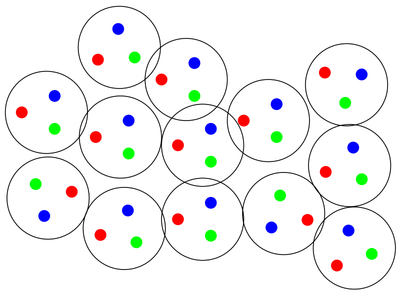
QCD at Finite Density

(Quark Matter)

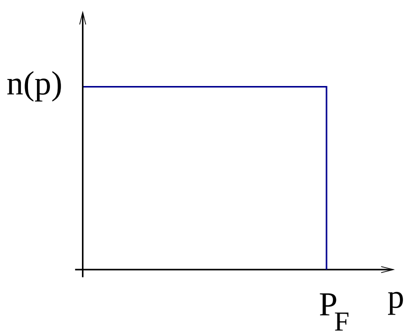
Very Dense Matter

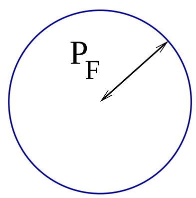
Consider baryon density $n_B \gg 1\,\mathrm{fm}^{-3}$



quarks expected to move freely

Ground state: cold quark matter (quark fermi liquid)





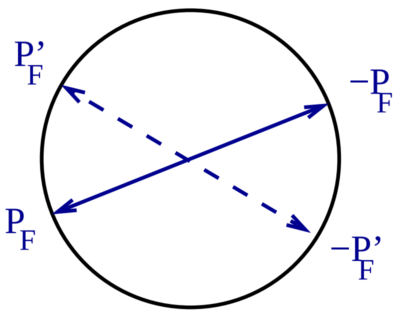
only quarks with $p \sim p_F$ scatter

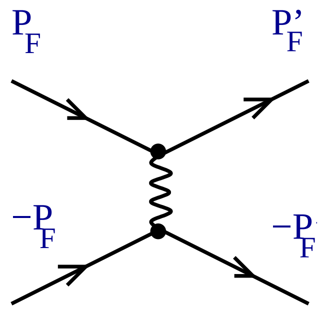
$p_F \gg \Lambda_{QCD} \rightarrow$ coupling is weak

No chiral symmetry breaking, confinement, or dynamically generated masses

Color Superconductivity

Is the quark liquid stable?





Dominant interaction:

Uses Fermi surface

coherently

Attractive interaction leads to instability

$\langle qq \rangle$ condensate, superfluidity/superconductivity, gap in fermion spectrum, transport without dissipation

QCD: gluon exchange attractive in $\bar{3}$ channel

$$3 \times 3 = 6_S + 3_A \qquad \text{flux reduced} \Rightarrow \text{attractive}$$

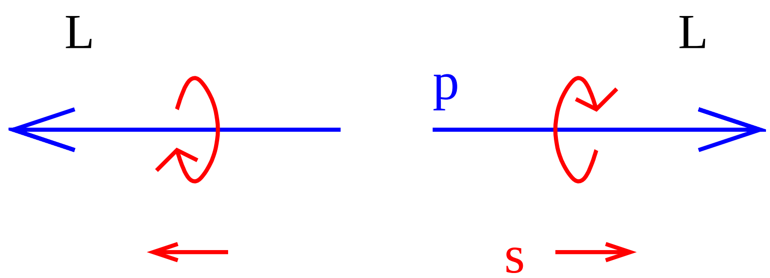
Spin-flavor-color wave function

$$(\uparrow\downarrow - \downarrow\uparrow) \times (ud - du) \times (rb - br) \qquad s = 0,\ I = 0,\ c = \bar{3}$$

Order parameter

$$\Phi^a = \epsilon^{abc}\langle q^b C\gamma_5\tau_2 q^c\rangle$$

Chiral symmetry is not broken



$$\Phi \sim \langle q_L q_L\rangle - \langle q_R q_R\rangle$$

Color symmetry broken by Higgs mechanism (Meissner effect)

$$\Phi^a \in [\bar{3}];\ \Rightarrow\ SU(3) \rightarrow SU(2)$$

5/8 gluons acquire mass via Higgs mechanism, $SU(2)$ is confined

QQ vs $\bar{Q}Q$ Condensation

Schematic interaction: $\mathcal{L} = G(\bar{\psi}\gamma_\mu\lambda^a\psi)^2$

$$\mathcal{L} = G_M(\bar{\psi}\psi)^2 + \ldots \qquad\qquad \mathcal{L} = G_D(\psi C\gamma_5\tau_2\lambda_2\psi)(h.c.) + \ldots$$

$\bar{Q}Q$ gap equation

$$M = m_0 + G_M\langle\bar{q}q\rangle \qquad \langle\bar{q}q\rangle = -\frac{3}{\pi^2}\int p^2 dp\,\frac{M}{E_p}(1-n_F)$$

