QCD at Finite Density

(Nuclear Matter)

QCD at Finite Density

Partition function

$$Z = \mathrm{Tr}\left[e^{-\beta(H-\mu N)}\right] \qquad \beta = 1/T \qquad N = \int d^3x\, \psi^\dagger \psi$$

Path integral representation (euclidean)

$$Z = \int dA_\mu \det(i\not{D} + i\mu\gamma_4)e^{-S} = \int dA_\mu e^{i\phi} |\det(i\not{D} + i\mu\gamma_4)| e^{-S}$$

Sign problem: importance sampling does not work

Also: No general theorems (a la Vafa-Witten)

Phase structure much richer

(breaking of translational, rotational, parity, isospin, . . . symmetry)

QCD at Very Low Density

QCD has a large mass gap in the $B \neq 0$ sector

$$\mu_{onset} = \min(E_i/B_i) \simeq 930 \text{ MeV}$$

$\mu > \mu_{onset}$: Dilute proton/neutron liquid

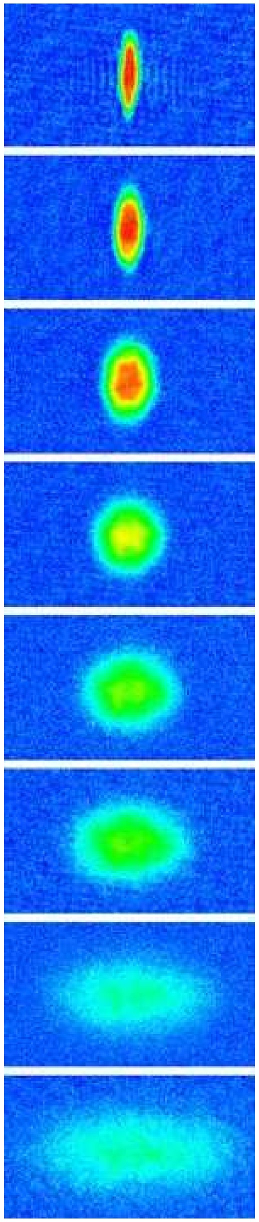
Dilute Neutron Matter

Relevant to neutron stars: $p + e^- \leftrightarrow n + \nu$

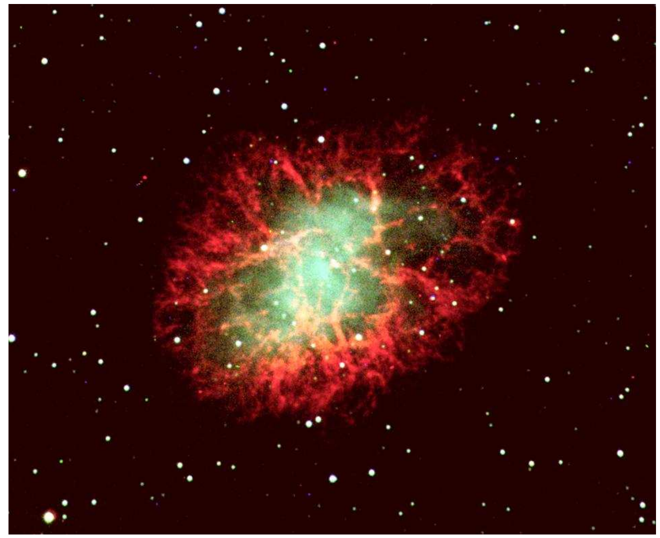
Neutron matter has positive pressure at all densities

Almost universal properties: neutron matter in the lab

Trapped Fermi Gas



Neutron Star (Crab)



Neutron Matter

$\rho r_{nn}^3 \ll 1$: EFT for pointlike, non-relativistic neutrons

$$\mathcal{L}_{\rm eff} = \psi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2M} \right) \psi - \frac{C_0}{2} (\psi^\dagger \psi)^2 + \frac{C_2}{16} \left[(\psi\psi)^\dagger (\psi \overleftrightarrow{\nabla}^2 \psi) + h.c. \right] + \ldots$$

Coupling constants determined by nn interaction

$$C_0 = \frac{4\pi a}{M} \qquad a = -18 \text{ fm} \qquad\qquad C_2 = \frac{4\pi a^2}{M} \frac{r}{2} \qquad r = 2.8 \text{ fm}$$

