QCD at High Temperature

(Theory)

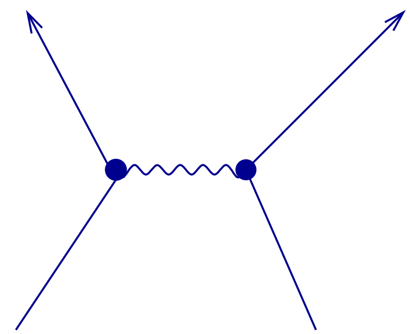
The High T Phase: Qualitative Argument

High T phase: Weakly interacting gas of quarks and gluons?

typical momenta $p \sim 3T$

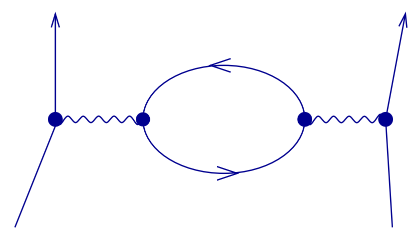
Large angle scattering involves large momentum transfer

effective coupling is small



Small angle scattering is screened (not anti-screened!)

coupling does not become large



Quark Gluon Plasma

Basic Thermodynamics

Massless particles, zero baryon density $(\zeta(3) = 1.2)$

$$n = g\frac{\zeta(3)}{\pi^2}T^3 \begin{cases} 1 \\ 3/4 \end{cases} \qquad \epsilon = g\frac{\pi^2}{30}T^4 \begin{cases} 1 & \text{bosons} \\ 7/8 & \text{fermions} \end{cases}$$

$$s/n = 2\pi^2/(45\zeta(3)) \simeq 3.6 \qquad P = \epsilon/3$$

massless quarks and gluons

$$g_{eff} = 2 \times 8 \times 1 + 4 \times 3 \times 2 \times 7/8 = 37$$

spin × color × boson + spin × color × flavors × fermion

massless pions

$$g = (N_f^2 - 1) \times 1 = 3$$

First Approach: Bag Model

Low temperature: Pions

$$\epsilon = \frac{3\pi^2}{30} T^4 \qquad\qquad P = \frac{3\pi^2}{90} T^4$$

High temperature: Quarks and gluons

$$\epsilon = \frac{37\pi^2}{30} T^4 \qquad\qquad P = \frac{37\pi^2}{90} T^4$$

Include vacuum energy $T_{\mu\nu} = B g_{\mu\nu}$ (QCD cosmological constant)

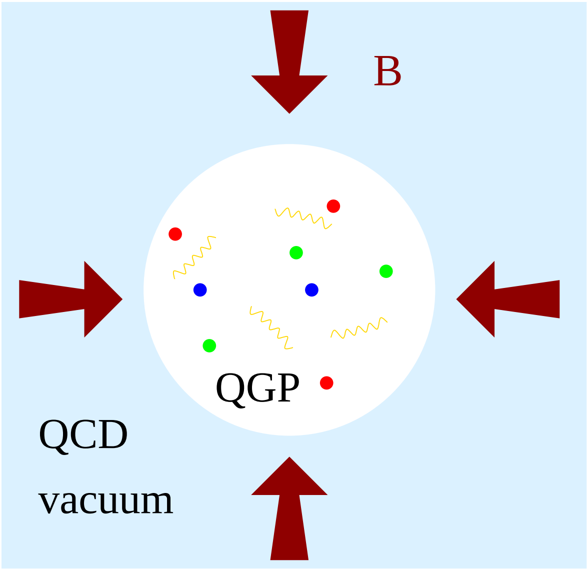
$$\epsilon_{vac} = -P_{vac} = +B \qquad\qquad \epsilon_{vac} = -\frac{b}{32}\langle \frac{\alpha}{\pi} G^2 \rangle \simeq 0.5 \text{ GeV}/\text{fm}^3$$

