} g^2 + a_2^{(1)} g^4 + \ldots \Big) \\
& + \sigma^2 e^{-2S/g^2} \Big(a_0^{(2)} + a_1^{(2)} g^2 + a_2^{(2)} g^4 + \ldots \Big) + \ldots
\end{aligned}
$$

Perturbative terms + instantons = trans-series

Ambiguities in (Borel sum) of perturbation theory canceled by ambiguities in multi-instanton effects = "resurgence"

Kinetic theory: Perturbative sum = gradient terms
instantons = non-hydrodynamic modes

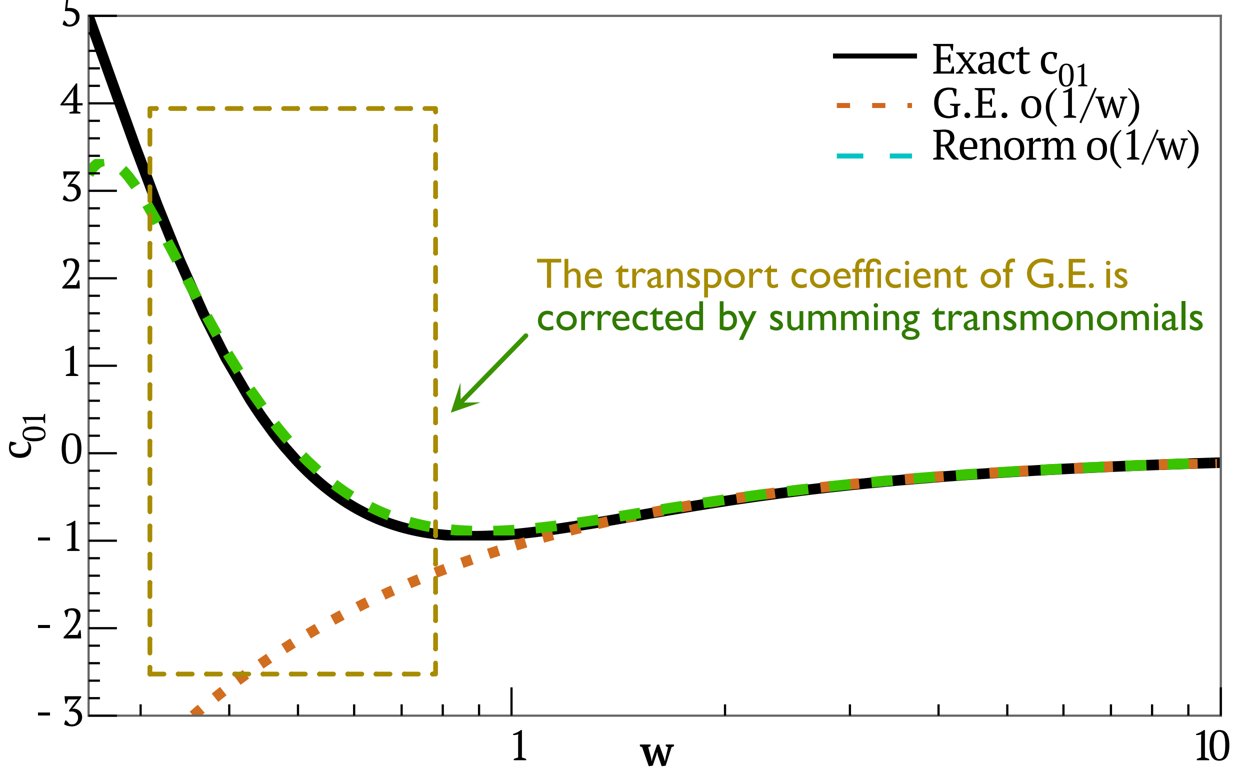
Expansion parameter $\quad w = \dfrac{4\pi s \tau T}{\eta}$

Resurgent kinetic theory: Bjorken expansion

$$c_{01} = \sum_{k=0} a_k w^{-k} \quad \rightarrow \quad c_{01} = \sum_{k=0} \left[a_k + \sum_{n=0} u_k^n \left(\sigma e^{-Sw} w^{-\beta}\right)^n \right] w^{-k}$$

Transport coefficient of gradient expansion (G.E.)