Finite density: $\mathcal{L} \to \mathcal{L} - \mu \psi^\dagger \psi \Rightarrow$ Modified propagator

$$G_0(k)_{\alpha\beta} = \delta_{\alpha\beta} \left(\frac{\theta(k - k_F)}{k_0 - k^2/2M + i\epsilon} + \frac{\theta(k_F - k)}{k_0 - k^2/2M - i\epsilon} \right) \qquad \frac{k_F^2}{2M} = \mu$$

particles holes

Perturbative Results

Neutron density

$$\rho = \int \frac{d^4k}{(2\pi)^4} S^0_{\alpha\alpha}(k)\; e^{ik_0\eta}\Big|_{\eta\to 0^+} = 2\int \frac{d^3k}{(2\pi)^3}\Theta(k_F - k) = \frac{k_F^3}{3\pi^2}$$

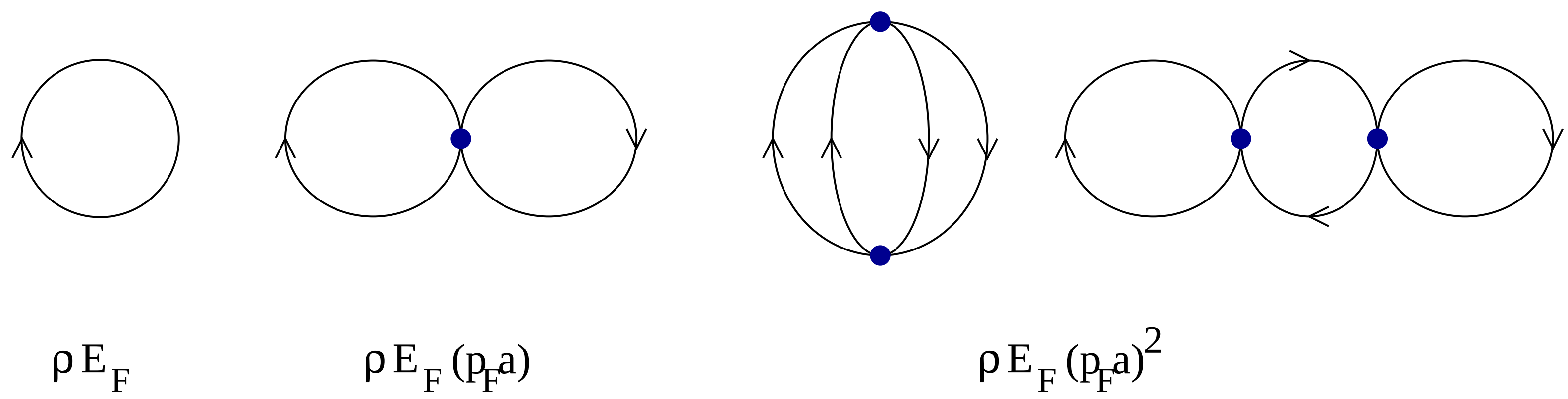
Energy density

$$\mathcal{E} = 2\int \frac{d^3k}{(2\pi)^3} E_k \Theta(k_F - k) = \frac{3}{5}\rho\frac{k_F^2}{2m}$$

First correction: $\mathcal{L}_i = -C_0/2(\psi^\dagger\psi)^2$

$$\mathcal{E}_1 = C_0\left(\frac{k_F^3}{6\pi^2}\right)^2$$

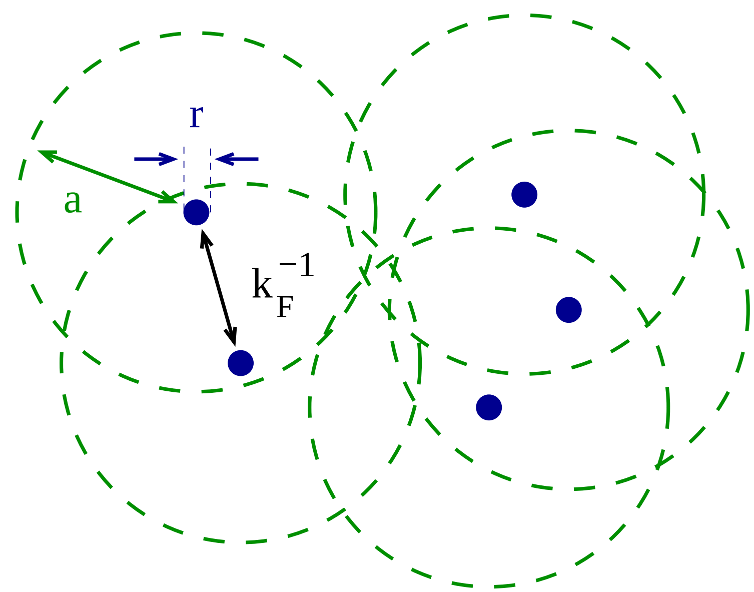
Higher orders: $(k_F a)$ expansion



$$\frac{E}{A} = \frac{k_F^2}{2M}\left[\frac{3}{5} + \left(\frac{2}{3\pi}(k_F a) + \frac{4}{35\pi^2}(11 - 2\log(2))(k_F a)^2\right) + \ldots\right]$$