QQ gap equation

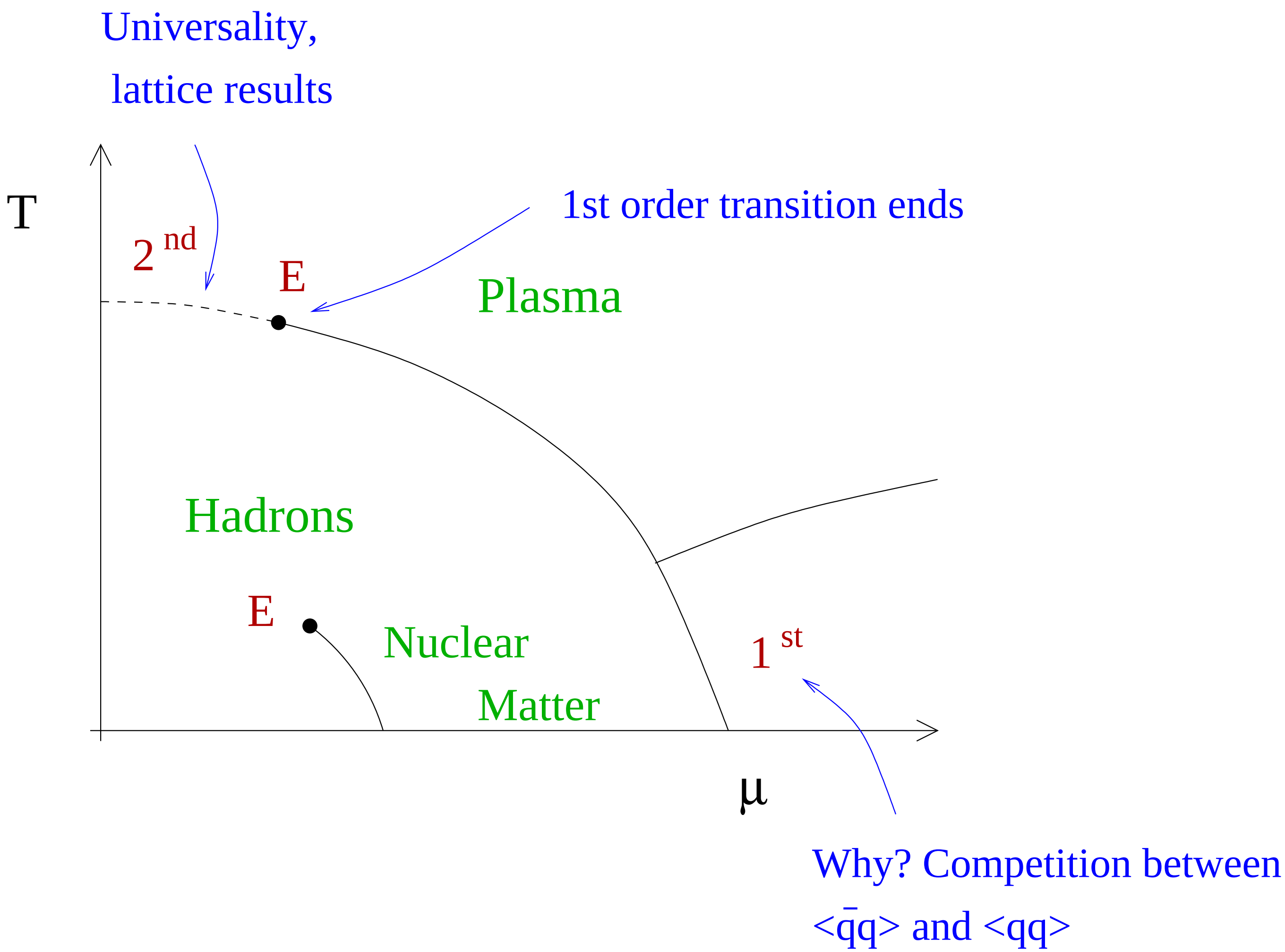
$$\Delta = G_D\langle qq\rangle \qquad \langle qq\rangle = \frac{1}{4\pi^2}\int p^2 dp\,\frac{\Delta}{[(E_p-\mu)^2+\Delta^2]^{1/2}}$$

Condensation energy

$$E_{\bar{Q}Q} = -f_\pi^2 M^2 \qquad\qquad E_{QQ} = \frac{\mu^2}{2\pi^2}\Delta^2$$

$G_M > G_D$ favors $\bar{Q}Q$ $\qquad$ $\mu > 0$ favors QQ

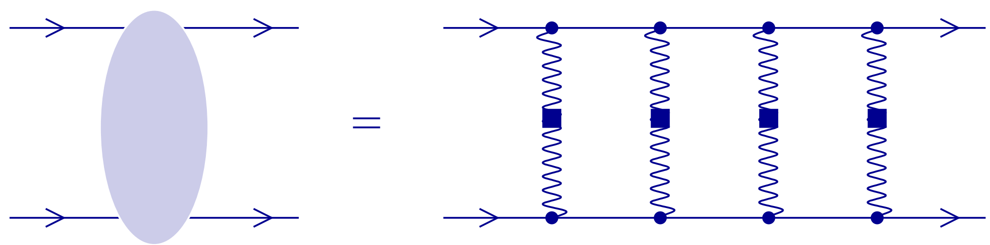
Phase Diagram: Second Revision



critical endpoint (E) persists even if $m \neq 0$

Very Large Density: Gap Equation

$\mu \gg \Lambda_{QCD}$: perturbative forces dominate



Small angle scattering dominates $\longrightarrow$ medium effects important

$D_E = \frac{1}{\vec{q}^2 + 2m^2}$ $\quad m^2 = \frac{N_f}{4\pi^2} g^2 \mu^2$ $\quad$ Debye screening

$D_M = \frac{1}{\vec{q}^2 + i\frac{\pi}{2} m^2 \frac{\omega}{q}}$ $\quad \omega < q$ $\quad$ Landau damping

Consider $\omega \simeq \Delta$. Typical momenta

$$q_E \simeq g\mu \quad q_M \simeq (g^2\mu^2\Delta)^{1/3}$$

Superconductivity driven by magnetic forces!