transmonomial corrections



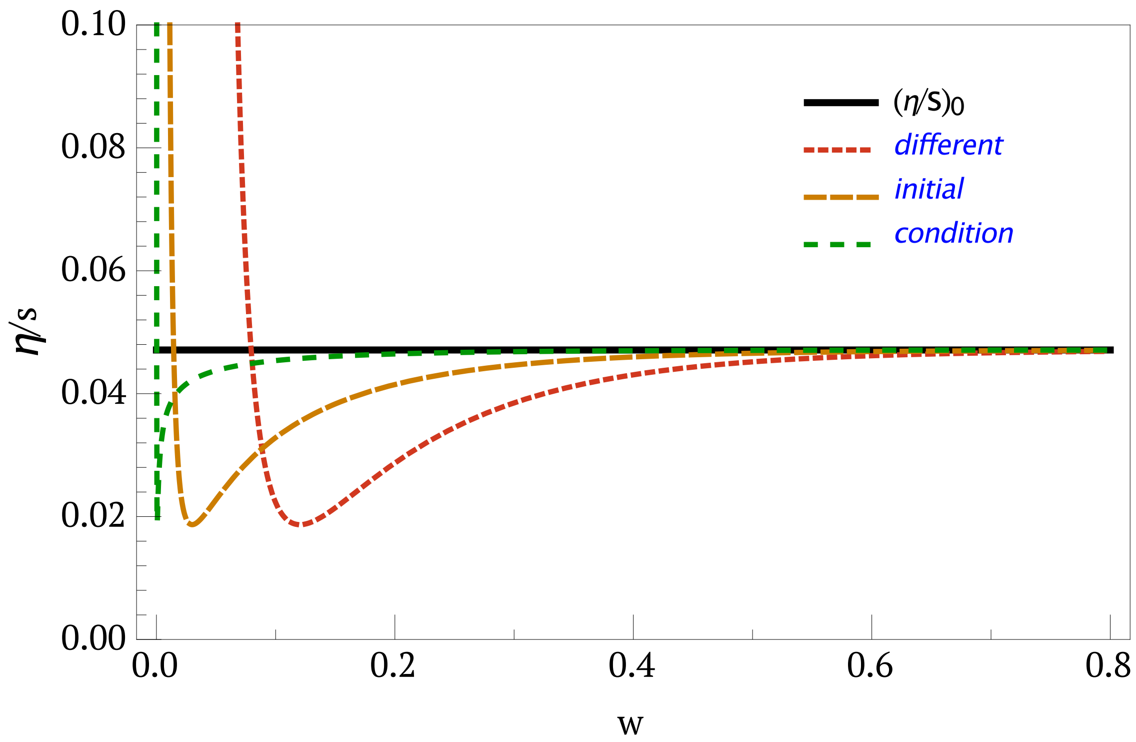
Transasymptotic matching: All-orders viscosity

Dynamical Renormalization of Transport Coefficient

First order transport coefficient $\frac{\eta}{s} \to \left(\frac{\eta}{s}\right)_0 + \sum_{n=0} u_1^n \left(\sigma e^{-Sw} w^{-\beta}\right)^n$

transmonomial & initial constant

Satisfies the transasymptotic matching which depends on its gradient size $\frac{d\frac{\eta}{s}(w, \sigma)}{d\log w} = \beta(w)$



Behtash, Kamata, Martinez, Shi, arxiv:1901.08632

2. Extended Hydrodynamic Theories: Maximum Entropy

Consider moment equations for

$$\rho_{(n)}^{\mu_1\mu_2\cdots\mu_l} \equiv \langle\, (u\cdot p)^n \; p^{\langle\mu_1}\, p^{\mu_2} \;\cdots p^{\mu_l\rangle} \rangle_\delta,$$

Need closure. Idea: Use the least-biased distribution that uses all of (and only) the information provided by hydro. This is the f that maximizes

$$s[f] = -\int dP \;(u\cdot p)\, f\ln(f),$$

subject to constraints

$$\int dP\,(u\cdot p)^2\; f = e, \quad -\frac{1}{3}\int dP\, p_{\langle\mu\rangle} p^{\langle\mu\rangle}\; f = P + \Pi, \quad \int dP\, p^{\langle\mu} p^{\nu\rangle}\; f = \pi^{\mu\nu}$$

The maximum-entropy distribution is

$$f_{\rm ME}(x,p) = \left[\exp\left(\Lambda\,(u\cdot p) - \frac{\lambda_\Pi}{u\cdot p} p_{\langle\alpha\rangle} p^{\langle\alpha\rangle} + \frac{\gamma_{\langle\alpha\beta\rangle}}{u\cdot p} p^{\langle\alpha} p^{\beta\rangle}\right) - a\right]^{-1},$$

where $(\Lambda, \lambda_\Pi, \gamma_{\alpha\beta})$ are Lagrange parameters.