Problem: $a_{nn} \simeq -20$ fm

$\Rightarrow (k_f a) \gg 1$



Universality

Consider limiting case ("Bertsch" problem)

$$(k_F a) \to \infty \qquad\qquad (k_F r) \to 0$$

Universal equation of state

$$\frac{E}{A} = \xi \left(\frac{E}{A}\right)_0 = \xi \frac{3}{5}\left(\frac{k_F^2}{2M}\right)$$

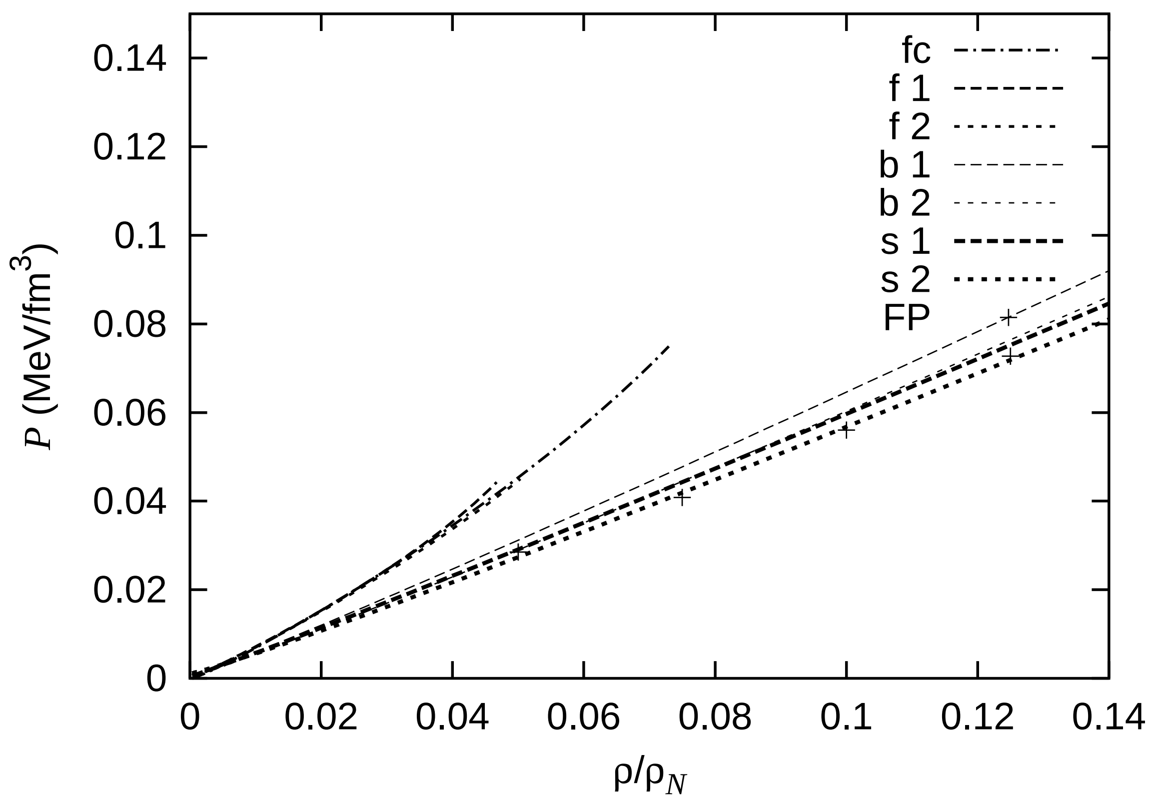
How to find ξ?