trace anomaly relation

Critical temperature: equate pressures

$$\frac{3\pi^2}{90}T^4 + B = \frac{37\pi^2}{90}T^4$$

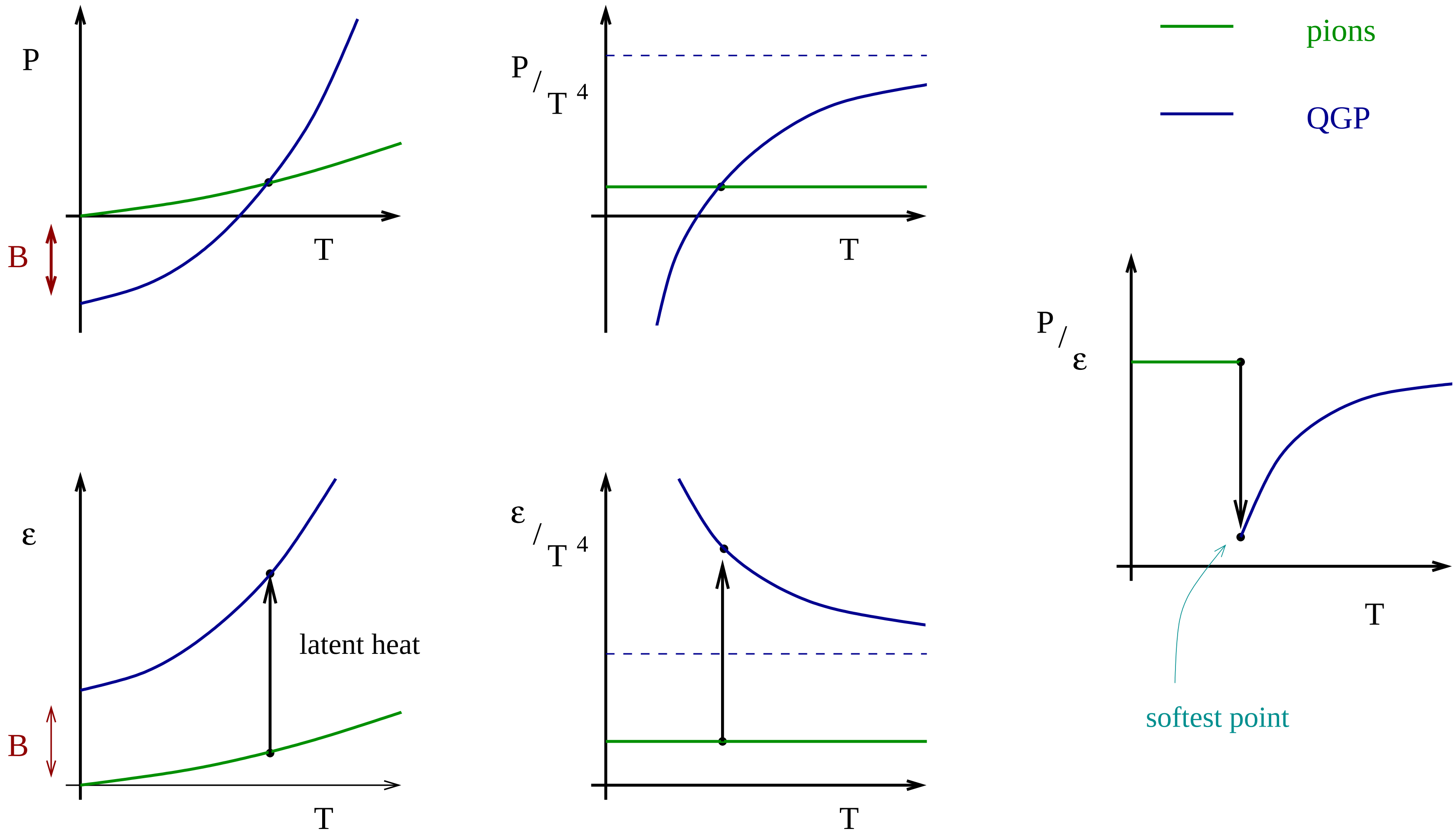
$$T_c = \left(\frac{45B}{17\pi^2}\right)^{1/4} \simeq 150 \text{ MeV}$$



Pressure is continuous, but energy density jumps

$$\begin{aligned} \epsilon(T_c^-) &= \frac{3\pi^2}{30}T_c^4 \simeq 100 \text{ MeV/fm}^3 \\ \epsilon(T_c^+) &= \frac{37\pi^2}{30}T_c^4 + B \simeq 1500 \text{ MeV/fm}^3 \end{aligned}$$