Maximum Entropy Fluid Dynamics: Bjorken Flow

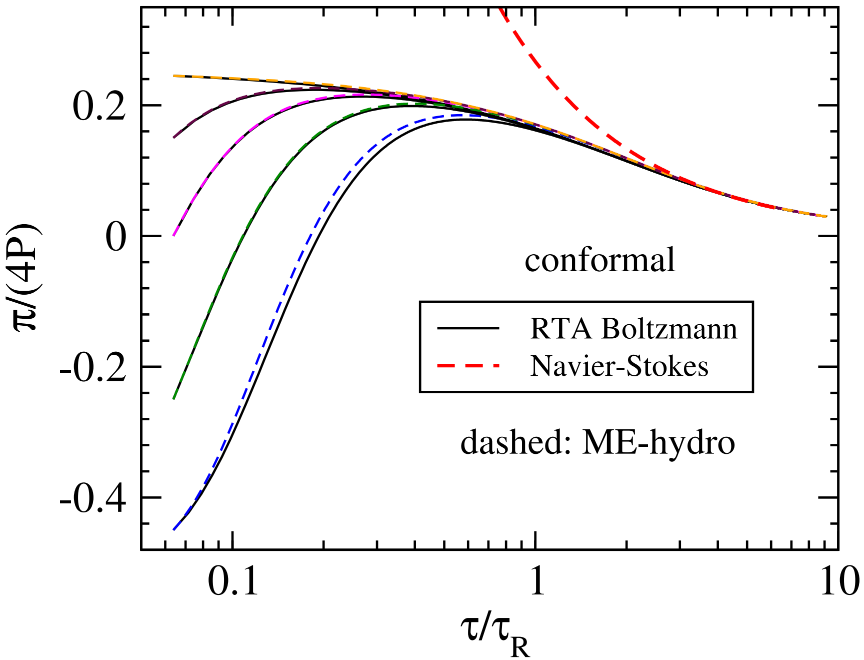
Evolution equations

$$\frac{de}{d\tau} = -\frac{e+P_L}{\tau}$$

$$\frac{dP_L}{d\tau} = -\frac{P_L - P}{\tau_R} + \frac{\bar{\zeta}_z^L}{\tau}$$

$$\frac{dP_T}{d\tau} = -\frac{P_T - P}{\tau_R} + \frac{\bar{\zeta}_z^\perp}{\tau}$$

where $\bar{\zeta}^{L,\perp}$ fixed by Max-Ent



Compare to exact RTA kinetic theory. Very good agreement.

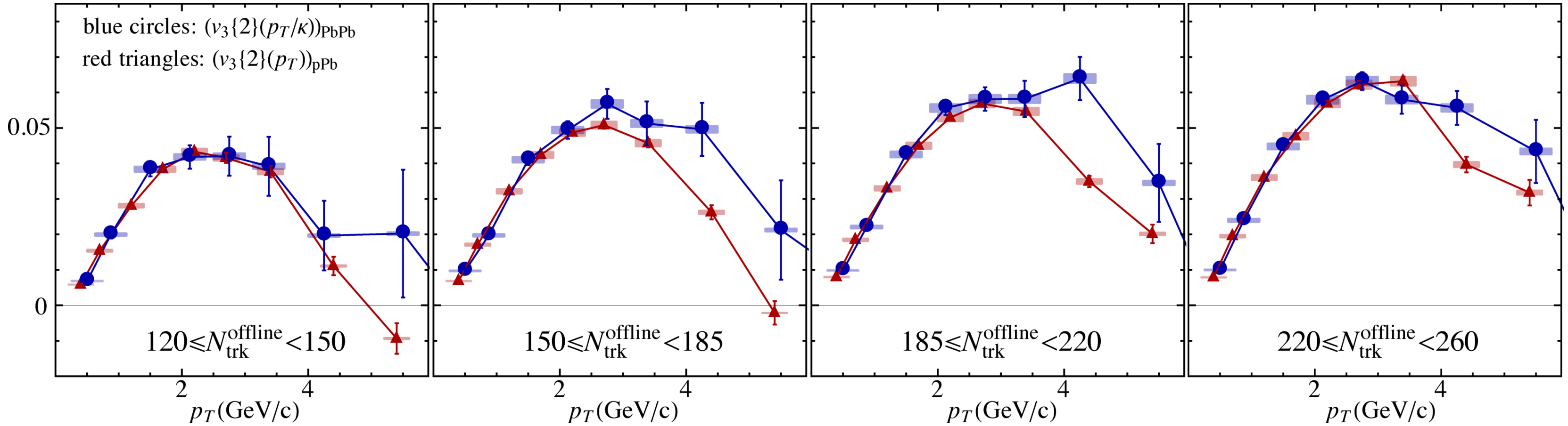
Chattopadhyay, Heinz, Schaefer, arxiv:2307.10769