Numerical Simulations

Experiments with trapped fermions

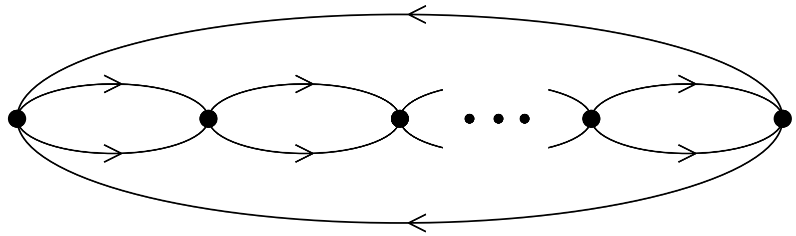
Analytic Approaches

Neutron Matter on the Lattice



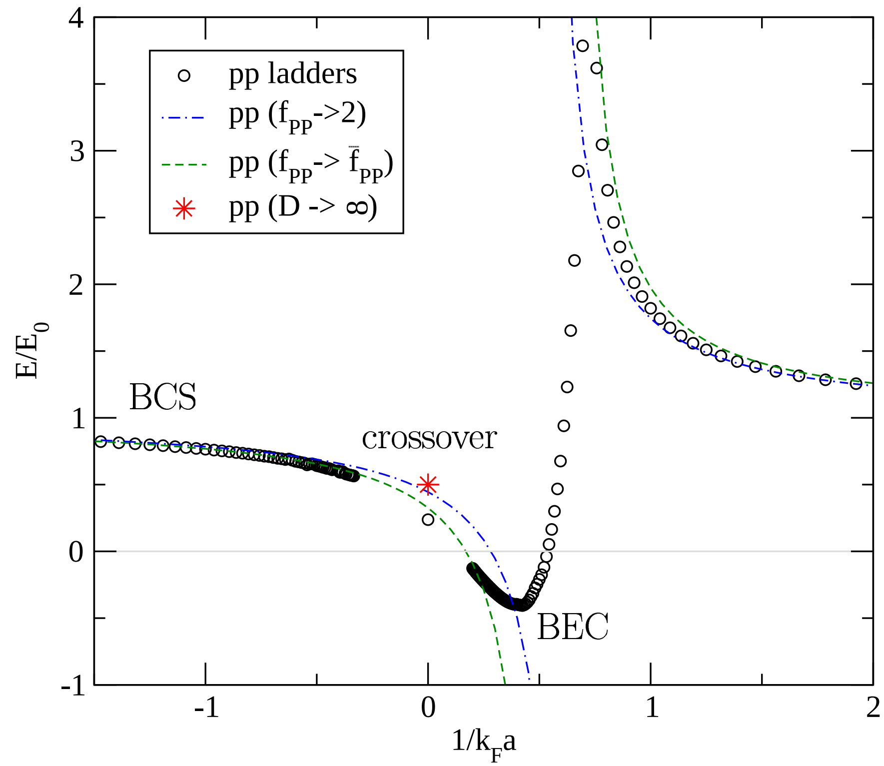
$$P \sim 0.5 P_0$$

Analytic Approaches



$$\frac{E}{A} = \frac{k_F^2}{2M} \times$$

$$\frac{2(k_F a)/(3\pi)}{1 - \frac{6}{35\pi}(11 - 2\log(2))(k_F a)}$$



Large D Expansion: $\xi = 0.5 + O(1/D)$

Chiral Restoration

Quark condensate in dilute nuclear matter

$$\langle \bar{q}q \rangle_\rho = T \frac{\partial}{\partial m_q} \log Z \qquad Z = 4 \int \frac{d^3p}{(2\pi)^3} \log\left(1 + e^{-(E_N-\mu)/T}\right)$$