Eliashberg Equation

Retardation important: Eliashberg theory

$$\Delta(p_0) = \frac{g^2}{18\pi^2} \int dq_0 \left\{ \log\left(\frac{b_M}{|p_0 - q_0|}\right) + \ldots \right\} \frac{\Delta(q_0)}{\sqrt{q_0^2 + \Delta(q_0)^2}}$$

collinear log BCS log

Double logarithmic behavior

$$\Delta_0 \sim \alpha_s \Delta_0 \left[\log\left(\frac{\mu}{\Delta_0}\right)\right]^2 \quad \rightarrow \quad \Delta_0 \sim \exp\left(-\frac{c}{\sqrt{\alpha_s}}\right)$$

More careful analysis (2SC phase)

$$\Delta_0 = 512\pi^4 \mu g^{-5} \exp\left(-\frac{\pi^2 + 4}{8}\right) \exp\left(-\frac{3\pi^2}{\sqrt{2}g}\right)$$

Condensation energy

$$\epsilon \simeq 4\Delta_0^2 \left(\frac{\mu^2}{4\pi^2}\right)$$

Phase structure: gap matrix is of the form

$$(\Delta_{ij}^{ab})_{\alpha\beta} \qquad\qquad ij \text{ flavor}, ab \text{ color}, \alpha\beta \text{ spin}$$

Have to minimize $F[(\Delta_{ij}^{ab})_{\alpha\beta}]$. In most cases

$$(\Delta_{ij}^{ab})_{\alpha\beta} = (C\gamma_5)_{\alpha\beta}\Delta_{ij}^{ab}$$

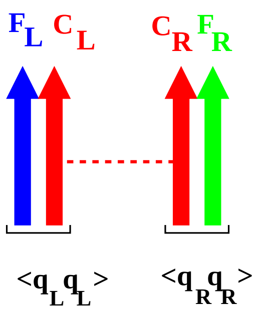
Beautiful Case: $N_f = 3 \ (m_q = 0)$

Color-Flavor Locking (CFL)

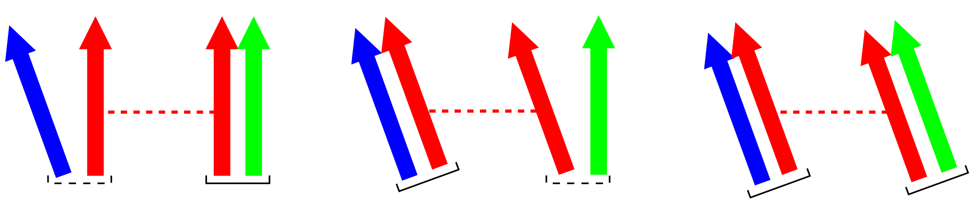
$$\langle q_i^a q_j^b \rangle = \phi \ \left(\delta_i^a \delta_j^b - \delta_j^a \delta_i^b \right)$$

Note that $\langle ud \rangle = \langle us \rangle = \langle ds \rangle$, $\langle rb \rangle = \langle rg \rangle = \langle bg \rangle$

Symmetry breaking pattern



$$SU(3)_L \times SU(3)_R \times [SU(3)]_C \times U(1)$$
$$\rightarrow SU(3)_{C+F}$$



Rotate left flavor

Compensate by rotating color

... have to rotate right flavor also !

Novel mechanism for chiral symmetry breaking:

$$\langle \psi_L \psi_L \rangle = -\langle \psi_R \psi_R \rangle$$

Breaks chiral $SU(3)_L \times SU(3)_R$ symmetry because

$$SU(3)_L \stackrel{Lock}{\longleftrightarrow} SU(3)_C \stackrel{Lock}{\longleftrightarrow} SU(3)_R$$

Gauge invariant order parameters

$\langle (\bar{q}q)^2 \rangle$	χSB
$\langle \bar{q}q \rangle$	χSB (instantons)
$\langle (uds)(uds) \rangle$	$U(1)$

CFL Phase: Excitations

Quark pairs are charged, but $U(1)_{EM}$ remains unbroken

$$Q^* = \alpha_{11} Q + \alpha_{12} T_3 + \alpha_{13} T_8$$

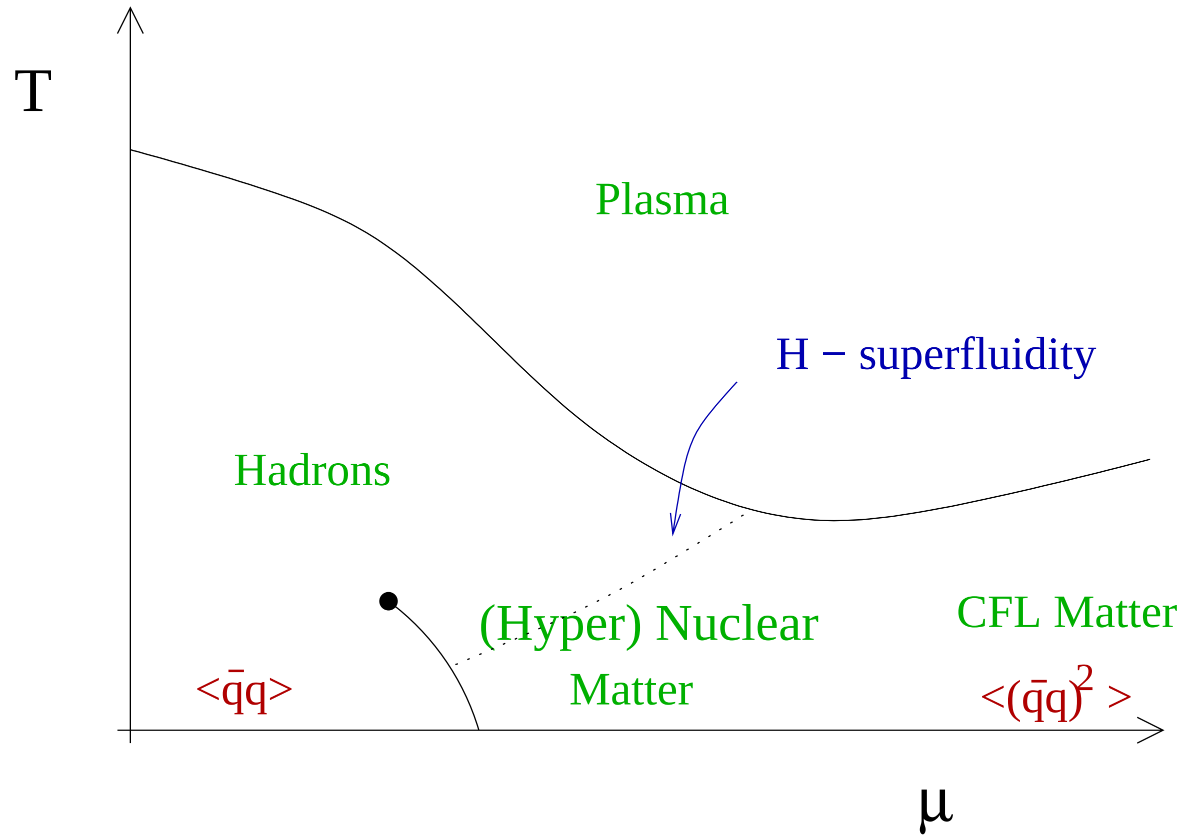
CFL quark matter is a transparent insulator