3. Phenomenology: Knudsen Scaling?

Consider scaling with Knudsen number in $Pb+Pb$ and $p+Pb$

$$Kn^{-1} \sim \frac{c_s}{S}\frac{dN}{dy} \qquad\qquad Kn^{-1} \sim \left(c_s \frac{dN}{dy}\right)^{1/3}$$

non-conformal fluid — conformal fluid

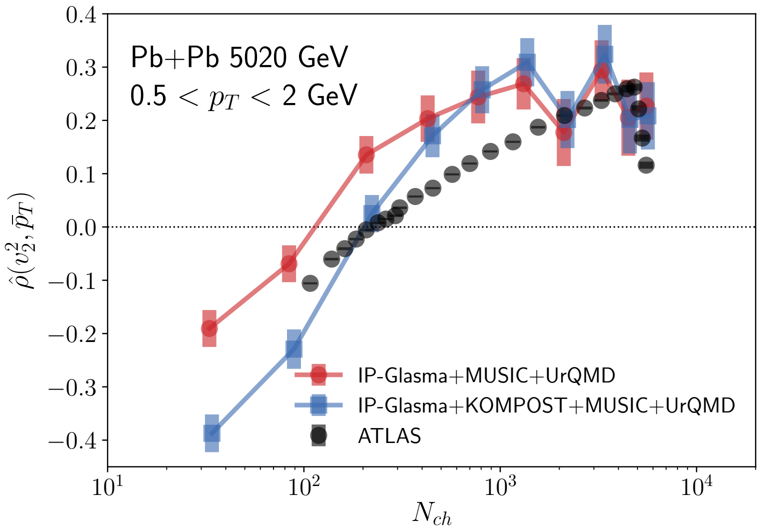


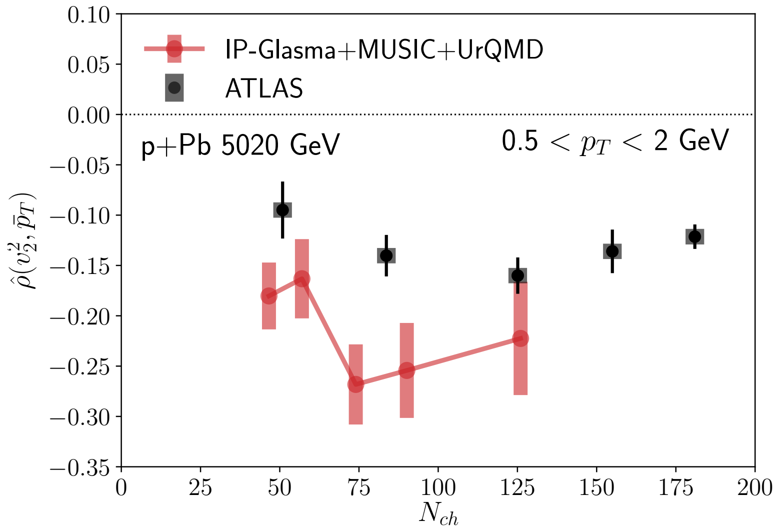
Triangular flow $v_3(p_T)$ in pPb (red) and PbPb (blue)

p_T dependence scaled by mean $\langle p_T \rangle$

Basar, Teaney, arxiv:1312.6770

Phenomenology: $p_T - v_2$ correlations?



