Emergence of Collectivity in Small Systems

Part 2: Ultra-cold atoms

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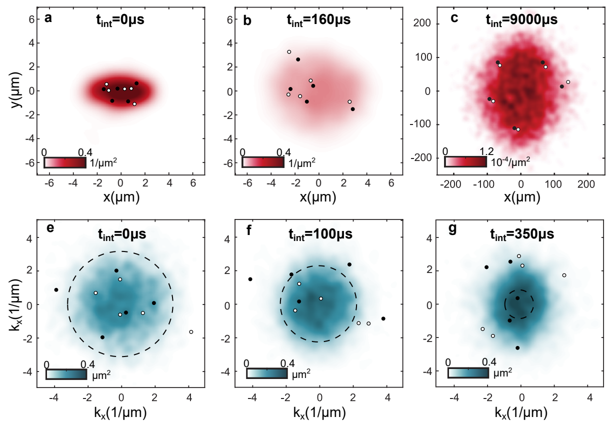
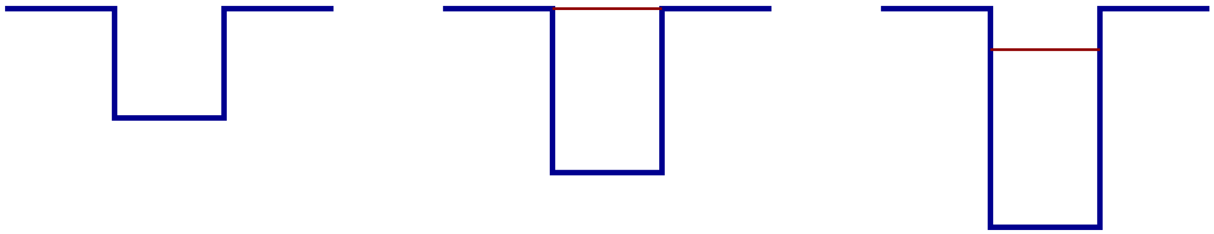


Non-relativistic fermions in unitarity limit

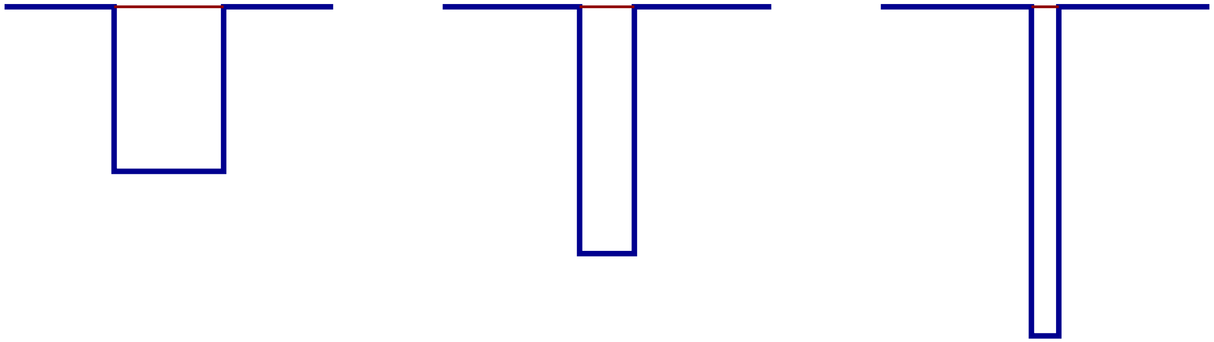
Two body interaction: Consider simple square well potential



$a < 0$ $\qquad$ $a = \infty,\ \epsilon_B = 0$ $\qquad$ $a > 0,\ \epsilon_B > 0$

Non-relativistic fermions in unitarity limit

Now take the range to zero, keeping $\epsilon_B \simeq 0$

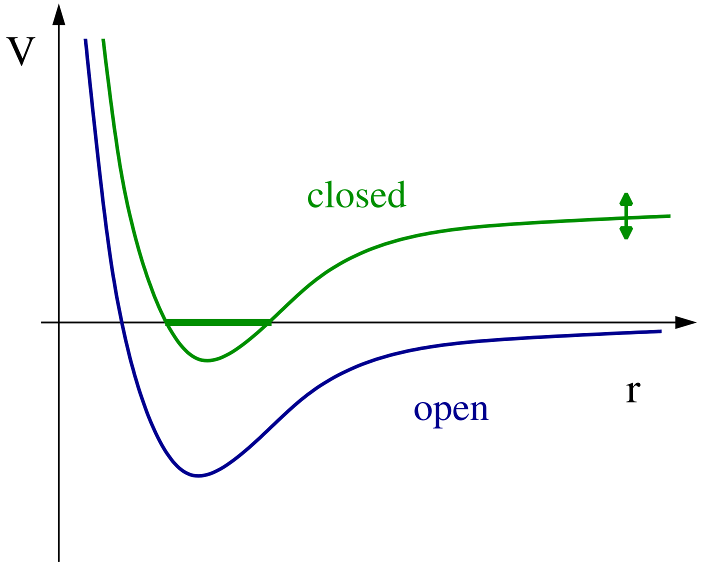


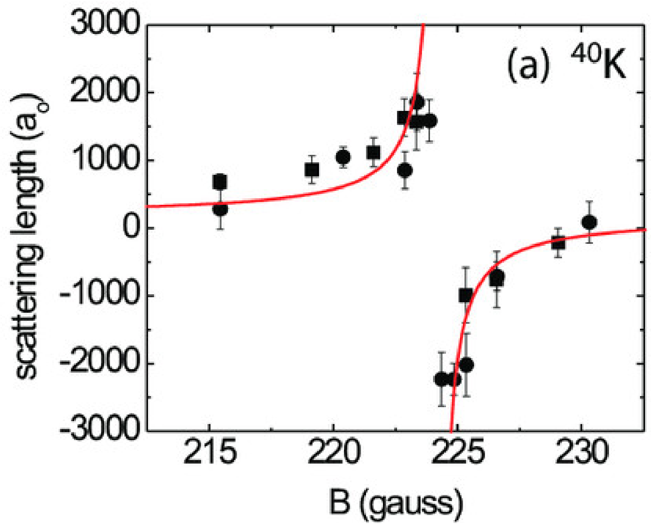
Universal relations

$$\mathcal{T} = \frac{1}{ik + 1/a} \qquad \epsilon_B = \frac{1}{2ma^2} \qquad \psi_B \sim \frac{1}{\sqrt{a}r}\exp(-r/a)$$

Feshbach resonances

Atomic gas with two spin states: "↑" and "↓"





Feshbach resonance $$a(B) = a_0\left(1 + \frac{\Delta}{B - B_0}\right)$$

Fermi Gas at Unitarity: Field Theory

Non-relativistic fermions at low momentum

$$\mathcal{L}_{\rm eff} = \psi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2M} \right) \psi - \frac{C_0}{2} (\psi^\dagger \psi)^2$$

Unitary limit: $a \to \infty$ (DR: $C_0 \to \infty$)

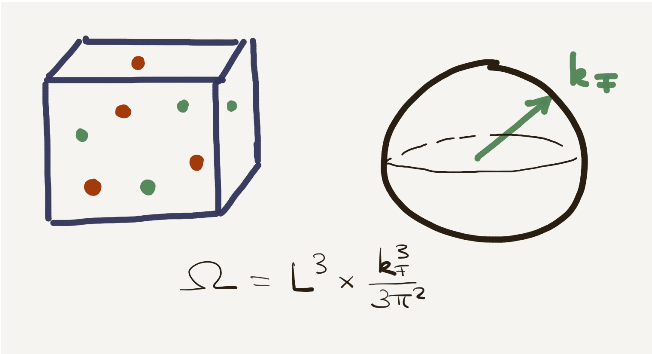
This limit is smooth (HS-trafo, $\Psi = (\psi_\uparrow, \psi_\downarrow^\dagger)$)

$$\mathcal{L} = \Psi^\dagger \left[i\partial_0 + \sigma_3 \frac{\vec{\nabla}^2}{2m} \right] \Psi + \left(\Psi^\dagger \sigma_+ \Psi \phi + h.c. \right) - \frac{1}{C_0} \phi^* \phi \ ,$$

$\phi \sim \psi_\uparrow \psi_\downarrow$ auxiliary "pair" or "dimer" field.

Equation of state

Free fermi gas at zero temperature



$$\frac{E}{N} = \frac{3}{5}\frac{k_F^2}{2m} \qquad \frac{N}{V} = \frac{k_F^3}{3\pi^2}$$

$$\mathcal{E} = E/V \sim (N/V)^{5/3}$$

Unitarity limit ($a \to \infty$, $r \to 0$). No expansion parameters.

$$\frac{E}{N} = \xi\,\frac{3}{5}\frac{k_F^2}{2m} \qquad k_F \equiv (3\pi^2 N/V)^{1/3}$$

Prize problem (George Bertsch, 1998): Determine ξ.

Is $\xi > 0$ (is the system stable)?