Bag Model Equation of State

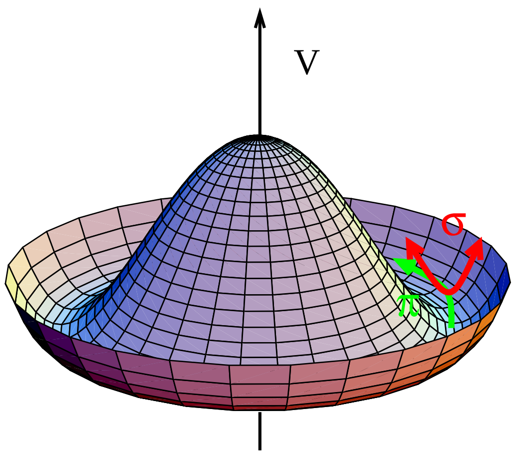


Second Approach: Sigma Model

Simple model based on linear representation of $SU(2)_L \times SU(2)_R$

$$\phi^a = (\sigma, \vec{\pi}) \qquad\qquad O(4) = SU(2)_L \times SU(2)_R$$

Chirally symmetric lagrangian



$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^a)^2 + V(\phi^a \phi^a)$$

$$V(\phi^a \phi^a) = -\frac{\mu^2}{2}(\phi^a \phi^a) + \frac{\lambda}{4}(\phi^a \phi^a)^2$$

Minimum of potential

$$\partial V / \partial \phi^a = \phi^a(-\mu^2 + \lambda \phi^a \phi^a) = 0 \qquad \phi^a_0 = (\sigma_0, \vec{0}) \;\; \sigma_0^2 = \mu^2/\lambda$$

Direction fixed by explicit breaking $\mathcal{L}_{SB} = -c\sigma$

σ_0 related to pion decay constant

$$\vec{A}_\mu = \sigma\partial_\mu\vec{\pi} + \vec{\pi}\partial_\mu\sigma \simeq \sigma_0\partial_\mu\vec{\pi} \qquad\qquad \sigma_0 = f_\pi = 93 \text{ MeV}$$

Consider small oscillations. Equation of motion

$$\delta\mathcal{L}/\delta\phi^a = -\Box\phi^a - \partial V/\partial\phi^a = 0$$

Write $\phi^a = \phi_0^a + \delta\phi^a$

$$\begin{aligned}\Box(\delta\phi^a) &= (\phi_0^a + \delta\phi^a)\left(-\mu^2 + \lambda(\phi_0^a + \delta\phi^a)^2\right) \\ &= (-\mu^2 + \lambda\phi_0^a\phi_0^a)\phi_0^a + \left(-\mu^2 + 2\lambda\phi_0^a\phi_0^b + \lambda\delta^{ab}\phi_0^c\phi_0^c\right)\delta\phi^b + \ldots\end{aligned}$$

Split in (σ, π) components

$$\begin{aligned}\Box(\delta\sigma) &= \left(-\mu^2 + 3\lambda\sigma_0^2\right)\delta\sigma \qquad & m_\sigma^2 = 2\mu^2 \\ \Box(\delta\vec{\pi}) &= \left(-\mu^2 + \lambda\sigma_0^2\right)\delta\vec{\pi} \qquad & m_\pi^2 = 0\end{aligned}$$

Thermal Fluctuations

Write $\phi^a = \langle\phi^a\rangle + \tilde{\phi}^a$ where $\tilde{\phi}^a$ is a thermal fluctuation. Use

$$\begin{aligned}\langle\tilde{\phi}^a\rangle &= 0\\ \langle\tilde{\phi}^a\tilde{\phi}^b\rangle &= (\delta^{ab}/4)\langle\tilde{\phi}^a\tilde{\phi}^a\rangle\\ \langle\tilde{\phi}^a\tilde{\phi}^b\tilde{\phi}^c\rangle &= 0\end{aligned}$$

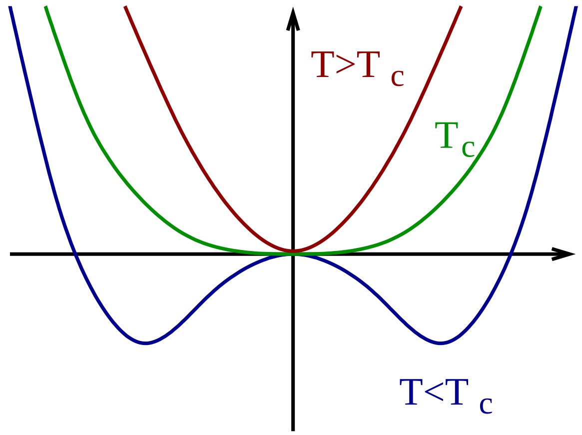
Equation of motion for $\langle\phi^a\rangle$ (use $1/N$)

$$\begin{aligned}\Box\langle\phi^a\rangle &= -\mu^2\langle\phi^a\rangle + \lambda\langle\left(\langle\phi^a\rangle + \tilde{\phi}^a\right)\left(\langle\phi^b\rangle + \tilde{\phi}^b\right)^2\rangle\\ &= -\mu^2\langle\phi^a\rangle + \lambda\langle\phi^a\rangle\left[\langle\phi^b\rangle^2 + \langle\tilde{\phi}^b\tilde{\phi}^b\rangle\right]\end{aligned}$$