Quark mass dependence of m_N: πN Sigma term $\Sigma_{\pi N}$

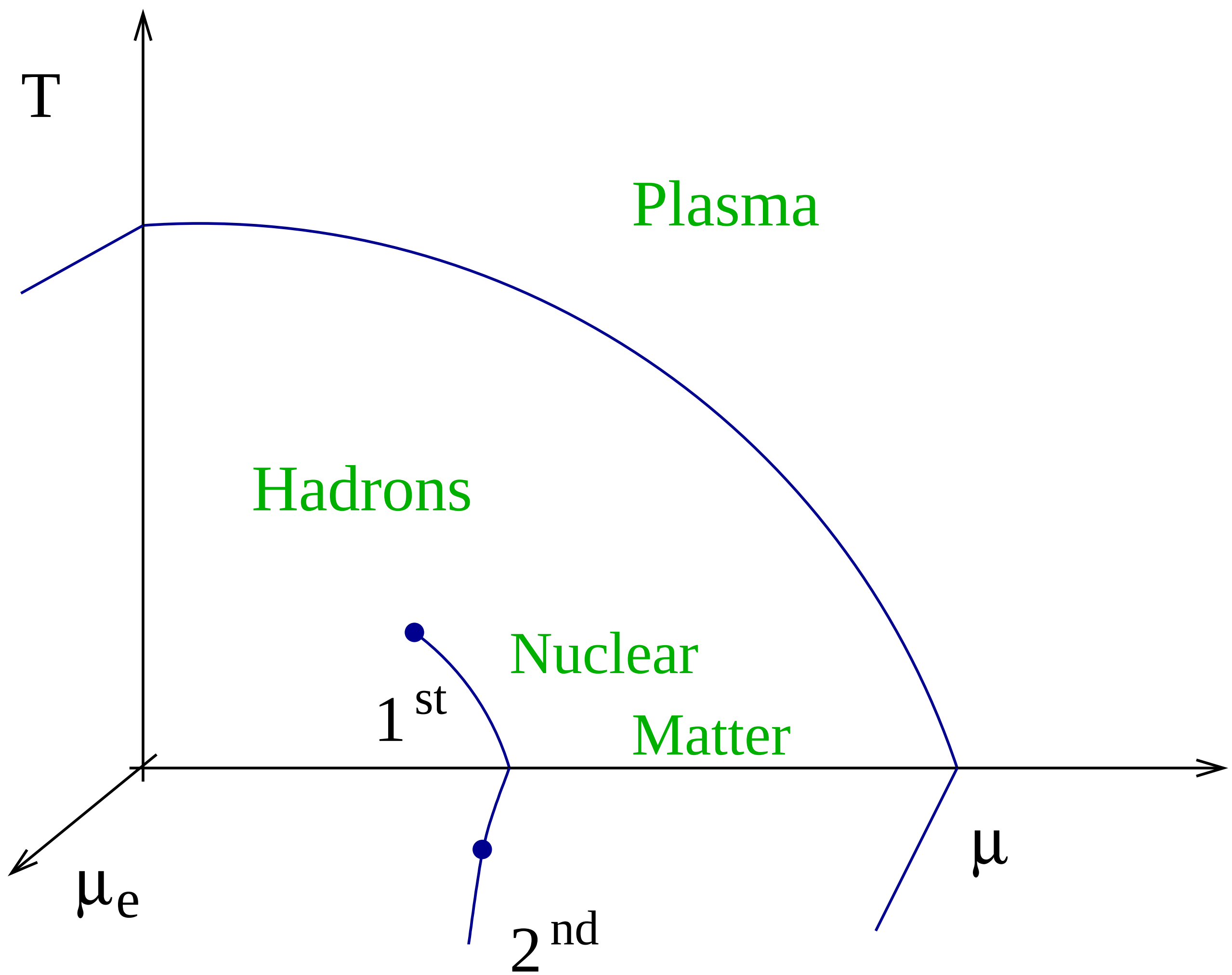
$$\langle \bar{q}q \rangle_\rho = 4 \int \frac{d^3p}{(2\pi)^3} \frac{M_N}{E_N} \left(\frac{\partial M_N}{\partial m_q}\right) = \rho_s \frac{\Sigma_{\pi N}}{m_q}$$

$$\langle \bar{q}q \rangle_\rho = \langle \bar{q}q \rangle_0 \left\{ 1 - \frac{\Sigma_{\pi N} \rho_0}{m_\pi^2 f_\pi^2} \left(\frac{\rho}{\rho_0}\right) \right\}$$

Using $\Sigma_{\pi N} \simeq 45$ MeV

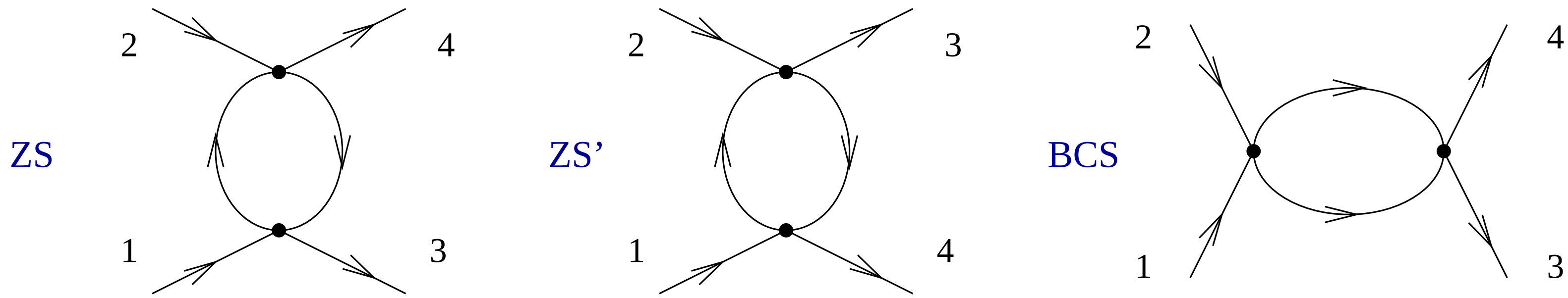
$$\langle \bar{q}q \rangle_\rho \simeq \langle \bar{q}q \rangle_0 \left\{ 1 - \frac{1}{3} \left(\frac{\rho}{\rho_0}\right) \right\}$$