Excitations classified by $SU(3)_F$ and Q^*

$[8]$	$Q=0,\pm 1$	Goldstones (π, K, η)	$(QQ)(QQ)^{-1},\ldots$
$[8]+[1]$	$Q=0,\pm 1$	baryons (p,n,Λ,Σ)	$(Q)(QQ),\ldots$
$[8]$	$Q=0,\pm 1$	vectors $(\rho, K^*,\ldots)$	$g, QQ^{-1},\ldots$
$[1]$	$Q=0$	$U(1)_A$ Goldstone (η')	$QQ^{-1},(QQ)(QQ)^{-1},\ldots$
$[1]$	$Q=0$	$U(1)_B$ Goldstone	$QQ^{-1},(QQ)(QQ)^{-1},\ldots$

Continuity between quark and hadron matter?

Phase Diagram: Third Revision



Questions

Less symmetric states

$$m_s \neq 0, \;\; \mu_e \neq 0$$

Spectrum of excitations

$$m_\pi, \, m_K, \;\; f_\pi, \ldots$$

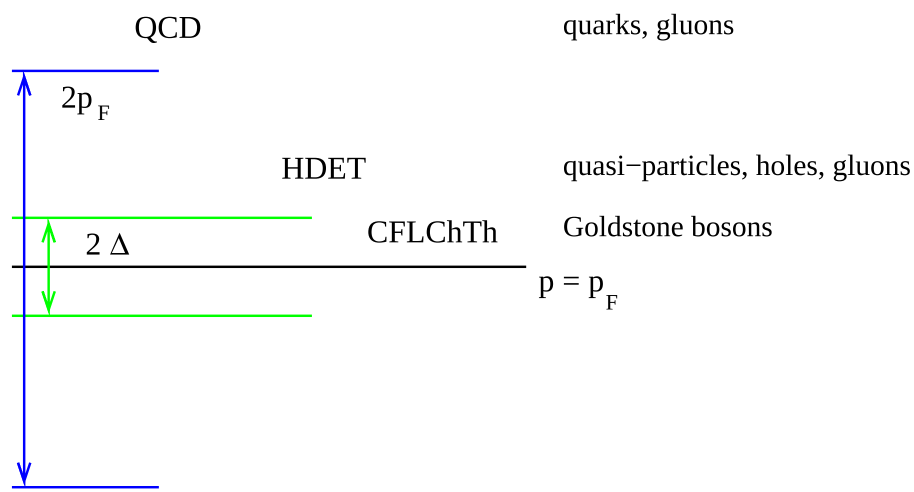
Transport properties

mean free path $(\nu, \gamma, \ldots)$

specific heat, thermal conductivity

neutrino emissivity

Effective Field Theories

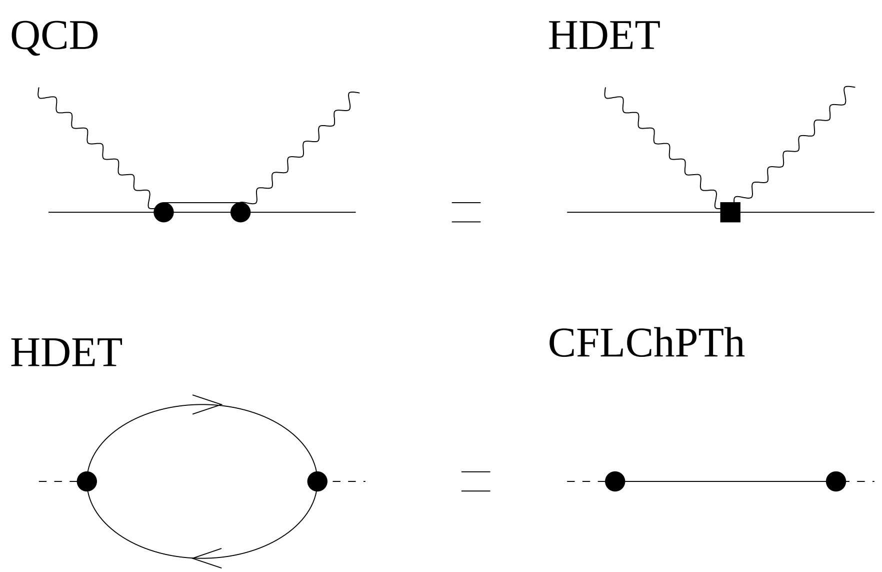


QCD $\mathcal{L} = \bar\psi(i\not{D} + \mu\gamma_0)\psi - \frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu}$

HDET $\mathcal{L} = \psi_v^\dagger(iv\cdot D)\psi_v - \frac{\Delta}{2}\psi^T_{-v}C\psi_v - \frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} + \ldots$

CFLχT $\mathcal{L} = \frac{f_\pi^2}{4}\mathrm{Tr}\left[\nabla_0\Sigma\nabla_0\Sigma^\dagger - v^2\vec\nabla\Sigma\vec\nabla\Sigma^\dagger\right] + \ldots$