Fluctuations tend to restore symmetry

Thermal averages

$$\begin{aligned} \vec{\pi}_T &= 0 \\ \sigma_T^2 &= f_\pi^2 \left(1 - \frac{1}{f_\pi^2} \langle \tilde{\phi}^a \tilde{\phi}^a \rangle \right) \end{aligned}$$



Gaussian fluctuations $(m = 0)$

$$\langle \tilde{\phi}^a \tilde{\phi}^a \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_k} \frac{1}{e^{\beta \omega_k} - 1} = \frac{T^2}{12}$$

Critical temperature (3 light d.o.f.)

$$\sigma_T^2 = f_\pi^2 \left(1 - \frac{T^2}{3 f_\pi^2} \right) \qquad T_c = 2 f_\pi \simeq 180 \text{ MeV}$$

Note: Chiral perturbation theory predicts

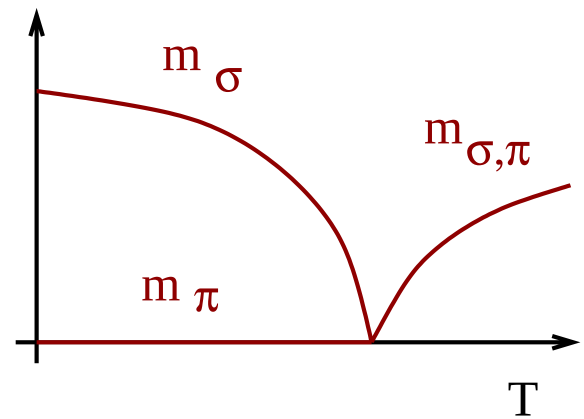
$$\langle\bar{\psi}\psi\rangle_T = \langle\bar{\psi}\psi\rangle_0 \left\{ 1 - \frac{N_f^2 - 1}{3N_F} \left(\frac{T^2}{4f_\pi^2} \right) + \ldots \right\}$$

this suggests $T_c \sim 1/\sqrt{N_f}$

also note $n_\pi \sim N_f^2$ but $n_q \sim N_f$

Thermal masses

m_σ, m_π become degenerate at T_c



also true for other chiral partners

$$m_\rho \leftrightarrow m_{a_1} \qquad\qquad m_N \leftrightarrow m_{N?}$$

Universality

Chiral phase transition might be continuous (2nd order)

Near T_c masses go to zero and correlation length diverges

Physics independent of microscopic details

Long distance behavior is universal

Only depends on symmetries of the order parameter

Landau-Ginzburg effective action

$$F = \int d^3x \left\{ \frac{1}{2}(\vec{\nabla}\phi^a)^2 + \frac{\mu^2}{2}(\phi^a\phi^a) + \frac{\lambda}{4}(\phi^a\phi^a)^2 + \ldots \right\}$$

Consider $\lambda > 0$, $\mu^2(T_c) = 0$

Universality

$SU(2)_L \times SU(2)_R$	QCD	$\equiv$	$O(4)$	magnet
$\langle\bar{\psi}\psi\rangle$	χ condensate		$\vec{M}$	magnetization
m_q	quark mass		H_3	magnetic field
$\vec{\pi}$	pions		$\vec{\phi}$	spin waves

Predictions

$$\begin{aligned} C &\sim t^{\alpha} & \alpha &= -0.19 \\ \langle\bar{\psi}\psi\rangle &\sim t^{\beta} & \beta &= 0.38 \\ m_\pi &\sim t^{\nu} & \nu &= 0.73 \end{aligned}$$