How to measure ξ with trapped atoms

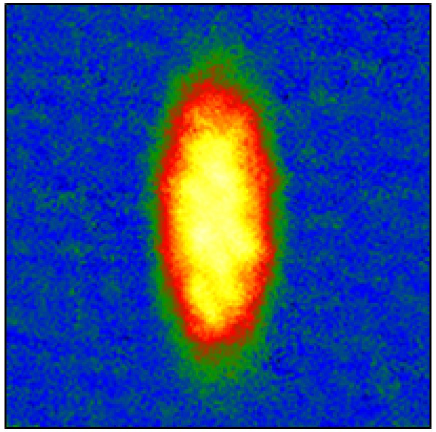
Trapped gas in hydrostatic equilibrium

$$\frac{1}{n}\vec{\nabla}P = -\vec{\nabla}V_{ext} \qquad P = \frac{2}{3}\mathcal{E}$$

Pressure determines size of the cloud ($V_{ext} = \frac{1}{2}m\omega^2 x^2$).

$$r(a=0) = \sqrt{\frac{2E_F}{m\omega^2}} \qquad r(a=\infty) = \xi^{1/4} r(0)$$

Cloud size can be measured with a CCD camera and a ruler

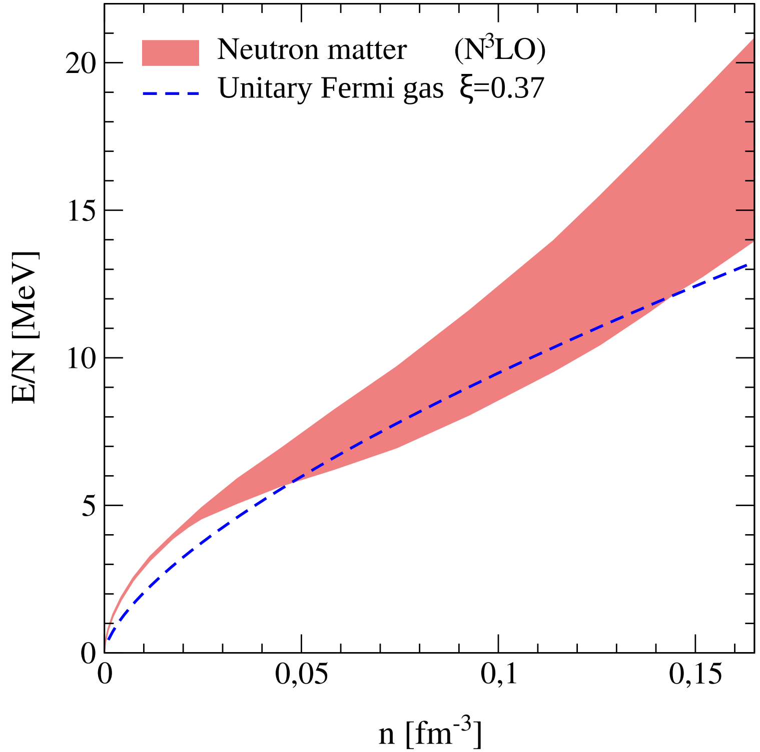


modern value

$$\xi = 0.37(5)$$

(MIT, Sommer et al.)

Neutron matter equation of state



$n \lesssim 0.1\,\mathrm{fm}^{-3}$: Unitary gas with a^{-1}, r corrections.

$n \gtrsim 0.1\,\mathrm{fm}^{-3}$: Repulsive 2-body, 3-body forces.

$n \gtrsim 0.2\,\mathrm{fm}^{-3}$: New degrees of freedom.

Outline

1. Transport coefficients: Theory

2. Transport: Viscosity from elliptic flow.

3. Transport: Linear response and small systems.

4. Outlook: External fields, OTOCs, etc.

1. Fluid dynamics

Simple fluid: Conservation laws for mass, energy, momentum

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\vec{j}^{\rho} = 0 \qquad \frac{\partial \mathcal{E}}{\partial t} + \vec{\nabla}\vec{j}^{\epsilon} = 0$$

$$\frac{\partial \pi_i}{\partial t} + \nabla_j \Pi_{ij} = 0 \qquad \vec{j}^{\rho} \equiv \rho\vec{v} = \vec{\pi}$$

Scale invariance: Ideal fluid dynamics

$$\Pi^0_{ij} = P g_{ij} + \rho v_i v_j , \qquad \boxed{P = \frac{2}{3}\mathcal{E}}$$

First order viscous hydrodynamics

$$\delta^{(1)}\Pi_{ij} = -\eta\sigma_{ij} - \zeta g_{ij}\langle\sigma\rangle \qquad \boxed{\zeta = 0}$$

$$\sigma_{ij} = \nabla_i v_j + \nabla_j v_i - \frac{2}{3}\delta_{ij}\nabla\cdot v \qquad \langle\sigma\rangle = \sigma_{ii}$$

Microscopic Theory

$P = P(\mathcal{E})$ fixed by conformal symmetry. $P(\mu, T) = P_0 f(T/\mu)$ can be computed from euclidean data

$$P = \log Z(\mu, T) \qquad Z = \int D\psi D\psi^\dagger e^{-S_E}$$

But: Transport coefficients determined by Kubo relations

$$\eta = -\lim_{\omega\to 0} \frac{1}{\omega} \mathrm{Im} \int dt d^3x \, e^{-i(\omega t - kx)} \, \Theta(t) \langle [\Pi_{xy}(0), \Pi_{xy}(t, x)] \rangle$$

Requires real time data.

Kinetic theory

Microscopic picture: Quasi-particle distribution function $f_p(x,t)$

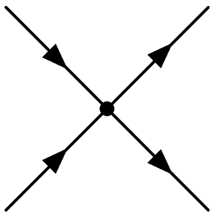
$$\rho(x,t) = \int d\Gamma_p \, m f_p(x,t) \qquad \pi_i(x,t) = \int d\Gamma_p \, p_i f_p(x,t)$$

$$\Pi_{ij}(x,t) = \int d\Gamma_p \, p_i v_j f_p(x,t)$$

Boltzmann equation

$$\left(\frac{\partial}{\partial t} + \frac{p^i}{m} \frac{\partial}{\partial x^i} - F^i \frac{\partial}{\partial p^i} \right) f_p(x,t) = C[f]$$

$$C[f] =$$



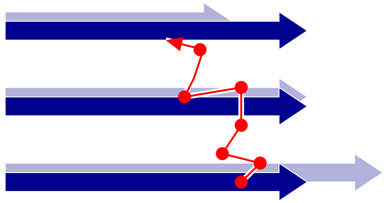
Solve order-by-order in Knudsen number $Kn = l_{mfp}/L$

Kinetic theory: Knudsen expansion

Chapman-Enskog expansion $f = f_0 + \delta f_1 + \delta f_2 + \ldots$

Gradient exp. $\delta f_n = O(\nabla^n)$

$\equiv$ Knudsen exp. $\delta f_n = O(Kn^n)$



First order result

Bruun, Smith (2005)

$$\delta^{(1)}\Pi_{ij} = -\eta\sigma_{ij} \qquad \delta^{(1)} j_i^{\epsilon} = -\kappa\nabla_i T \qquad \eta = \frac{15}{32\sqrt{\pi}}(mT)^{3/2} \qquad \kappa = \frac{2}{3}c_P\eta$$

Second order result

Chao, Schaefer (2012), Schaefer (2014)

$$\begin{aligned}\delta^{(2)}\Pi^{ij} \;=\; & \frac{\eta^2}{P}\left[\langle D\sigma^{ij}\rangle + \frac{2}{3}\sigma^{ij}(\nabla\cdot v)\right] \\ & + \frac{\eta^2}{P}\left[\frac{15}{14}\sigma^{\langle i}{}_k\sigma^{j\rangle k} - \sigma^{\langle i}{}_k\Omega^{j\rangle k}\right] + O(\kappa\eta\nabla^i\nabla^j T)\end{aligned}$$

relaxation time $\tau_\pi = \eta/P$

Frequency dependence, breakdown of kinetic theory

Consider harmonic perturbation $h_{xy}e^{-i\omega t+ikx}$. Use schematic collision term $C[f_p^0 + \delta f_p] = -\delta f_p/\tau_0$.

$$\delta f_p(\omega, k) = \frac{1}{2T} \frac{-i\omega p_x v_y}{-i\omega + i\vec{v}\cdot\vec{k} + \tau_0^{-1}} f_p^0 \, h_{xy} \, .$$

Leads to Lorentzian line shape of transport peak

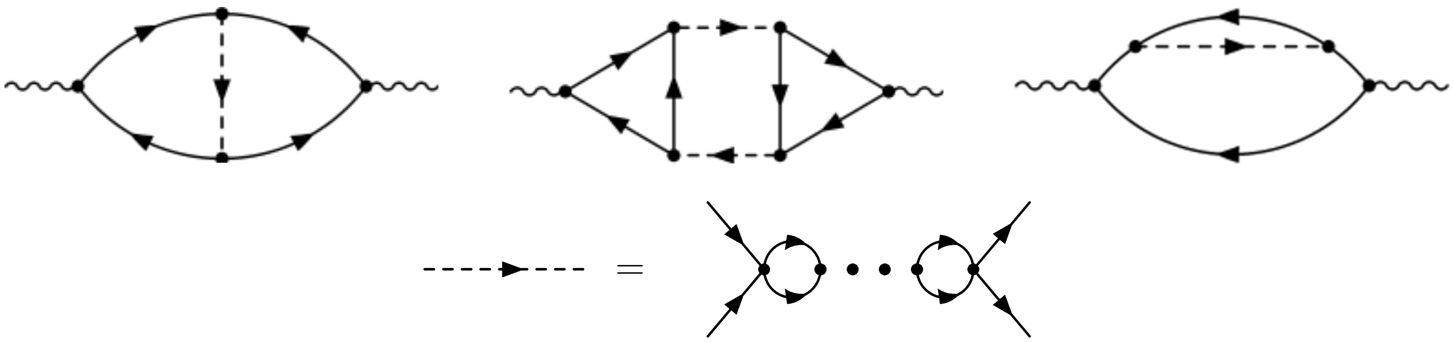
$$\eta(\omega) = \frac{\eta(0)}{1+\omega^2\tau_0^2}$$

Pole at $\omega = \dfrac{i}{\tau_0} = \dfrac{isT}{\eta}$ controls range of convergence of gradient expansion.