Phase Diagram: First Version



BCS Instability

Loop corrections to scattering near Fermi surface



BCS graph is special: Consider $\vec{p}_{1,2} = \pm\vec{p}$

$$\Gamma = C_0^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{(E+q_0-\epsilon_q)(E-q_0-\epsilon_q)} = -C_0^2 \left(\frac{p_F m}{2\pi^2}\right) \log\left(\frac{\Lambda}{E}\right)$$

Effective, energy dependent coupling

$$E\frac{dC_0}{dE} = C_0^2 \left(\frac{p_F m}{2\pi^2}\right)$$

Evolution of effective coupling ($N = (p_F m)/(2\pi^2)$)

$$C_0(E) = \frac{C_0(\Lambda)}{1 + N C_0(\Lambda) \log(E_0/E)}$$

Effective coupling

$C_0(\Lambda) > 0 \qquad C_0(E \to 0) \to 0$

$C_0(\Lambda) < 0 \qquad C_0(E \to E_{crit}) \to \infty \qquad E_{crit} \sim \Lambda \exp(-1/(N|C_0(\Lambda)|))$

What happens when C_0 reaches Landau pole?

Pair Condensate $\langle \psi(-\vec{p})\psi(\vec{p}) \rangle$

BCS Calculation of Pair Condensate

Step 1: Fierz rearrange

$$\frac{C_0}{2}(\psi^\dagger\psi)^2 = \frac{C_0}{4}(\psi^\dagger\sigma_2\psi^\dagger)(\psi\sigma_2\psi)$$

Step 2: Hubbard-Stratonovich trick

$$1 = Z^{-1}\int D\Delta\,\exp((\Delta^*\Delta)/C_0))$$

Step 3: Shift $\Delta \to \Delta - C_0(\psi\sigma_2\psi)$

$$\mathcal{L}_I = \psi\sigma_2\Delta\psi + h.c. + (\Delta^*\Delta)/C_0$$

Step 4: Nambu-Gorkov field $\Psi = (\psi, \psi^\dagger\sigma_2)$

$$\mathcal{S} = \frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\Psi^\dagger\begin{pmatrix} p_0 - \epsilon_p & \Delta \\ \Delta^* & p_0 + \epsilon_p \end{pmatrix}\Psi.$$

Step 5: Integrate out Ψ

$$\mathcal{L} = \frac{1}{2}\mathrm{Tr}\Big[\log\Big(G_0^{-1}G\Big)\Big] + \frac{1}{C_0}|\Delta|^2$$

$$G(p) = \frac{1}{p_0^2 - \epsilon_p^2 - |\Delta|^2}\begin{pmatrix} p_0 + \epsilon_p & \Delta^* \\ \Delta & p_0 - \epsilon_p \end{pmatrix}$$

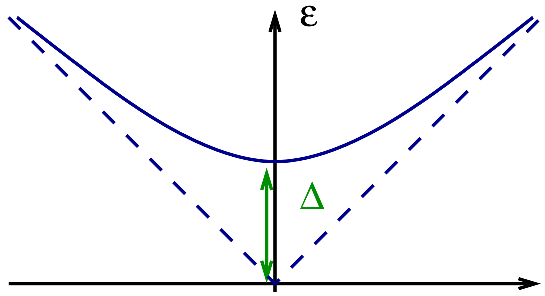
Step 5: Mean Field (Classical) Approximation $(\delta S)/(\delta\Delta) = 0$

$$1 = \frac{|C_0|}{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{\epsilon_p^2 + \Delta^2}}$$