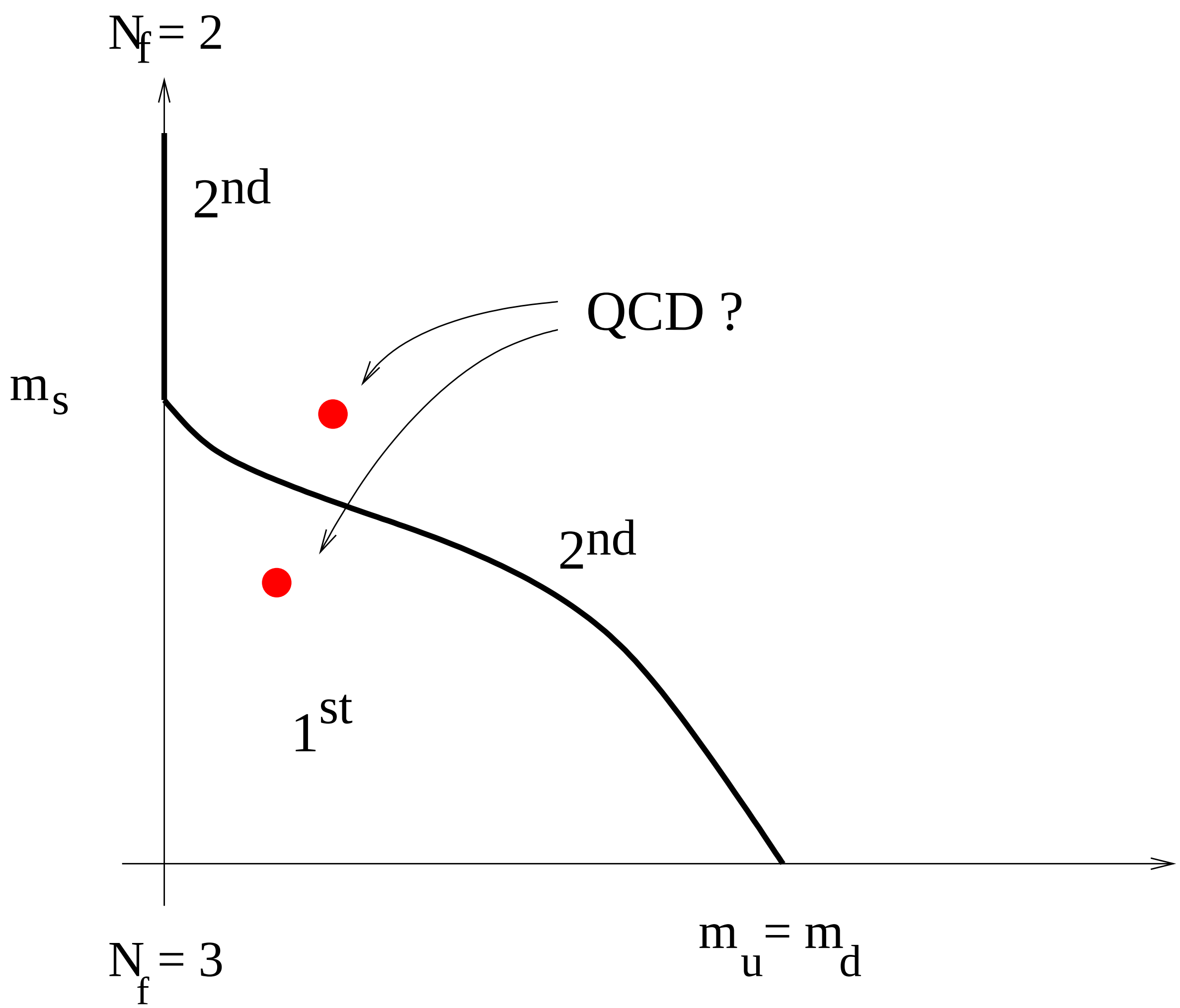
$t = (T - T_c)/T$

from ϵ expansion,

numerical simulations

$N_f = 3$: extra cubic invariant $\det(\phi)$, 2nd order transition unstable

$N_f = 3$ transition is 1st order

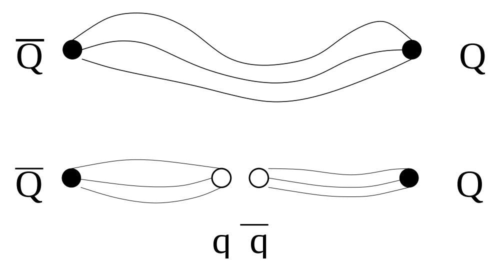


Universality: Confinement

Confinement characterized by heavy quark potential

$$V(r) \sim kr$$

$$k \sim 1 \text{ GeV/fm}$$

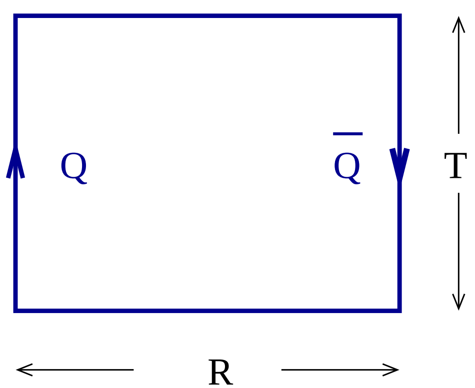


Propagator for heavy quark

$$\left(i\partial_0 + gA_0 + \vec{\alpha}(i\vec{\nabla} + g\vec{A}) + \gamma_0 M \right) \psi = 0$$

$$S(x, x') \simeq \exp\left(ig \int A_0 dt \right) \left(\frac{1+\gamma_0}{2} \right) e^{im(t-t')} \delta(\vec{x} - \vec{x}')$$

Potential related to Wilson loop



$$W(R,T) = \exp\left(ig \oint A_\mu dz_\mu \right)$$

Have $W(R,T) = \exp(-E \cdot T) = \exp(-V(R)T)$