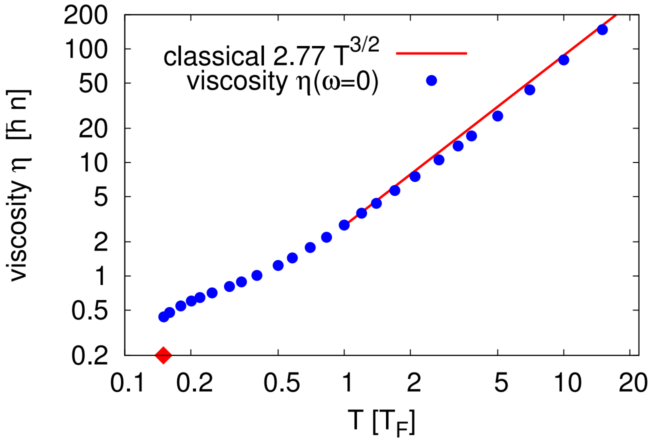
High frequency behavior misses short range correlations for $\omega > T$.

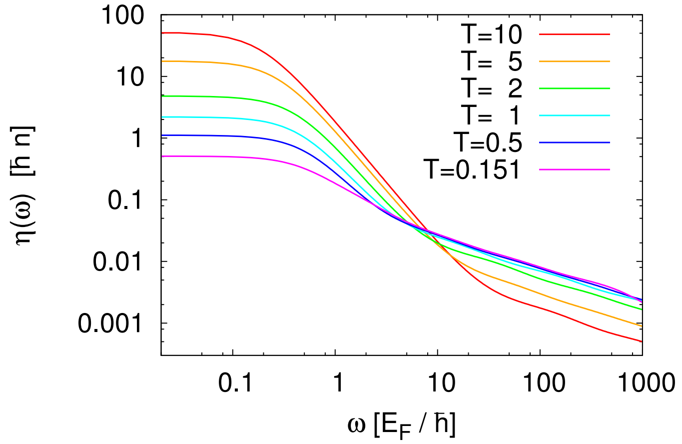
Quantum Field Theory

The diagrammatic content of the Boltzmann equation is known: Kubo formula with "Maki-Thompson" + "Azlamov-Larkin" +"Self-energy"



Limits subtle ($\omega \to 0$ and $n\lambda^3 \to 0$ don't commute). Can be used to extrapolate Boltzmann result to $T \sim T_F$





Enss, Zwerger (2011), see also Levin (2014)

Short time behavior: OPE

Operator product expansion (OPE)

$$\eta(\omega) = \sum_n \frac{\langle \mathcal{O}_n \rangle}{\omega^{(\Delta_n - d)/2}} \qquad \mathcal{O}_n(\lambda^2 t, \lambda x) = \lambda^{-\Delta_n} \mathcal{O}_n(t, x)$$

Leading operator: Contact density (Tan)

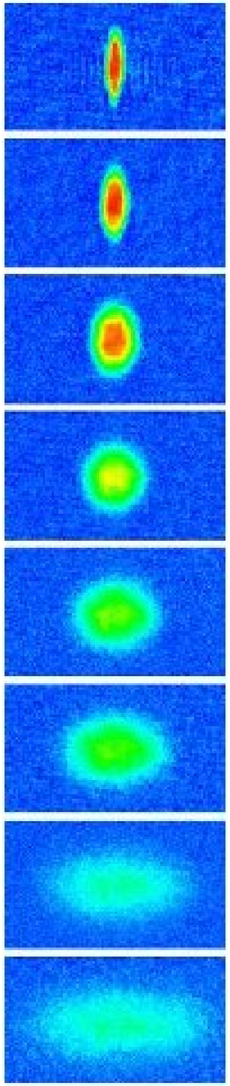
$$\mathcal{O}_{\mathcal{C}} = C_0^2 \psi\psi\psi^\dagger\psi^\dagger = \Phi\Phi^\dagger \qquad \Delta_{\mathcal{C}} = 4$$

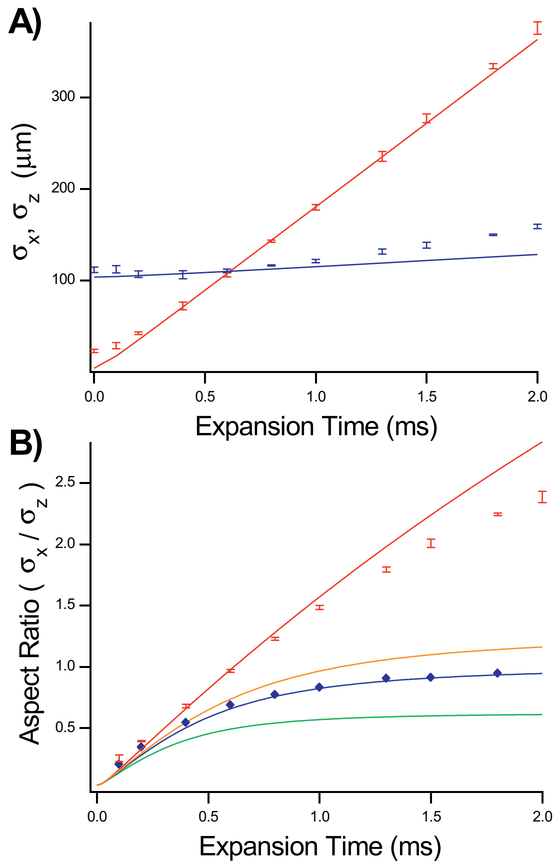
$\eta(\omega) \sim \langle \mathcal{O}_{\mathcal{C}} \rangle / \sqrt{\omega}$. Asymptotic behavior + analyticity gives sum rule

$$\frac{1}{\pi} \int dw \left[\eta(\omega) - \frac{\langle \mathcal{O}_{\mathcal{C}} \rangle}{15\pi\sqrt{m\omega}} \right] = \frac{\mathcal{E}}{3}$$

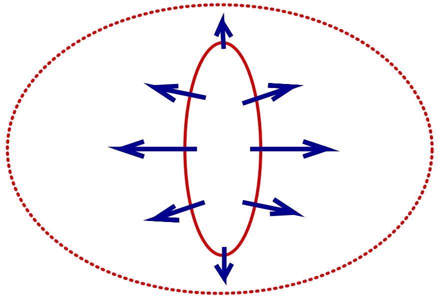
Randeria, Taylor (2010), Enss, Zwerger (2011), Hoffman (2013)

2. Elliptic flow in the unitary Fermi gas





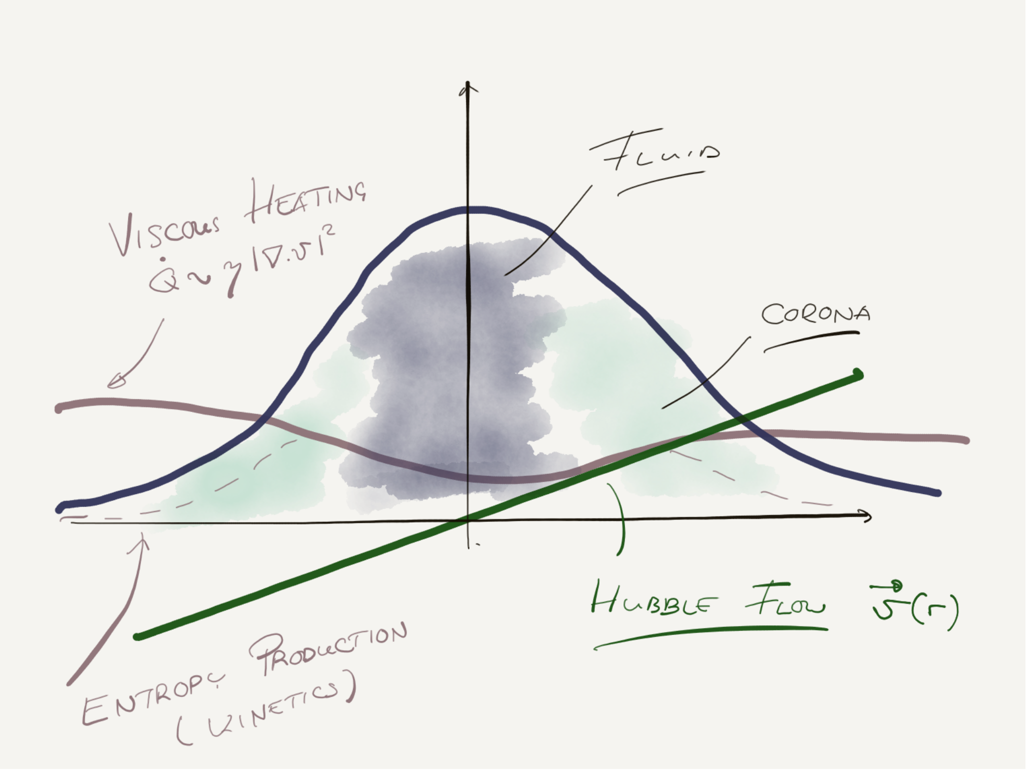
Hydrodynamic expansion converts coordinate space anisotropy to momentum space anisotropy



O'Hara et al. (2002)

Determination of $\eta(n, T)$

Measurement of $A_R(t, E_0)$ determines $\eta(n, T)$. But:



Fluid dynamics breaks down in the dilute corona. Causes large artifacts.