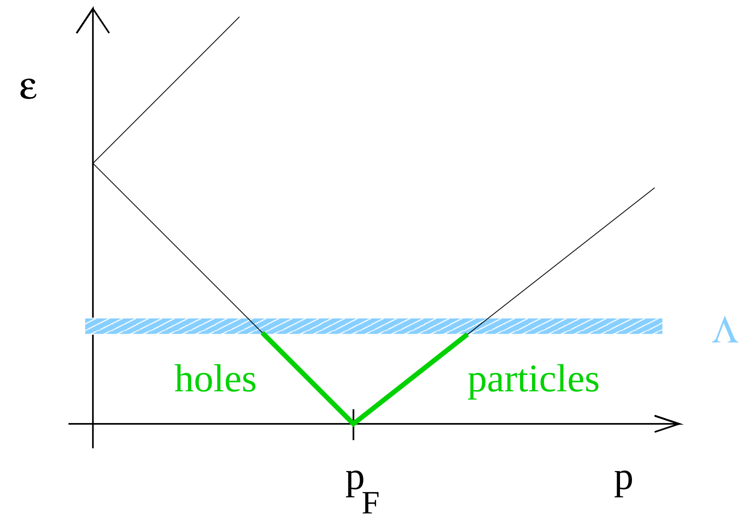
Coefficients determined by matching Green functions



High density effective theory

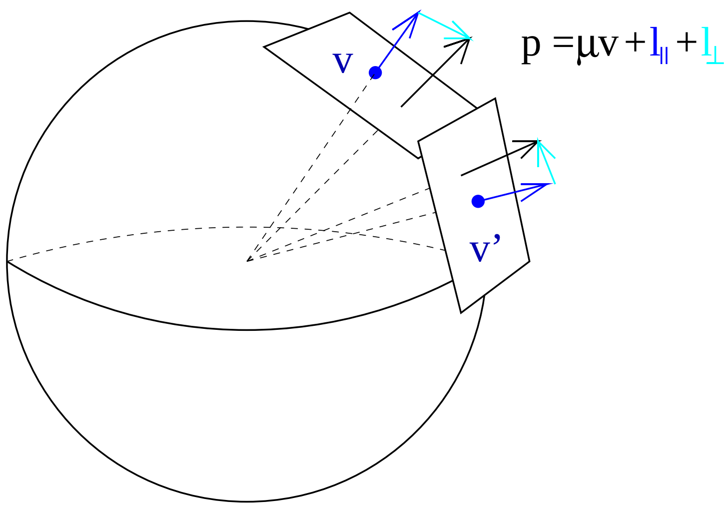
Quasi-particles (holes)

$$E_\pm = -\mu \pm \sqrt{\vec{p}^2 + m^2}$$
$$\simeq -\mu \pm |\vec{p}|$$



Effective field theory on v-patches

$$\psi_{v\pm} = e^{-i\mu v\cdot x}\left(\frac{1 \pm \vec{\alpha}\cdot\vec{v}}{2}\right)\psi$$

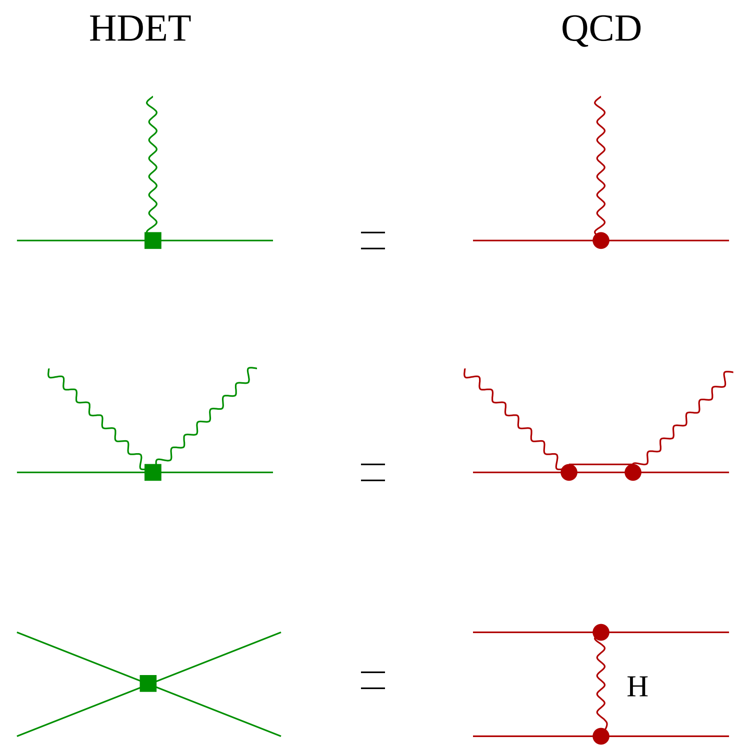


Effective lagrangian for ψ_{v+}

$$\mathcal{L} = \psi_v^\dagger (iv \cdot D)\psi_v$$

$$+ \frac{1}{2p_F}\psi_v^\dagger \left[(\vec{\alpha}_\perp \cdot \vec{D})^2 + MM^\dagger \right] \psi_v$$

$$+ \frac{\Gamma_{\vec{v}\cdot\vec{v}'}}{p_F^2}(\psi_{v'}^\dagger \psi_v)(\psi_{v'}^\dagger \psi_v) + \ldots$$



Effective Chiral Theory

Oscillations of the order parameter

$$X_i^a = \epsilon_{ijk}\epsilon^{abc}(\psi_L)_j^b(\psi_L)_k^c, \qquad \langle X_i^a \rangle \sim \delta_i^a$$

$$Y_i^a = \epsilon_{ijk}\epsilon^{abc}(\psi_R)_j^b(\psi_R)_k^c, \qquad \langle Y_i^a \rangle \sim \delta_i^a$$