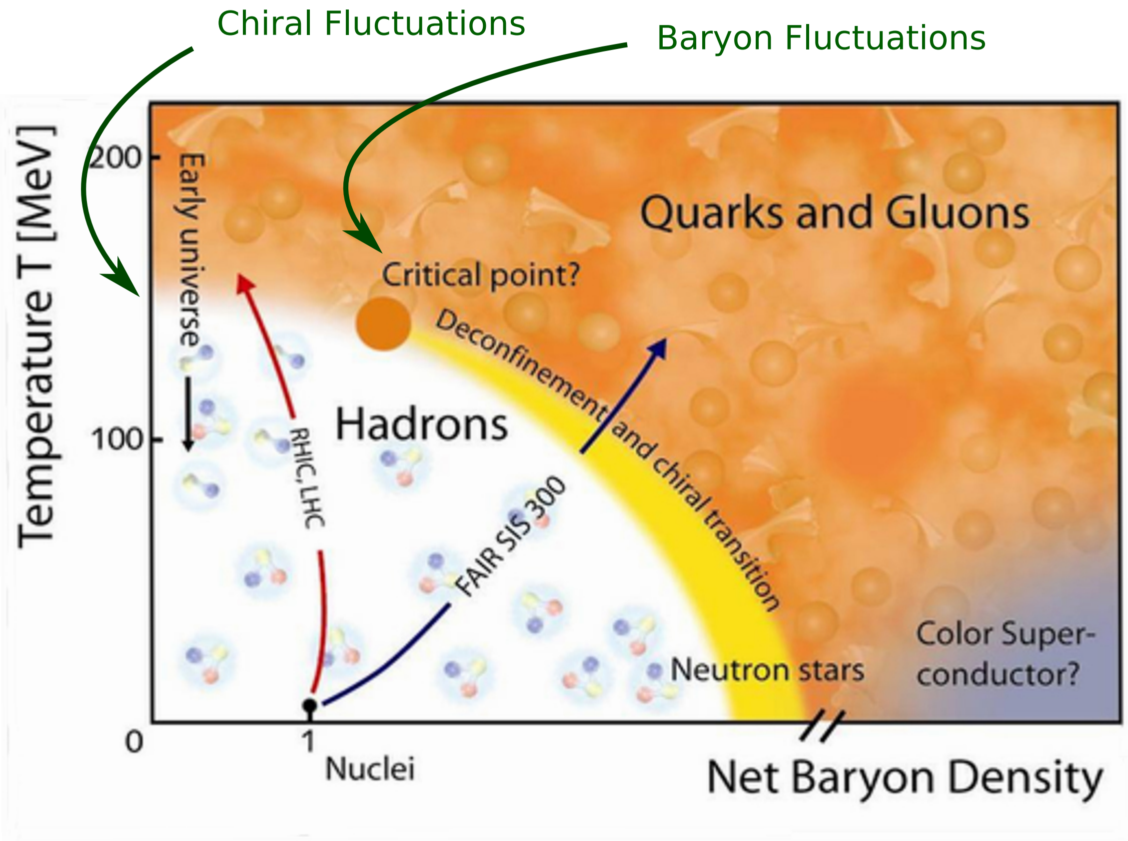


$$\hat{\rho}\left(v_2^2, \bar{p}_T\right) = \frac{\langle \hat{\delta} v_2^2 \hat{\delta} \bar{p}_T \rangle}{\sqrt{\langle (\hat{\delta} v_2^2)^2 \rangle \langle (\hat{\delta} \bar{p}_T)^2 \rangle}}$$

p+Pb qualitatively similar to very peripheral $Pb + Pb$

Schenke, Shen, Teaney, arXiv:2004.00690.

III. Phase Transitions and Fluctuations



Can we locate the chiral phase transition, or the endpoint of a first-order QGP-hadron gas transition?

Dynamical Theory

What is the dynamical theory near the critical point?

The basic logic of fluid dynamics still applies. Important modifications:

- Critical equation of state.
- Possible Goldstone modes (chiral field in QCD?)
- Stochastic fluxes, fluctuation-dissipation relations.

Digression: Diffusion

Consider a Brownian particle

$$\dot{p}(t) = -\gamma_D p(t) + \zeta(t) \qquad \langle \zeta(t)\zeta(t')\rangle = \kappa\delta(t-t')$$

drag (dissipation) white noise (fluctuations)

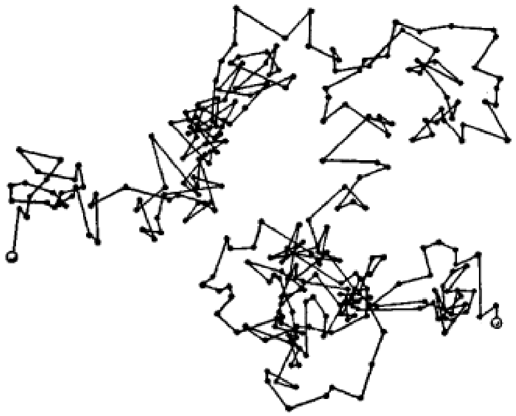
For the particle to eventually thermalize

$$\langle p^2 \rangle = 2mT$$

drag and noise must be related

$$\kappa = \frac{mT}{\gamma_D}$$

Einstein (Fluctuation-Dissipation)



Hydrodynamic equation for critical mode

Equation of motion for critical mode ϕ ("model H")

$$\frac{\partial \phi}{\partial t} = \kappa \, \nabla^2 \frac{\delta \mathcal{F}}{\delta \phi} - g \left(\vec{\nabla} \phi \right) \cdot \frac{\delta \mathcal{F}}{\delta \vec{\pi}^T} + \zeta \qquad (g = 1)$$

Diffusion Advection Noise

Equation of motion for momentum density π

$$\frac{\partial \vec{\pi}^T}{\partial t} = \eta \, \nabla^2 \frac{\delta \mathcal{F}}{\delta \vec{\pi}^T} + g \left(\vec{\nabla} \phi \right) \cdot \frac{\delta \mathcal{F}}{\delta \phi} - g \left(\frac{\delta \mathcal{F}}{\delta \vec{\pi}^T} \cdot \vec{\nabla} \right) \vec{\pi}^T + \vec{\xi}$$

Free energy functional: Order parameter ϕ, momentum density $\vec{\pi} = w\vec{v}$