Not a fundamental problem. Corona described by Boltzmann equation near ballistic limit.

Possible Solutions

Combine hydrodynamics & Boltzmann equation. Not straightforward.

Hydrodynamics $+$ non-hydro degrees of freedom ($\mathcal{E}_a$; $a = x, y, z$)

$$\frac{\partial \mathcal{E}_a}{\partial t} + \vec{\nabla} \cdot \vec{j}_a^{\,\epsilon} = -\frac{\Delta P_a}{2\tau} \qquad \Delta P_a = P_a - P$$

$$\frac{\partial \mathcal{E}}{\partial t} + \vec{\nabla} \cdot \vec{j}^{\,\epsilon} = 0 \qquad \mathcal{E} = \sum_a \mathcal{E}_a$$

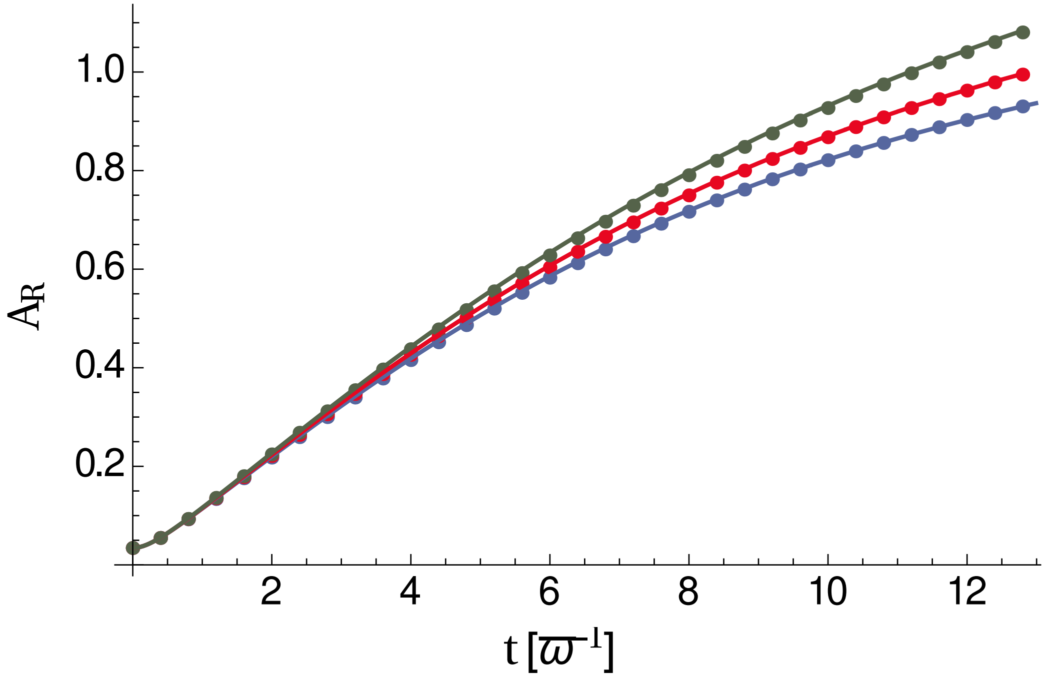
τ small: Fast relaxation to Navier-Stokes with $\tau = \eta / P$

τ large: Additional conservation laws. Ballistic expansion.

Can be derived from kinetics with strongly anisotropic distribution functions, "A-Hydro"

Anisotropic Hydrodynamics: Comparison with Boltzmann

Aspect ratio $A_R(t) = (\langle r_\perp^2 \rangle / \langle r_z^2 \rangle)^{1/2}$ $\quad (T/T_F = 0.79, 1.11, 1.54)$



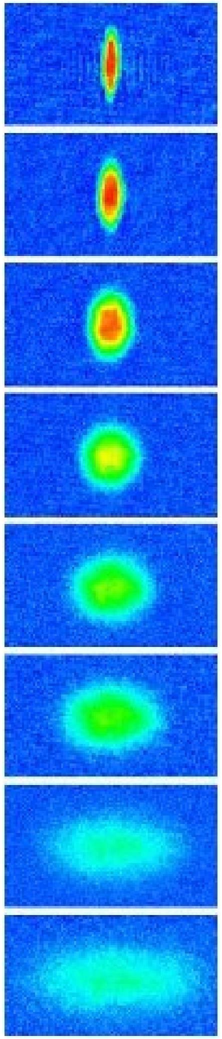
Dots: Two-body Boltzmann equation with full collision kernel

Lines: Anisotropic hydro with η fixed by Chapman-Enskog

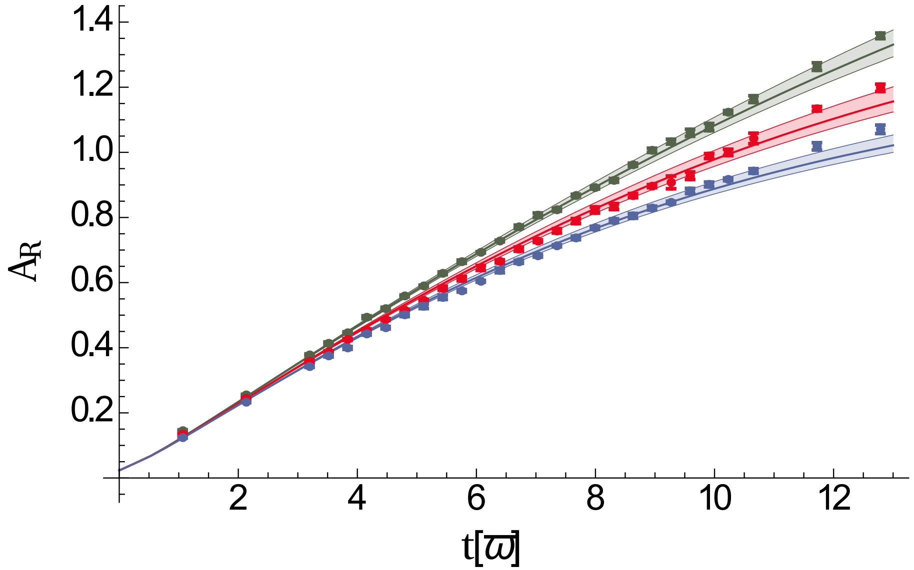
High temperature (dilute) limit: Perfect agreement!

Boltzmann, Pantel et al (2015); AVH1 hydro code, M. Bluhm & T.S. (2015)

Elliptic flow: High T limit



Quantum viscosity $\eta = \eta_0 \frac{(mT)^{3/2}}{\hbar^2}$



Cao et al., Science (2010)

Bluhm et al., PRL (2016)