Low energy degrees of freedom

$$V\Sigma = XY^\dagger = \exp\left(\frac{i\phi^a\lambda^a}{f_\pi}\right) \qquad \phi^a = (\pi, K, \eta, \eta')$$

$$\text{e.g. } K^0 \sim \epsilon^{abc}\epsilon_{ade}(\bar{u}_R^b C \bar{s}_R^c)(d_L^d C u_L^e)$$

Charges under $SU(3)_L \times SU(3)_R \times U(1)_V \times U(1)_A$

$$\Sigma: \quad (3,\bar{3})_{0,0} \qquad V: \quad (1,1)_{0,4} \qquad M: \quad (3,\bar{3})_{0,-2}$$

Effective theory

$$\begin{aligned}\mathcal{L} &= \tfrac{f_\pi^2}{4}\mathrm{Tr}\left(\nabla_0\Sigma\nabla_0\Sigma^\dagger - v^2\vec{\nabla}\Sigma\vec{\nabla}\Sigma^\dagger\right) \\ &\quad + \frac{f^2}{4}\left(\nabla_0 V\nabla_0 V^\dagger - v^2\vec{\nabla}V\vec{\nabla}V^\dagger\right) \\ &\quad + A\mathrm{Tr}(M\Sigma^\dagger)Ve^{i\Theta} \qquad\qquad \leftarrow \;\; U_A(1) \text{ anomaly} \\ &\quad + V^\dagger\left(B_1\left[\mathrm{Tr}(M\Sigma)\right]^2 + B_2\mathrm{Tr}\left[(M\Sigma)^2\right]\right)\end{aligned}$$

Chiral expansion

$$\mathcal{L} \sim f_\pi^2\Delta^2\left(\frac{\vec{\partial}}{\Delta}\right)^{N_1}\left(\frac{\partial_0 + MM^\dagger/p_F}{\Delta}\right)^{N_2}\left(\frac{MM}{p_F^2}\right)^{N_3}(\Sigma)^{N_4}(\Sigma^\dagger)^{N_4}$$

Note:

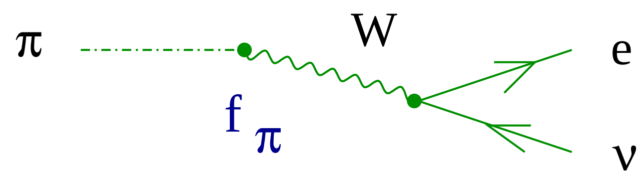
$$\Lambda_{\chi SB} \sim \Delta \ll 4\pi f_\pi$$

$$\frac{M^2}{p_F^2} \ll \frac{M^2}{p_F\Delta}$$

Matching, part I

Compute f_π: Gauge $SU(3)_L \times SU(3)_R$ flavor symmetry

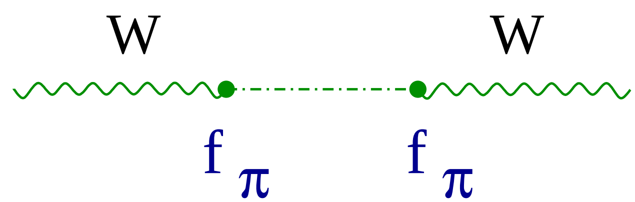
$$\nabla_\mu \Sigma = \partial_\mu \Sigma - iW_\mu^L \Sigma + i\Sigma W_\mu^R$$



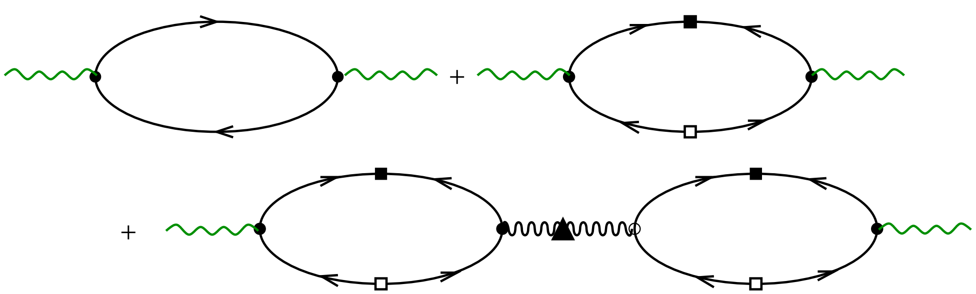
Higgs phenomenon

$$\mathcal{L} = \frac{f_\pi^2}{4} \mathrm{Tr}\left[(W_0^L - W_0^R)^2\right] + \ldots$$

$$m_W^2 = f_\pi^2$$



Microscopic theory



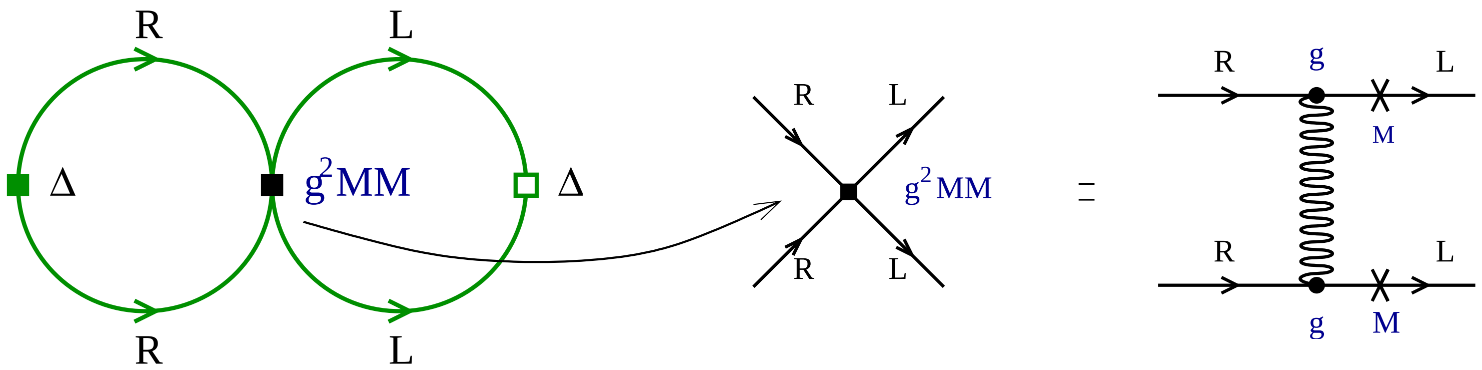
$$f_\pi^2 = \frac{21 - 8\log(2)}{18} \left(\frac{\mu^2}{2\pi^2}\right)$$