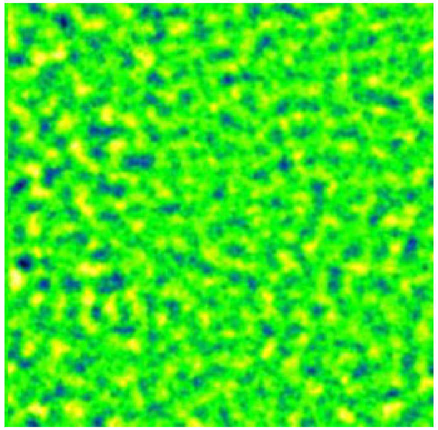
$$\mathcal{F} = \int d^3x \left[\frac{1}{2w} \vec{\pi}^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{m^2}{2} \phi^2 + \lambda \phi^4 \right] \qquad D = m^2 \kappa$$

Fluctuation-Dissipation relation

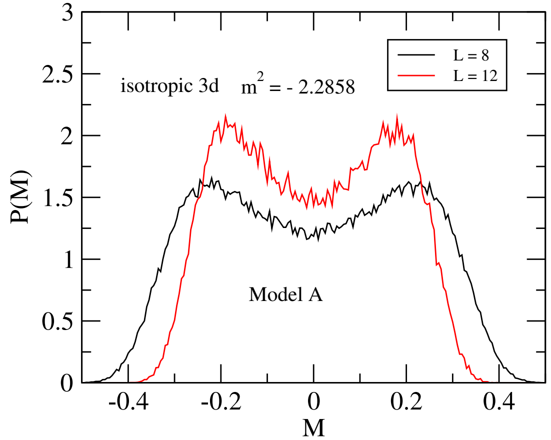
$$\langle \zeta(x,t)\zeta(x',t')\rangle = -2\kappa T \nabla^2 \delta(x-x')\delta(t-t')$$

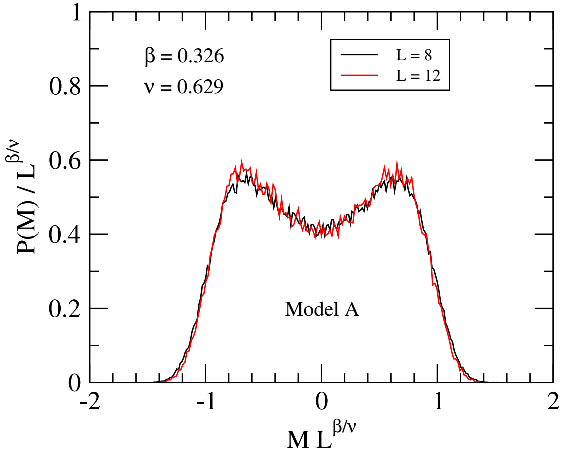
$$\langle \xi_i(x,t)\xi_j(x',t')\rangle = -2\eta T \nabla^2 P^T_{ij} \delta(x-x')\delta(t-t')$$

ensures $P[\phi,\vec{\pi}] \sim \exp(-\mathcal{F}[\phi,\vec{\pi}]/T)$



Tune m^2 to critical point $m^2 = m_c^2$ (Ising critical point)



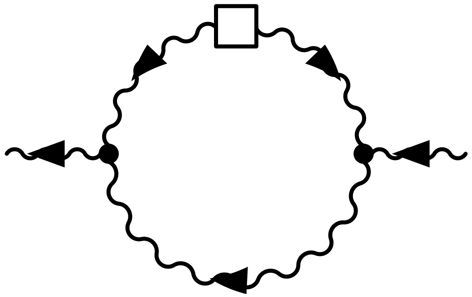


Linearized analysis (non-critical fluid)

Navier-Stokes equation: $\partial_0 \vec{\pi} + \nu \nabla^2 \vec{\pi} = mode\ couplings + noise$

Linearized propagator: $\langle \delta\pi_i^T \delta\pi_j^T \rangle_{\omega,k} = \dfrac{-\nu\rho k^2 P_{ij}^T}{-i\omega + \nu k^2}$ $\qquad \nu = \dfrac{\eta}{\rho}$

Fluctuation correction:



Renormalized viscosity:

$$\eta = \eta_0 + c_\eta \frac{T\rho\Lambda}{\eta_0} - c_\tau \sqrt{\omega}\frac{T\rho^{3/2}}{\eta_0^{3/2}}$$

Hydro is a renormalizable stochastic theory

governed by "Schwinger-Keldysh" effective field theory

Linearized analysis (non-critical fluid)

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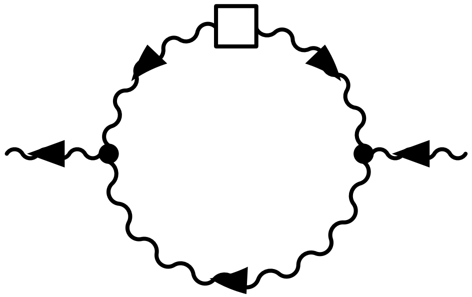


Renormalized viscosity:

$$\eta = \eta_0 + c_\eta \frac{T\rho\Lambda}{\eta_0} - c_\tau \sqrt{\omega}\frac{T\rho^{3/2}}{\eta_0^{3/2}}$$

Non-analytic term leads to long-time tail

and breakdown of naive gradient expansion.