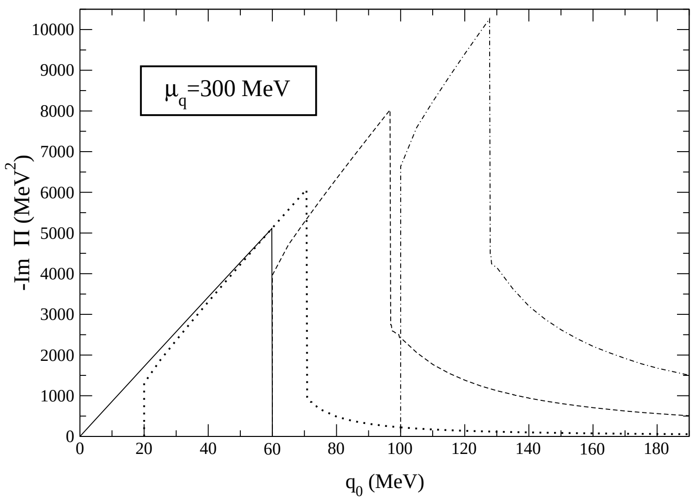
Step 6: Solve gap equation

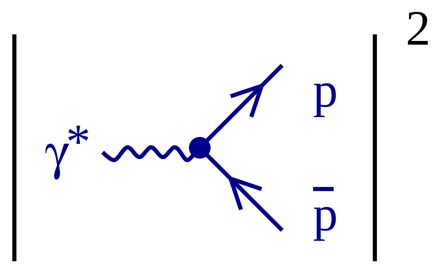
$$\Delta = \frac{8E_F}{e^2}\exp\Big(-\frac{\pi}{2p_F|a|}\Big) \qquad \langle\psi\psi\rangle = \Big(\frac{mp_F}{2\pi^2}\Big)\Delta$$

Pairing: Gap in Excitation Spectrum

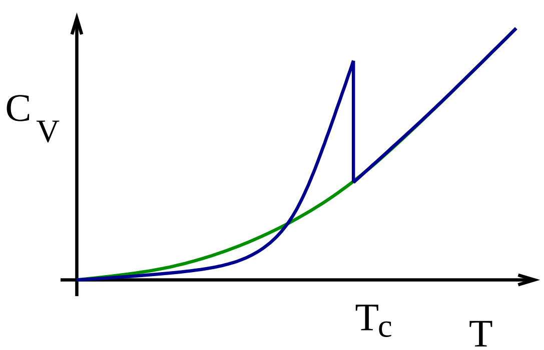


$$\epsilon_p = \sqrt{(p - p_F)^2 + \Delta^2}$$





$$\sim \mathrm{Im}\ \Pi(q_0, \vec{q})$$



specific heat

$$C_V \sim \exp(-\Delta/T)$$

Pairing Gap: Numerical Estimate

Nuclear matter saturation density $\rho_0 \simeq 0.15 \text{ fm}^{-3}$

$$p_F \simeq 250 \text{ MeV} \qquad\qquad E_F \simeq 35 \text{ MeV}$$

This suggest that $\Delta_{nn} \sim 30$ MeV!

Higher order effects cut this down by $\sim 1/2$.

More importantly, have to go beyond the scattering length

$$\Delta = \frac{8E_F}{e^2}\exp\left(-\frac{\pi}{2}\cot\delta(p_F)\right)$$

$$\Delta_{nn} \simeq (1-2) \text{ MeV}$$

Charged Fermions: Superconductivity

Order parameter $\Phi = \langle \epsilon^{\alpha\beta}\psi_\alpha\psi_\beta \rangle$ transforms as

$$\Phi \to \exp(2ie\Lambda)\Phi \qquad A_\mu \to A_\mu + \partial_\mu \Lambda$$

Define Goldstone boson field $\phi(x)$

$$\Phi(x) = \exp(2ie\phi(x))\tilde{\Phi}(x) \qquad \phi(x) \to \phi(x) + \Lambda(x)$$

Gauge invariance determines structure of the effective lagrangian

$$L = -\frac{1}{4}\int d^3x\, F_{\mu\nu}F_{\mu\nu} + L_s(A_\mu - \partial_\mu \phi)$$

Satbility requires L_s to have a minimum at the origin

Action is minimized by $A_\mu = \partial_\mu \phi$

This explains the two main properties of superconductors!