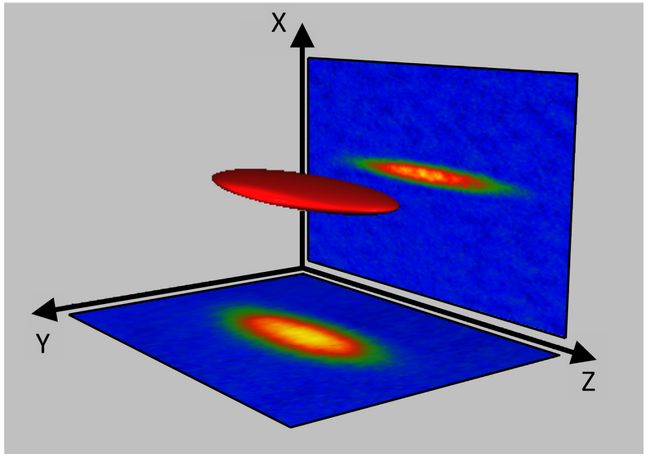
$T/T_F = 0.79, 1.11, 1.54$

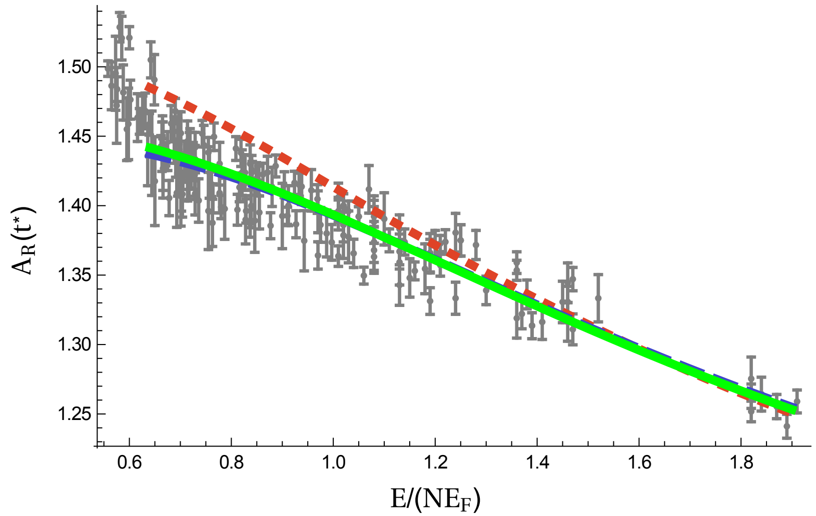
fit: $\eta_0 = 0.28 \pm 0.02$

theory: $\eta_0 = \frac{15}{32\sqrt{\pi}} = 0.269$

Analysis based on AHydro method.

Fluid dynamics analysis: Lower temperatures



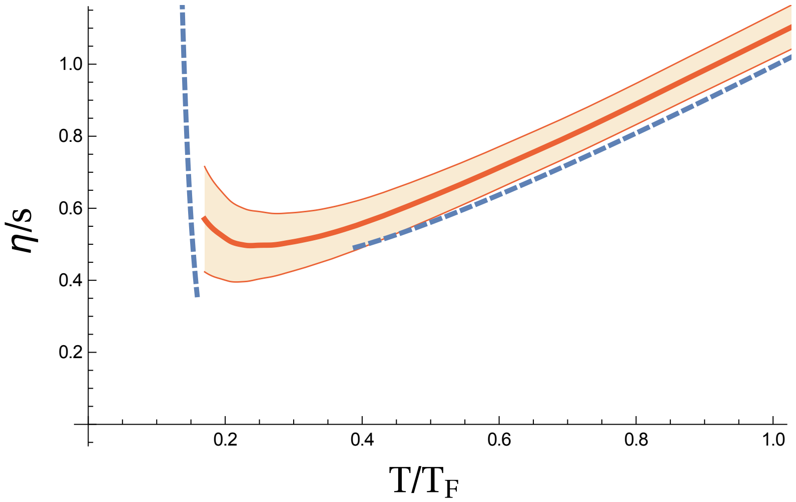


$A_R = \sigma_x/\sigma_y$ as function of total energy. Data: Joseph et al (2016). $E/(NE_F) \sim 0.6$ is the superfluid transition.

Grey, Blue, Green: LO, NLO, NNLO fit.

$$\eta = \eta_0 (mT)^{3/2} \left\{ 1 + \eta_2 n\lambda^3 + \eta_3 (n\lambda^3)^2 + \ldots \right\}$$

Reconstruct η/s (normal fluid)



$T_c \sim 0.17 T_F$. Kinetic theory at low and high T (blue dashed)

Consistency check: $T \gg T_c$

$$\eta|_{T \gg T_c} = (0.265 \pm 0.02)(mT)^{3/2}$$

$$\eta_0(th) = \frac{15}{32\sqrt{\pi}} = 0.269$$

Phenomenology (normal phase): Two-term virial expansion works well

$$\eta \simeq \eta_0 (mT)^{3/2} + \eta_1 \hbar n$$

$$\eta/s|_{T_c} = 0.56 \pm 0.20$$

Also find: $\eta/n|_{T_c} = 0.41 \pm 0.15$ and $\eta_1 = 0.25 \pm 0.07$

Superfluid hydrodynamics

Spontaneous symmetry breaking: $\langle\Psi\rangle = v_0 e^{i\theta}$.

Goldstone boson is a new hydro mode: $\vec{v}_s = \frac{\hbar}{m}\vec{\nabla}\theta$

$$\partial_t \vec{v}_s + \frac{1}{2}\vec{\nabla}(v_s^2) = -\vec{\nabla}\mu_s$$

Momentum density: $\vec{\pi} = \rho_n \vec{v}_n + \rho_s \vec{v}_s$

$$\rho = \rho_n + \rho_s \qquad \rho_n = 2\left.\frac{\partial P}{\partial w^2}\right|_{\mu_s,T} \qquad \vec{w} = \vec{v}_n - \vec{v}_s$$

Stress tensor and energy current

$$\begin{aligned} \Pi_{ij} &= P\delta_{ij} + \rho_n v_{n,i} v_{n,j} + \rho_s v_{s,i} v_{s,j} \\ \vec{j}^{\epsilon} &= sT\vec{v}_n + \left(\mu_s + \frac{1}{2}v_s^2\right)\vec{\pi} + \rho_n \vec{v}_n \vec{v}_n \cdot \vec{w} \end{aligned}$$

Superfluid hydrodynamics

Dissipative stresses

$$\delta\Pi_{ij} = -\eta\left(\nabla_i v_{n,j} + \nabla_j v_{n,i} - \frac{2}{3}\delta_{ij}\vec{\nabla}\cdot\vec{v}_n\right)$$

$$-\,\delta_{ij}\Big(\zeta_1\vec{\nabla}\left(\rho_s\left(\vec{v}_s - \vec{v}_n\right)\right) + \zeta_2\left(\vec{\nabla}\cdot\vec{v}_n\right)\Big)$$

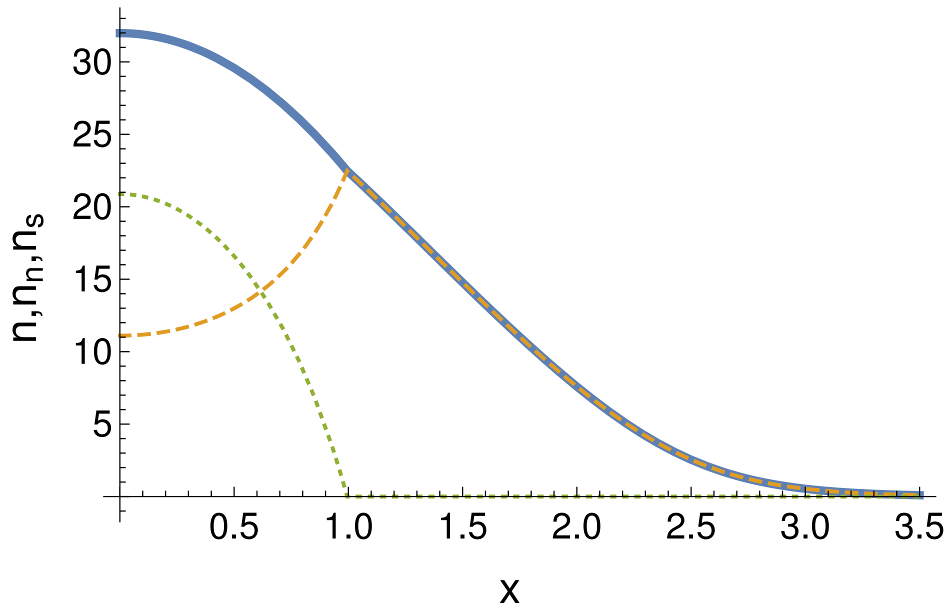
Equation of motions for v_s: $\dot{v}_s + \frac{1}{2}\nabla(v_s^{\,2}) = -\nabla(\mu_s + H)$ with

$$H = -\zeta_3\vec{\nabla}\left(\rho_s\left(\vec{v}_s - \vec{v}_n\right)\right) - \zeta_4\vec{\nabla}\cdot\vec{v}_n$$

Conformal symmetry: $\boxed{\zeta_1 = \zeta_2 = \zeta_4 = 0}$

Son (2007)