$$W(R,T) \sim \exp(-kA) \qquad \text{Confinement} \equiv \text{AreaLaw}$$

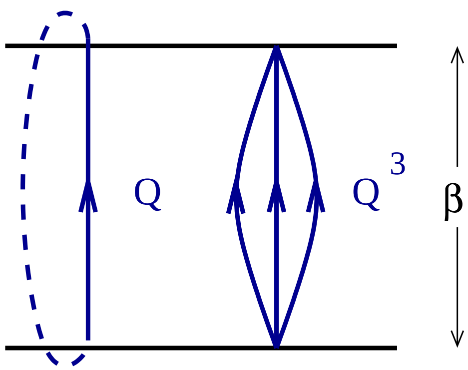
Local order parameter? Polyakov line

$$P(\vec{x}) = \frac{1}{N_c}\mathrm{Tr}[L(\vec{x})] = \frac{1}{N_c} P\mathrm{Tr}\left[\exp\left(ig\int_0^\beta A_0 dt\right)\right]$$

Naive Interpretation: $\langle P \rangle \sim \exp(-m_Q \beta)$

$$\langle P \rangle = 0 \quad \text{confined} \qquad \langle P \rangle \neq 0 \quad \text{deconfined}$$

Symmetry: Consider $L \to zL \quad z = \exp(2\pi k i/N_c) \in Z_{N_c}$



$$\mathrm{Tr}[L(\vec{x})] \rightarrow z\mathrm{Tr}[L(\vec{x})]$$
$$\mathrm{Tr}[L(\vec{x})^3] \rightarrow \mathrm{Tr}[L(\vec{x})^3]$$

Polyakov line: $P \to zP$

$\langle P \rangle = 0$ $\quad$ Z_{N_c} unbroken $\quad$ $T < T_c$

$\langle P \rangle \neq 0$ $\quad$ Z_{N_c} broken $\quad$ $T > T_c$

Landau-Ginzburg Theory (cubic invariant: $SU(3)$ only)