Numerical realization

Stochastic relaxation equation ("model A")

$$\partial_t \psi = -\Gamma \frac{\delta \mathcal{F}}{\delta \psi} + \zeta \qquad \langle \zeta(x,t)\zeta(x',t')\rangle = \Gamma T \delta(x-x')\delta(t-t')$$

Naive discretization

$$\psi(t+\Delta t) = \psi(t) + (\Delta t)\left[-\Gamma \frac{\delta \mathcal{F}}{\delta \psi} + \sqrt{\frac{\Gamma T}{(\Delta t)a^3}}\theta\right] \qquad \langle \theta^2 \rangle = 1$$

Noise dominates as $\Delta t \to 0$, leads to discretization ambiguities in the equilibrium distribution.

Idea: Use Metropolis update

$$\psi(t+\Delta t) = \psi(t) + \sqrt{2\Gamma(\Delta t)}\theta \qquad p = min(1, e^{-\beta \Delta \mathcal{F}})$$

Numerical realization

Central observation

$$\begin{aligned}\langle \psi(t+\Delta t,\vec{x}) - \psi(t,\vec{x})\rangle &= -(\Delta t)\,\Gamma\,\frac{\delta\mathcal{F}}{\delta\psi} + O\left((\Delta t)^2\right)\\ \langle [\psi(t+\Delta t,\vec{x}) - \psi(t,\vec{x})]^2\rangle &= 2(\Delta t)\,\Gamma\,T + O\left((\Delta t)^2\right)\,.\end{aligned}$$

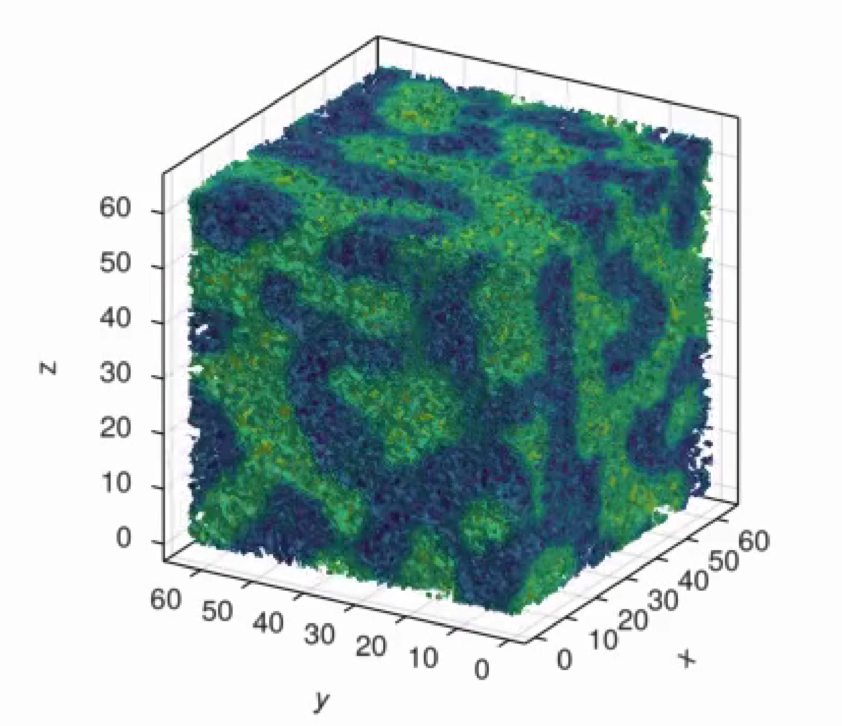
Metropolis realizes both diffusive and stochastic step. Also