Two-fluid hydro for an expanding cloud

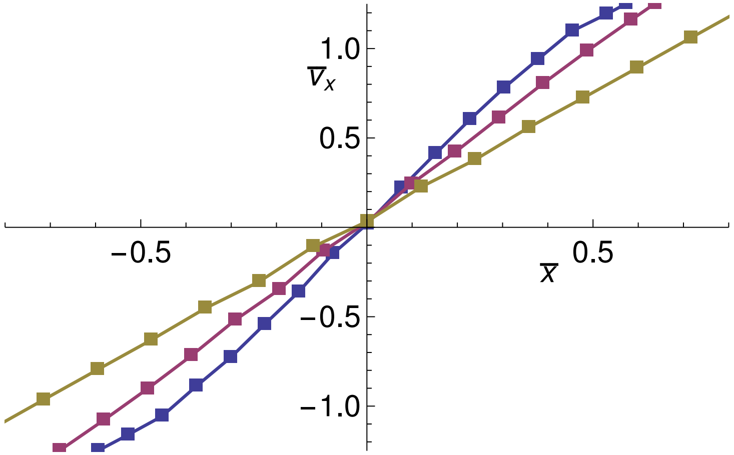


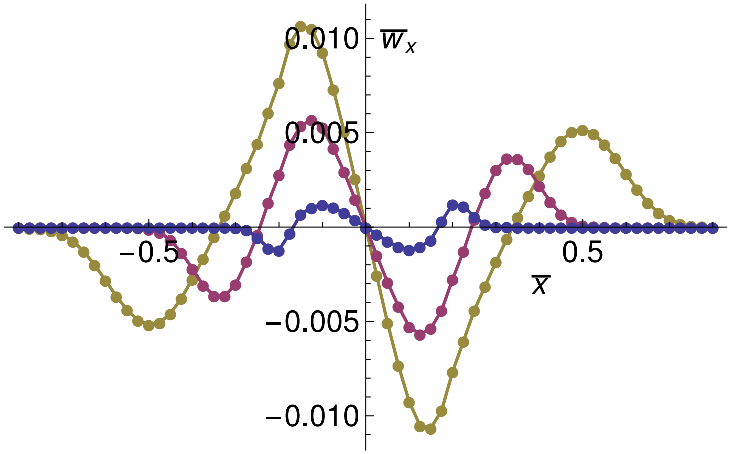
$\rho = \rho_s + \rho_n$ (solid), ρ_n (dashed), ρ_s (dotted)

Gibbs-Duhem relation

$$dP = n d\mu_s + s dT + \frac{\rho_n}{2} dw^2$$

Two-fluid hydro for an expanding cloud



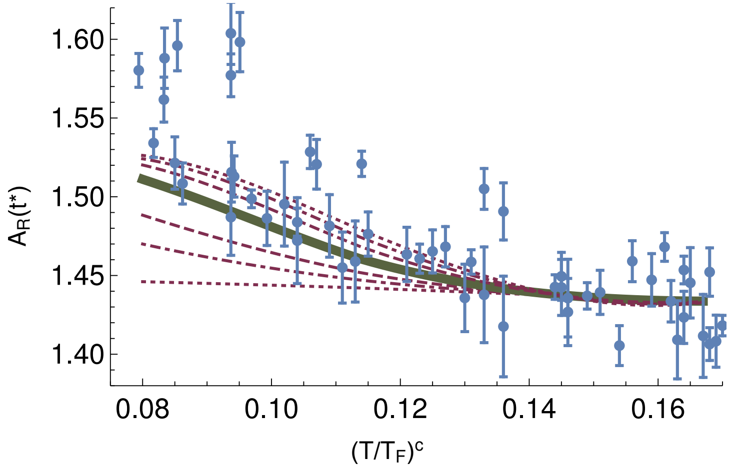


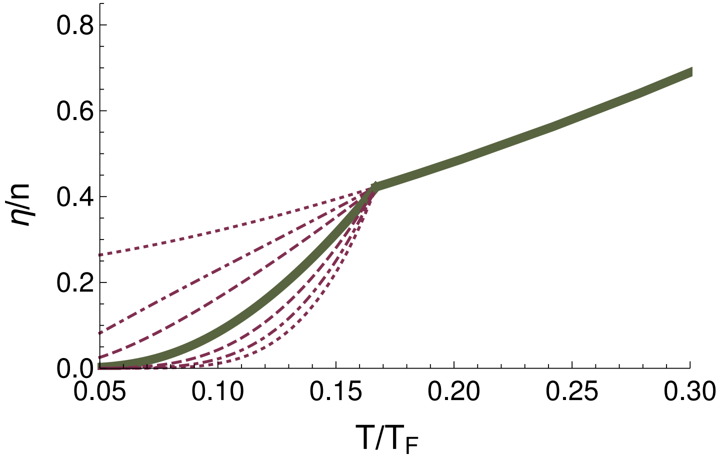
Average fluid velocity $v_x(x,t)$. Superfluid $w_x(x,t) = v^n_x(x,t) - v^s_x(x,t)$

Superfluid $\vec{w} = \vec{v}^n - \vec{v}^s$ can be computed perturbatively.

$$\left(\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} \right) \vec{w} = -\frac{s}{\rho_n} \vec{\nabla} T + O(w^2) \, .$$

Two-fluid hydro analysis of expanding cloud





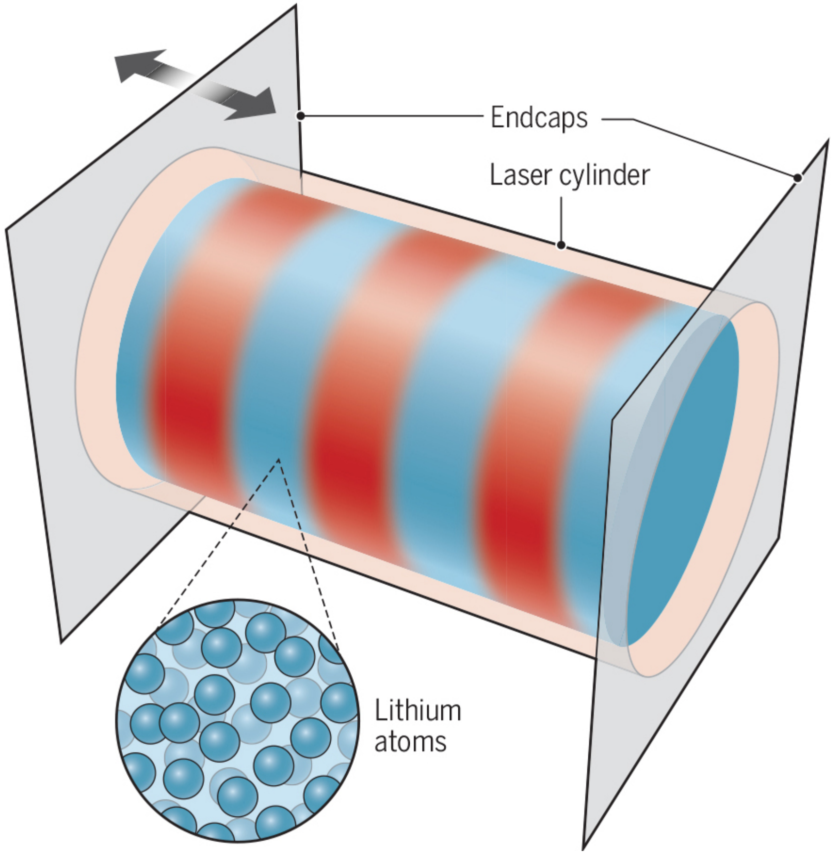
A_R in low T regime.

Small η corresponds to large A_R.

Fits for $\eta(T < T_c)$:

$$\eta \simeq \eta_0 \exp\left[-2\frac{T_c - T}{T}\right]$$

3. Linear Response: Sound attenuation

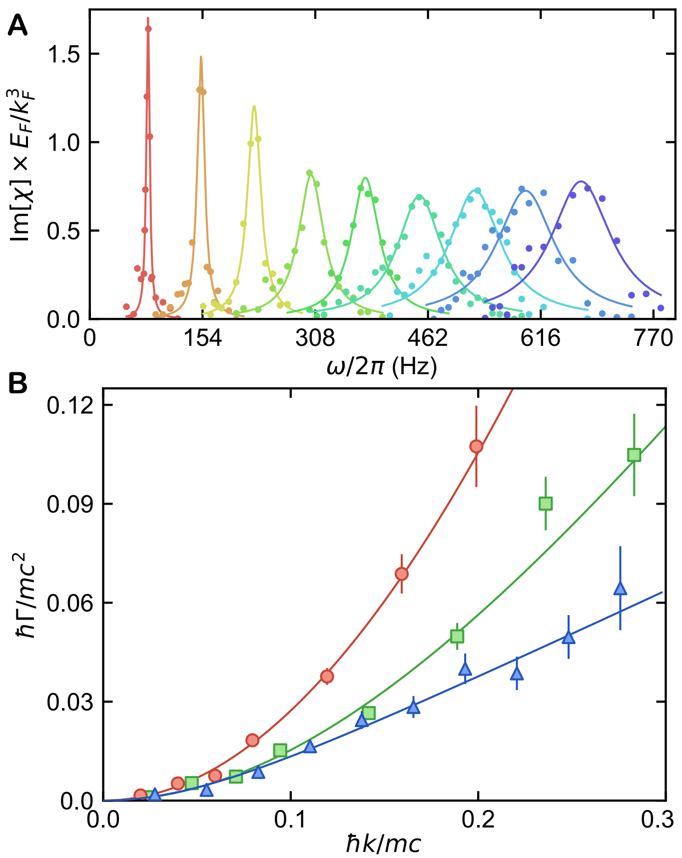


Cylindrical box, response to small harmonic drive.