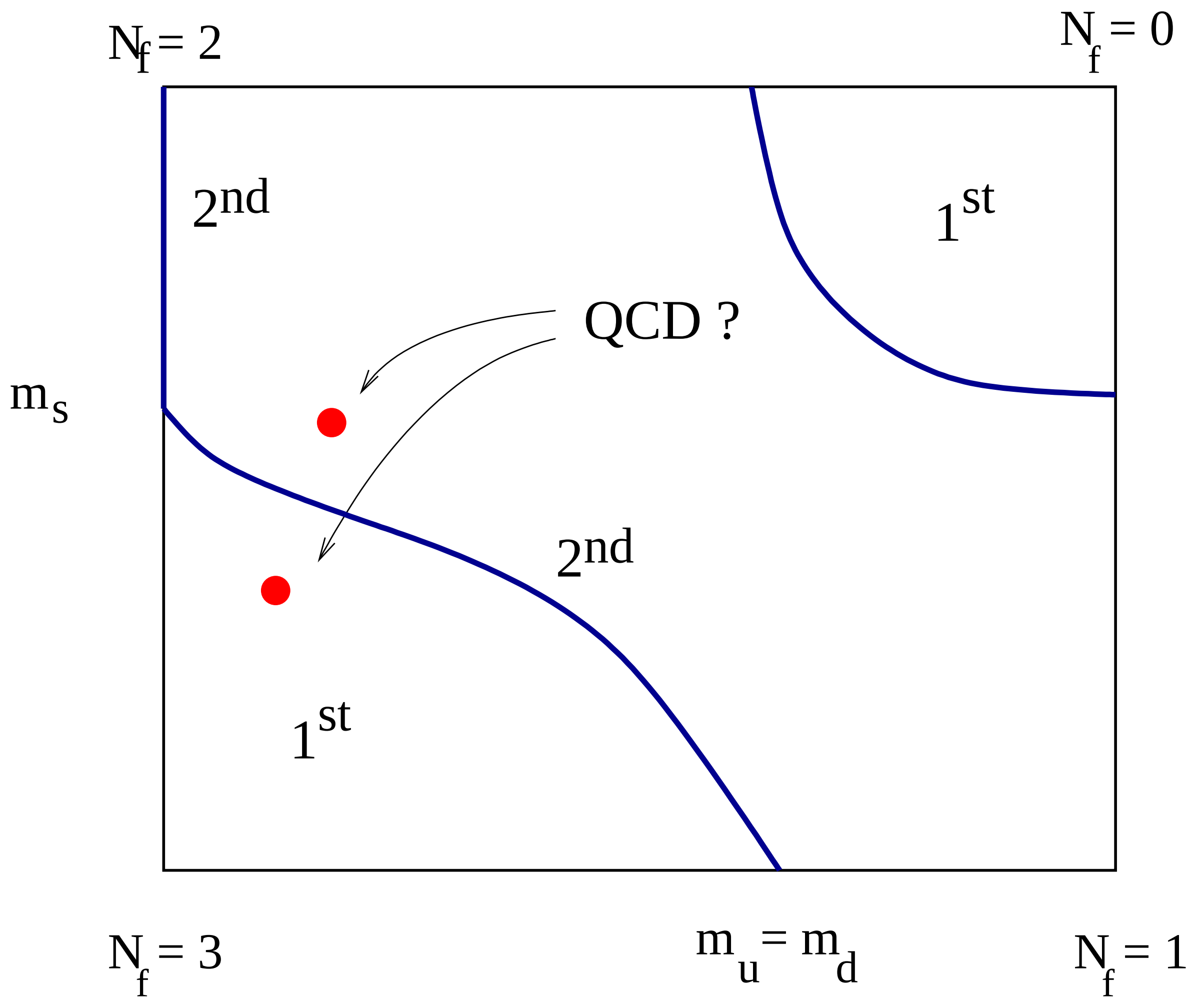
$$F = \int d^3x \left\{ \frac{1}{2} |\vec{\nabla} P|^2 + \mu^2 |P|^2 + g \mathrm{Re}(P^3) + \lambda |P|^4 + \ldots \right\}$$

Predictions

$SU(2)$-color: 2nd order

$SU(3)$-color: 1st order

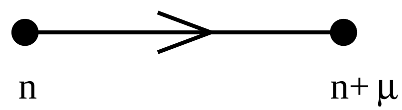
Summary: Universality



Lattice QCD

Euclidean partition function

$$Z = \int dA_\mu d\psi \exp(-S) = \int dA_\mu \det(i\not{D}\,) \exp(-S_G)$$

Lattice discretization: 