$$P[\psi] \sim \exp(-\beta\mathcal{F}[\psi])$$

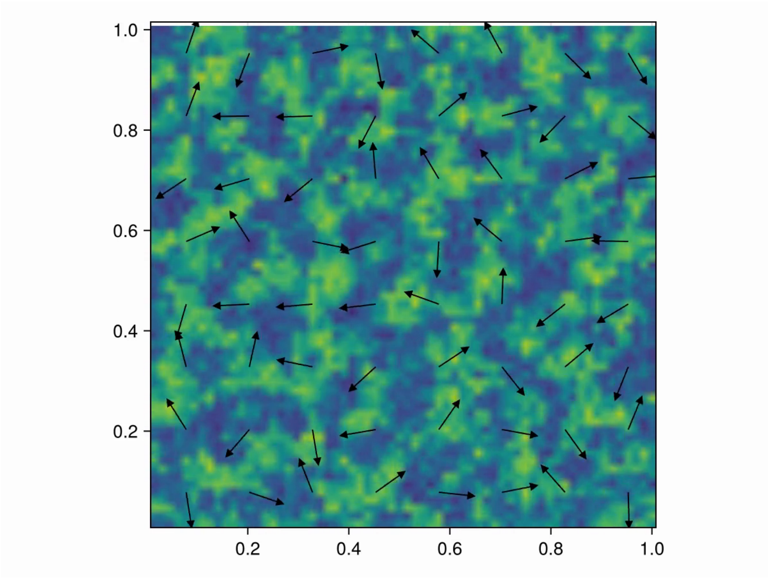
Note: Still have short distance noise; need to adjust bare parameters such as Γ, m^2, λ to reproduce physical quantities.

Numerical results (critical Navier-Stokes)

Order parameter (3d)



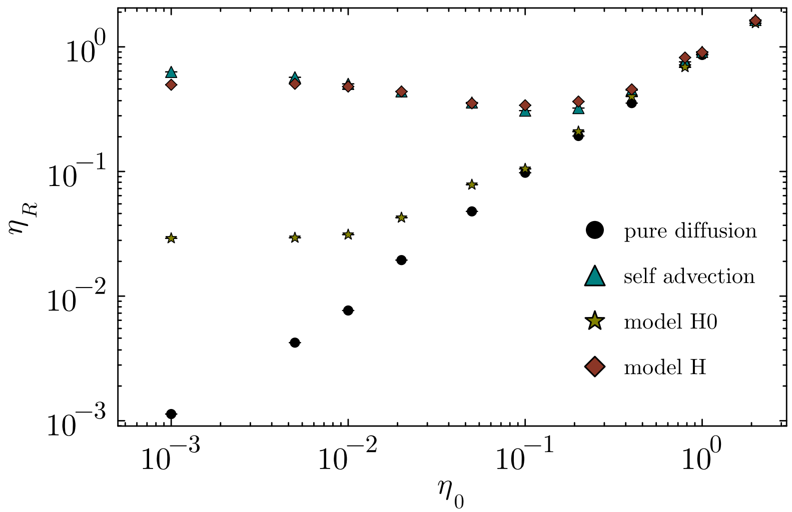
Order parameter/velocity field (2d)



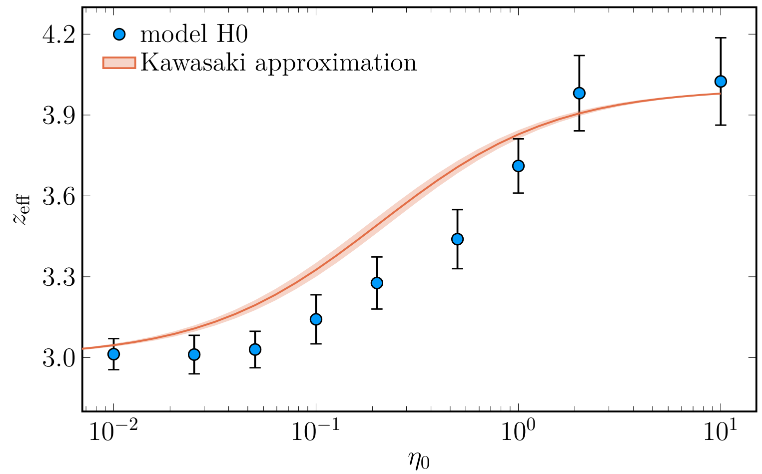
Ott, Chattopadhyay, Schaefer, Skokov, arxiv:2403.10608

Critical Navier-Stokes (model H)

Renormalized viscosity



Dynamic exponent $\tau \sim \xi^z$



Top: Model H

Middle: No self-advection

Bottom: No advection

small η/large ξ $\leftrightarrow$ large η/small ξ

Stickiness of shear waves

Shear waves speed up relaxation

Outlook

Opportunity: Discover QCD critical point by observing critical fluctuations in heavy ion collisions. Observe chiral transitions using soft pions and multi-pion correlations.

Challenge: Propagate fluctuations of conserved charges in relativistic fluid dynamics. Describe initial state fluctuations and final state freezeout.

Opportunity: Observe breakdown of hydrodynamics in small systems. Learn about initial state, sub-nucleonic degrees of freedom, and non-hydrodynamic modes.

Challenge: Disentangle initial state and fluid dynamic evolution.

Many interesting lessons about fluid dynamics along the way.