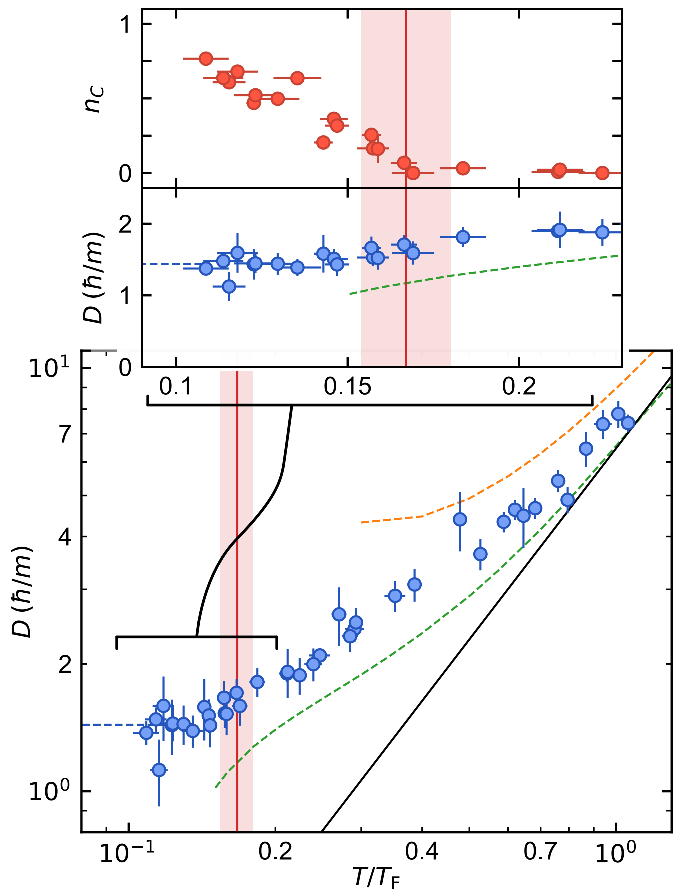
Patel et al., Science (2021), Schaefer, Science (2021)

Sound attenuation (MIT)

Spectral response $\rho_k(\omega)$.



Sound diffusivity $D_s(T)$



Damping rate $\Gamma(k)$

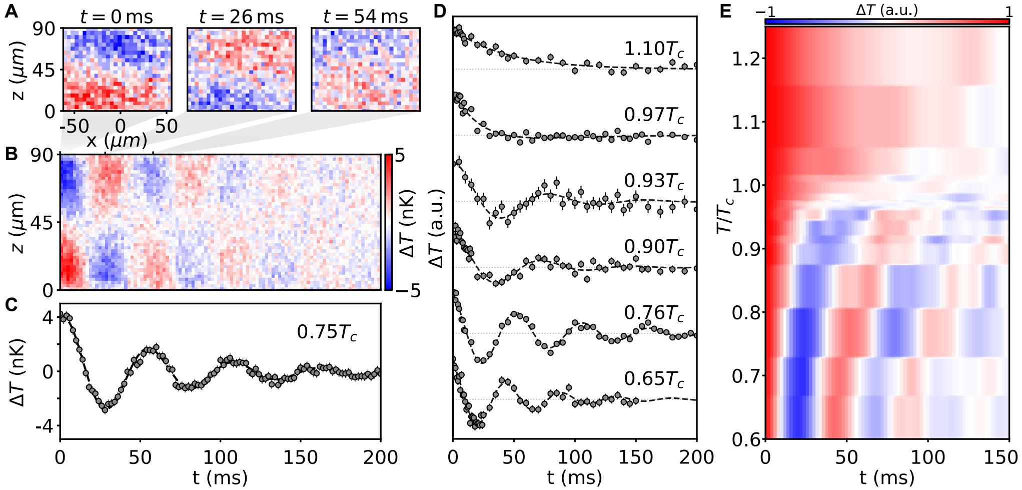
$$D_s = \frac{4\eta}{3\rho} + \frac{4\kappa T}{15P}.$$

$(T/T_F = 0.36, 0.21, 0.13)$.

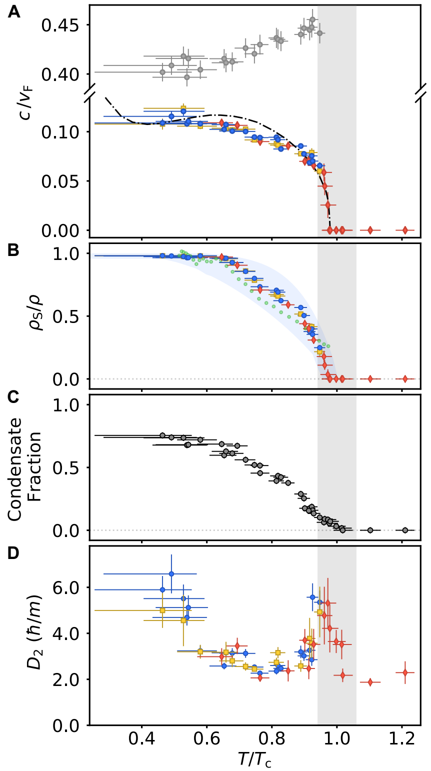
Patel et al., Science (2021)

MIT: Thermography and second sound

Heat propagation above and below T_c



2^{nd} sound diffusivity



c_2

ρ_s/ρ

f_c

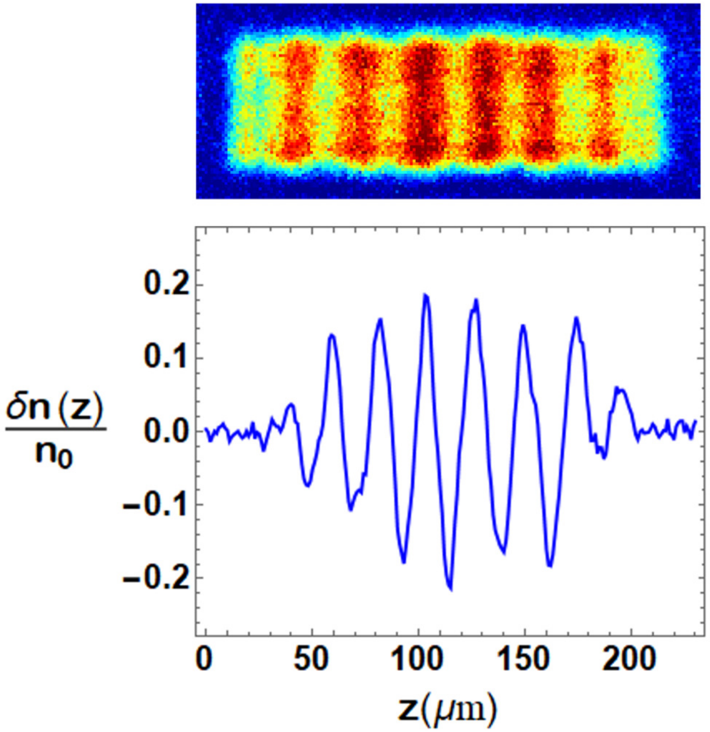
D_2

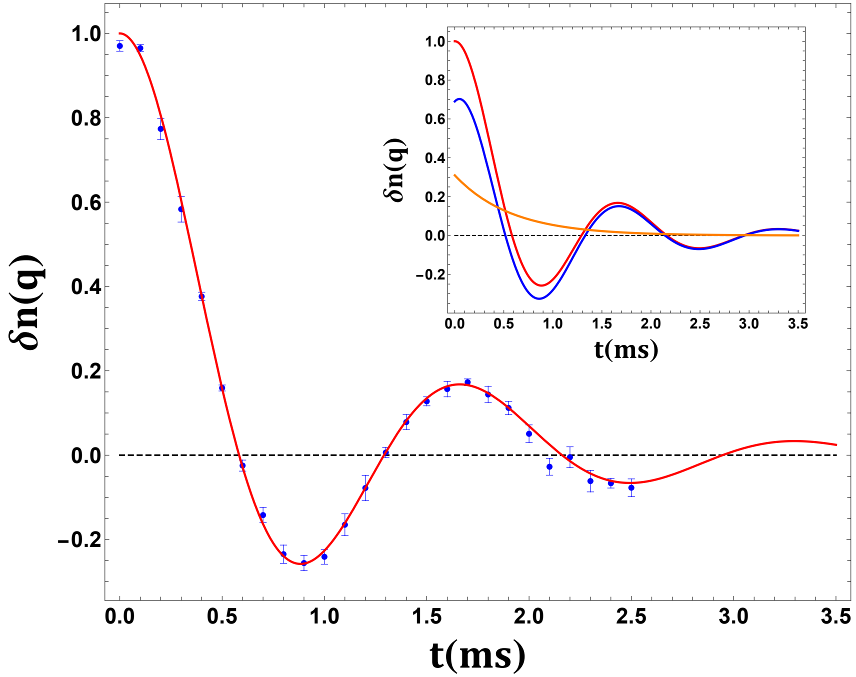
From diffusion to second sound

Yan et al., Science (2024)

Linear Response (NC State)

Baird et al., PRL 2019; Wang et al, PRL 2022.