Matching, part II

Compute quark mass dependence of vacuum energy

$$\Delta\mathcal{E} = -B_1\left[\mathrm{Tr}(M)\right]^2 - B_2\mathrm{Tr}(M^2) \qquad \Sigma = 1$$

Microscopic theory



$$\Delta\mathcal{E} \sim \left(\frac{g^2}{p_F^4}\right)\left[p_F^2\Delta\log(\Delta)\right]^2\left\{[\mathrm{Tr}(M)]^2 - \mathrm{Tr}(M^2)\right\}$$

$$\sim \Delta^2\left\{[\mathrm{Tr}(M)]^2 - \mathrm{Tr}(M^2)\right\}$$

Find

$$B_1 = -B_2 = \frac{3\Delta^2}{4\pi^2} \qquad \left[\sim f_\pi^2 \Delta^2 \left(\frac{m^2}{p_F^2} \right) \right]$$

Meson masses $m_{GB}^2 \sim m^2$

$$m_\pi^2 = \frac{3\Delta^2}{4f_\pi^2}(m_u + m_d)m_s$$

$$m_{K^\pm}^2 = \frac{3\Delta^2}{4f_\pi^2}(m_u + m_s)m_d$$

Note: $m_{GB} < \Delta$, spectrum inverted

$$m_{GB} \sim 10 \text{ MeV} \qquad m_K < m_\pi$$

$$f_\pi \sim 100 \text{ MeV}$$

Matching, part III

Consider $1/p_F$ expansion

$$\mathcal{L} = \psi_L^\dagger \left(p_0 - \epsilon_p - \frac{MM^\dagger}{2p_F} \right) \psi_L + \frac{\Delta}{2} \psi_L C \psi_L + O(M^2/p_F^2)$$

$MM^\dagger$ and $M^\dagger M$ enter as gauge fields

$$W_L = \frac{MM^\dagger}{2p_F} \qquad \psi_L \to L\psi_L, \;\; W_L \to L W_L L^\dagger + iL\partial_0 L^\dagger$$

$$W_R = \frac{M^\dagger M}{2p_F} \qquad \psi_R \to R\psi_R, \;\; W_R \to R W_R R^\dagger + iR\partial_0 R^\dagger, \ldots$$

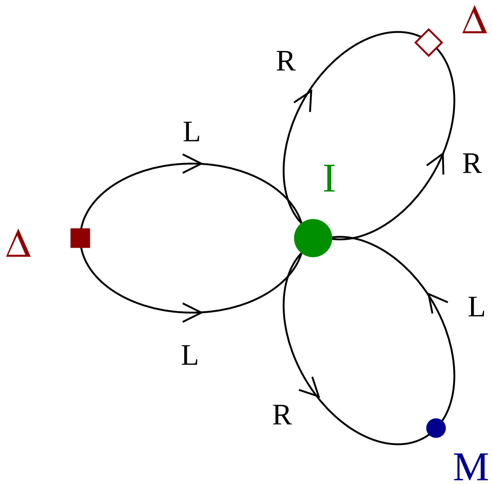
Implement gauge symmetry in effective lagrangian

$$\mathcal{L} = \frac{f_\pi^2}{4} \mathrm{Tr} \left(\nabla_0 \Sigma \nabla_0 \Sigma^\dagger \right) + \ldots \qquad \nabla_0 \Sigma = \partial_0 \Sigma + iW_L \Sigma - i\Sigma W_R$$

Acts like an effective chemical potential $\hat{\mu} = W_L = W_R$

Matching, anomalous part

Linear term $\mathrm{Tr}(M\Sigma)$ in vacuum energy related to instantons



Instanton size

$$\rho \sim \mu^{-1} \ll \Lambda_{QCD}^{-1}$$

Instanton contribution to vacuum energy: $\mathcal{E} = A\mathrm{Tr}(M)$

$$A = C_{N_c}^{N_f} \langle \psi\psi \rangle^2 \left(b \log \left(\frac{\mu}{\Lambda_{QCD}} \right) \right)^6 \left(\frac{\Lambda_{QCD}}{\mu} \right)^1 2\Lambda_{QCD}^{-3} e^{i\Theta}$$

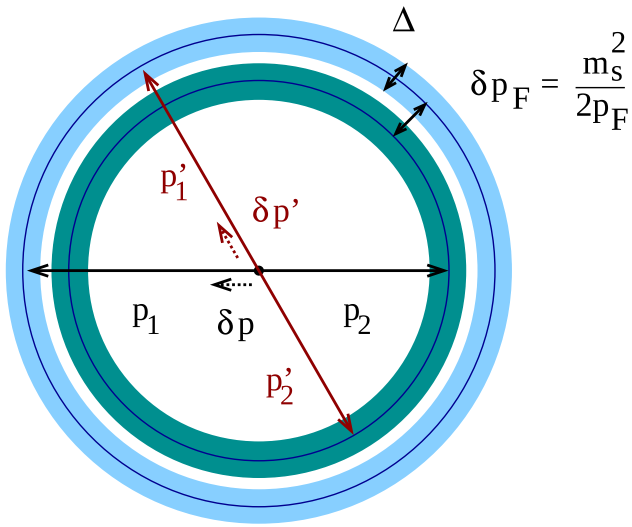
$$\langle \bar{\psi}\psi \rangle \sim \left(\frac{\Lambda_{QCD}}{\mu} \right)^8 \Lambda_{QCD}^{-3} \ll \Lambda^{-3}$$

Quark Mass Expansion: $(m/p_F), (m/\Delta), \ldots$?