Meissner effect: Magnetic field

$$\vec{B} = \vec{\nabla} \times \vec{A} = 0$$

Perfect Conductor: Potential

$$V(x) = \dot{\phi}(x)$$

stationary current $\vec{j} \sim \vec{\nabla}\phi$ requires $V = const$

Landau-Ginzburg Theory

Consider time-independent, small, slowly varying $\Phi(x)$

$$L_s = \int d^3x \left\{ -\frac{1}{2} \left| \left(\nabla - 2ie\vec{A} \right) \Phi \right|^2 + \frac{1}{2} m_H^2 \left(\Phi^* \Phi \right)^2 - \frac{1}{4} g \left(\Phi^* \Phi \right)^4 + \dots \right\}$$

Decompose $\Phi = \rho \exp(2ie\phi)$. Effective potential for ρ

$$V(\rho) = -\frac{1}{2} m_H^2 \rho^2 + \frac{1}{4} g \rho^4$$

$$\text{Parameters } m_H, g \leftrightarrow \langle \Phi \rangle, E_0$$

Equations of motion

$$\begin{aligned} \vec{\nabla} \times \vec{B} &= 4e^2 \rho^2 \left(\nabla \phi - \vec{A} \right) \\ \nabla^2 \rho &= -m_H^2 \rho^2 + g \rho^3 + 4e^2 \rho \left(\vec{\nabla} \phi - \vec{A} \right) \end{aligned}$$

This implies

$$\nabla^2 \vec{B} = -4e^2\rho^2\vec{B} \qquad B(z) = B_0 e^{-z/\lambda}$$

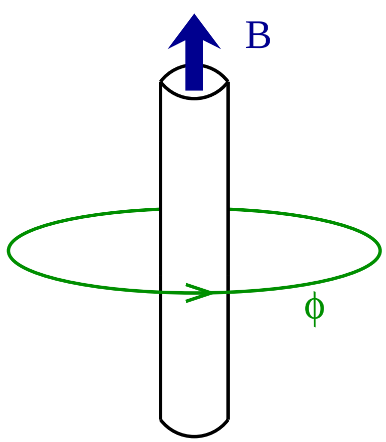
penetration depth λ

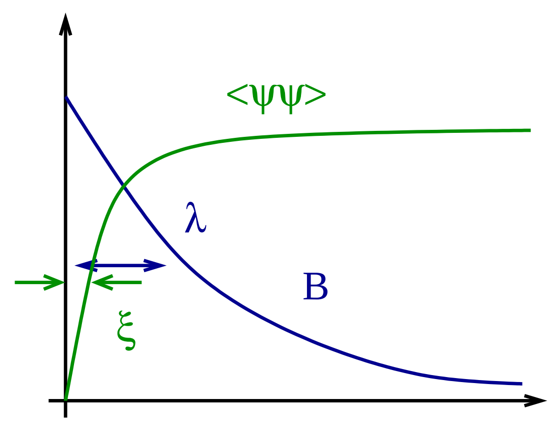
$$\nabla^2\rho = -m_H^2\rho + \ldots \qquad \rho(z) = \rho_0 e^{-z/\xi}$$

coherence length ξ

Type II materials: $\xi < \lambda$. Magnetic flux goes through vortices

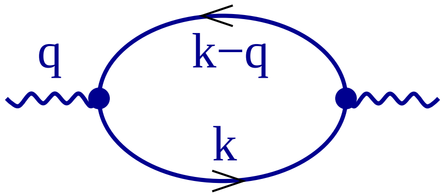
$$\int_A \vec{B}\cdot d\vec{S} = \oint_{\partial A}\vec{A}\cdot d\vec{l} = \oint_{\partial A}\vec{\nabla}\phi\cdot d\vec{l} = \frac{n\pi\hbar}{e}$$





Charged Fermions: Screening

Photon polarization function



$$\Pi_{00}(q) = e^2 \int \frac{d^4p}{(2\pi)^4} \frac{1}{(q_0 + p_0 - \epsilon_{p+q})(p_0 - \epsilon_p)}$$

Perform p_0 integral: particle-hole contribution

$$\Pi_{00}(q) = e^2 \int \frac{d^3p}{(2\pi)^3} \frac{n_{p+q} - n_p}{E_{p+q} - E_p}$$

Static polarization function, long distance

$$\Pi_{00}(q_0 = 0, \vec{q} \to 0) = e^2 \int \frac{d^3p}{(2\pi)^3} \frac{\partial n_p}{\partial E_p} = e^2 \frac{p_F m}{2\pi^2}$$

Screened potential

$$V(r) = -\frac{e}{r} \exp(-m_D r) \qquad\qquad m_D^2 = e^2 (p_F m)/(2\pi^2)$$