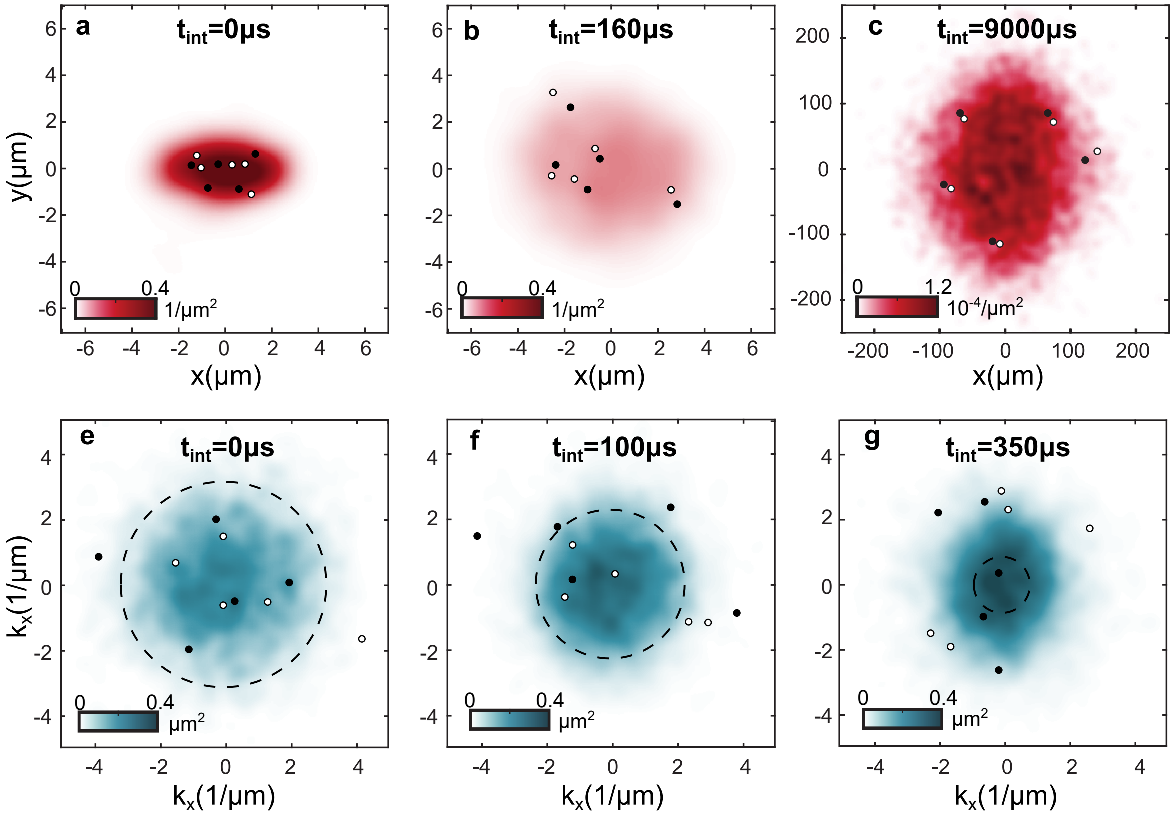




$$\left.\frac{\kappa}{\eta}\right|_{T \gg T_c} = 0.93(14)\,\frac{15k_B}{4m}$$

Flow in (very) small systems

S. Brandstetter et al., Nature (2025)



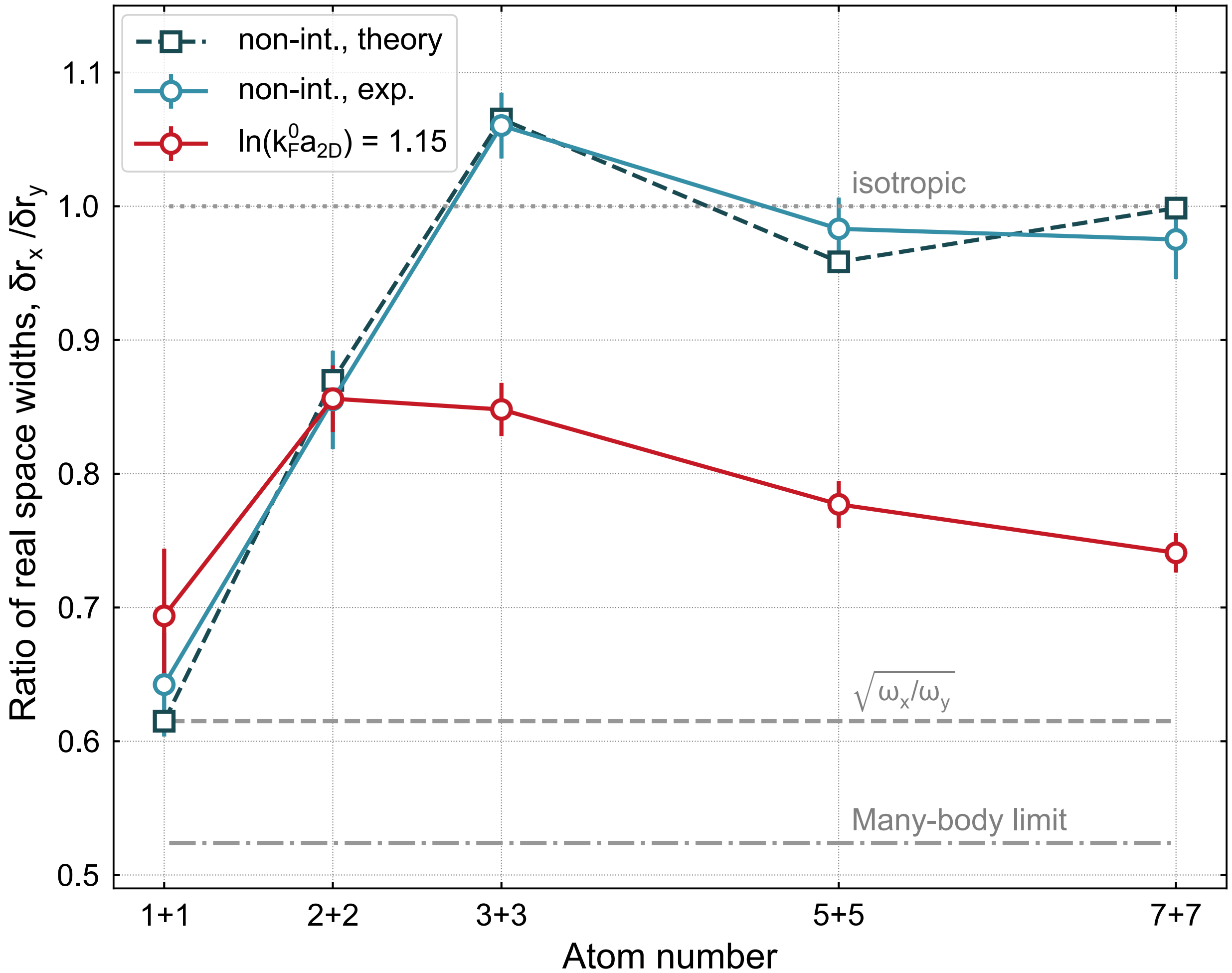
real space

momentum space

5+5 particles in two dimensions $(\log(k_F a) = 1.15)$

Flow in (very) small systems

S. Brandstetter et al., Nature (2025)



Free Fermi gas

One-body

Ideal Fluid

Dependence on particle number

4. Outlook: External fields, OTOCs, etc.

Can realize response to $A_0(x,t)$, as well as spatial/time variation of scattering length

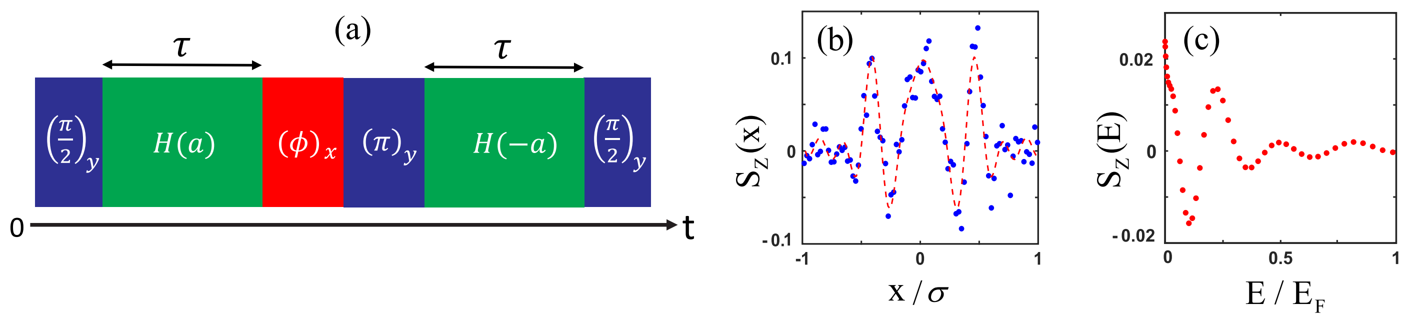
$$H' = \psi^\dagger \psi A_0(x,t), \qquad\qquad H' = C_0(x,t)(\psi^\dagger\psi)^2$$

We would like to realize non-trivial metric perturbations

$$H' = \frac{g_{xy}(x,t)}{m}\psi^\dagger \nabla_x \nabla_y \psi$$

We would also like to realize out-of-time-order correlators

$$C(t) = \langle [V(t), W(0)]^2 \rangle$$



Pegahan et al., PRL 2021.