Two parameters appear in matching procedure

$$\left(\frac{m^2}{p_F\Delta}\right), \qquad \left(\frac{m^2}{p_F^2}\right) = \left(\frac{\Delta}{p_F}\right)\left(\frac{m^2}{p_F\Delta}\right)$$

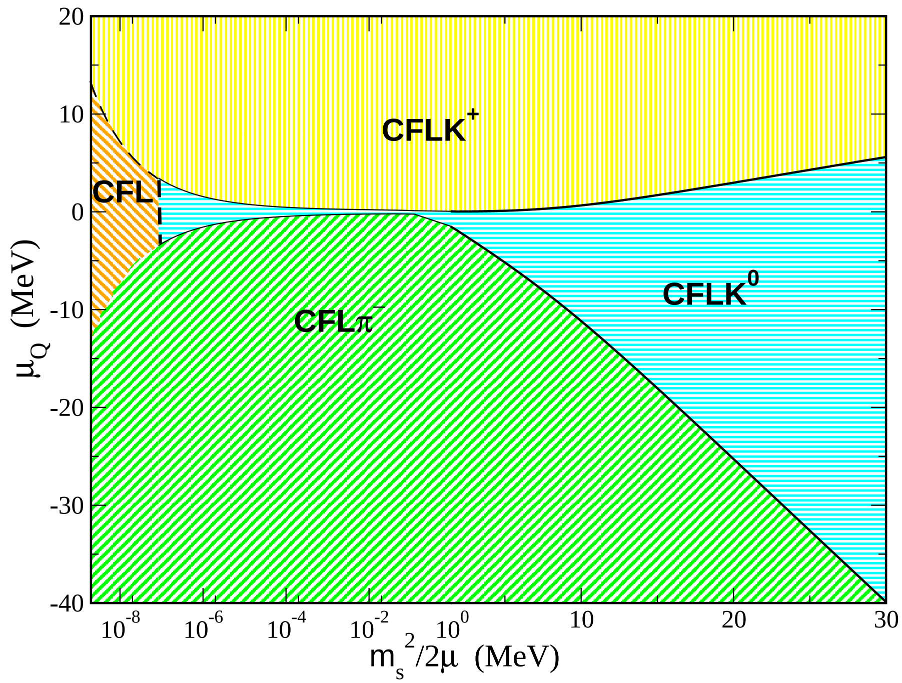
BCS pairing between two species
with different Fermi momenta
$\delta p_f \sim \Delta \rightarrow$ unlocking transition



$\Rightarrow$ mass expansion breaks down if $\quad \dfrac{\delta m^2}{p_F\Delta} \sim 1$

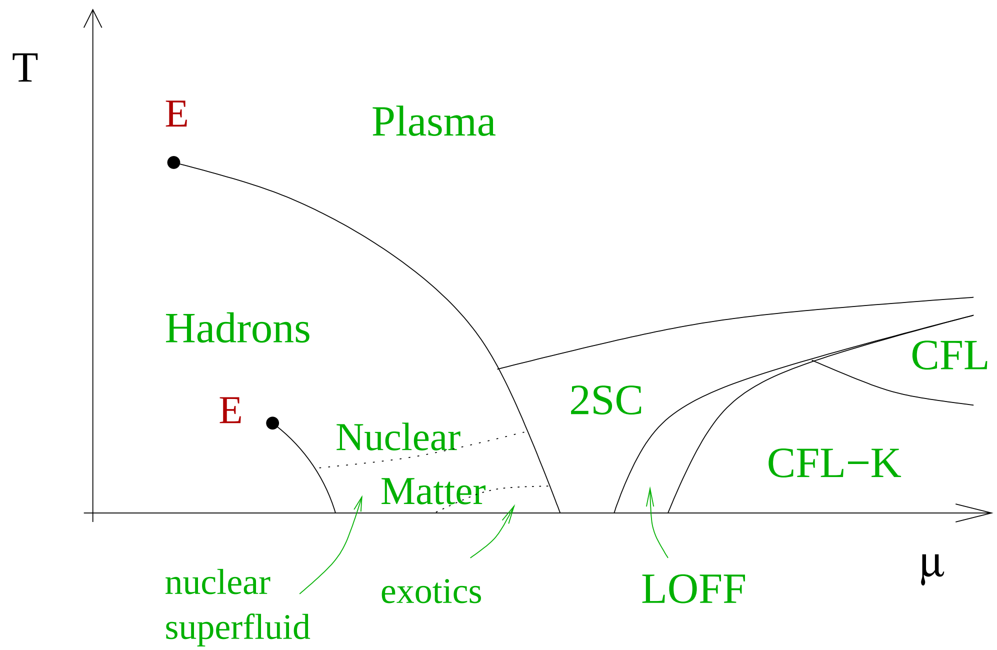
Phase Diagram of CFL phase

$$V(\Sigma) = \frac{f_\pi^2}{2} \mathrm{Tr}\left(W_L \Sigma W_R \Sigma^\dagger\right) - A \mathrm{Tr}(M\Sigma^\dagger) - B_1 \left[\mathrm{Tr}(M\Sigma^\dagger)\right]^2 + \ldots$$



T. Schäfer, P. Bedaque (2002); D. Kaplan, S. Reddy (2002)

Phase Diagram: $m_s \neq 0$



Phase structure at moderate μ (and $m_s, \mu_e \neq 0$) complicated and poorly understood.Systematic calculations

$$m_s^2 \ll \mu^2, m_s\Delta \ll \mu^2, g \ll 1$$

Sse neutron stars to rule out certain phases

What are the most useful observables?