$$U_\mu(n) = \exp(igaA_\mu(n))$$

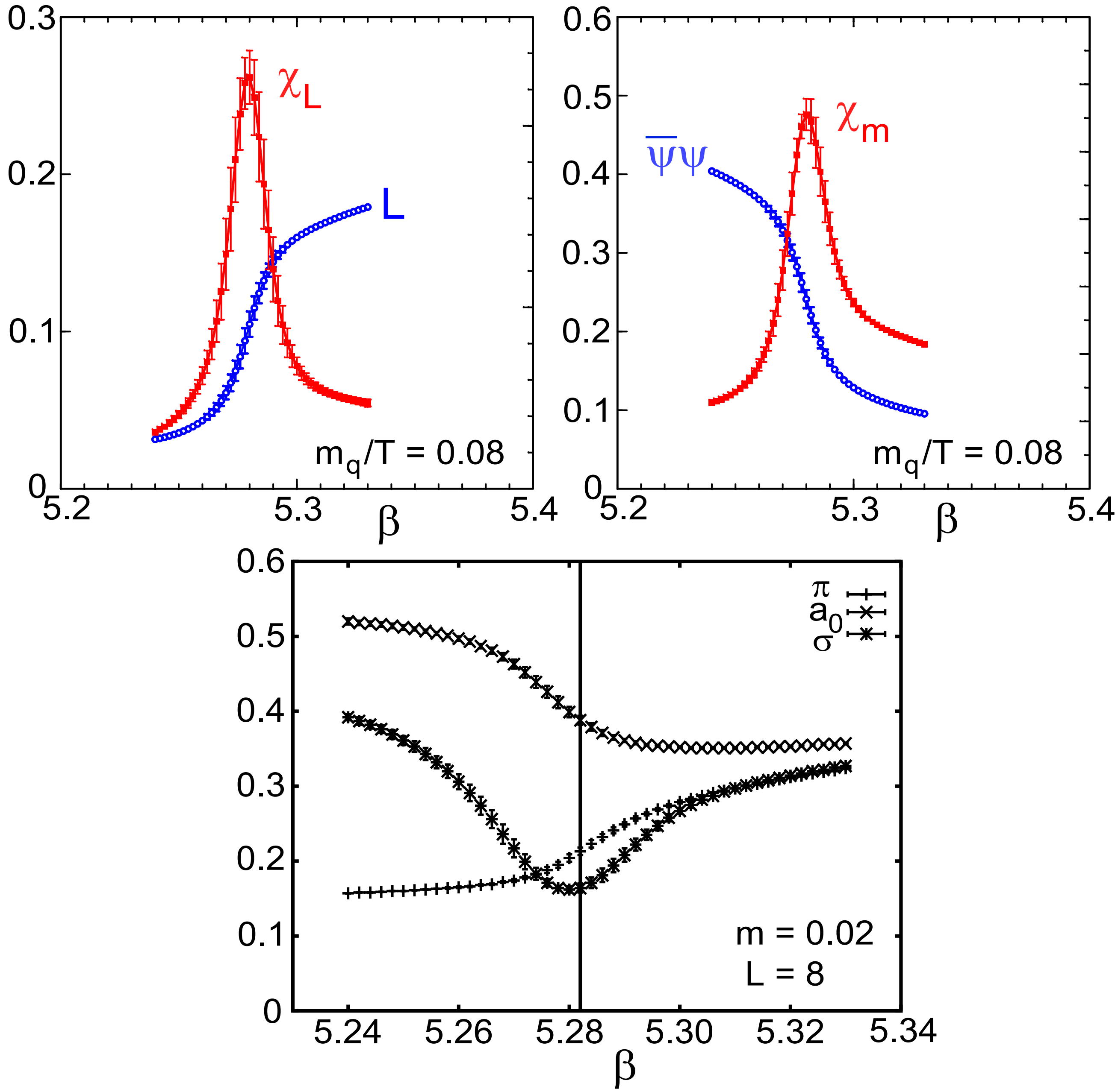
$$D_\mu \phi \;\to\; \frac{1}{a}[U_\mu(n)\phi(n+\mu) - \phi(n)]$$

$$(G^a_{\mu\nu})^2 \;\to\; \frac{1}{a^4}\mathrm{Tr}[U_\mu(n)U_\nu(n+\mu)U_{-\mu}(n+\mu+\nu)U_{-\nu}(n+\nu) - 1]$$

Monte Carlo:

$$\int dA_\mu \; e^{-S} \to \{U_\mu^{(1)}(n), U_\mu^{(2)}(n), \ldots\}$$

Lattice Results



Weakly coupled QGP

Basic object: Partition function

$$Z = \mathrm{Tr}[e^{-\beta H}], \quad \beta = 1/T \qquad\qquad F = T\log(Z)$$

Basic trick

$$Z = \mathrm{Tr}[e^{-i(-i\beta)H}] \qquad\qquad \text{imaginary time evolution}$$

Path integral representation

$$Z = \int dA_\mu d\psi \exp\left(-\int_0^\beta d\tau \int d^3x \, \mathcal{L}_E\right)$$

$$A_\mu(\vec{x},\beta) = A_\mu(\vec{x},0);\ \psi(\vec{x},\beta) = -\psi(\vec{x},0)$$

Fourier representation

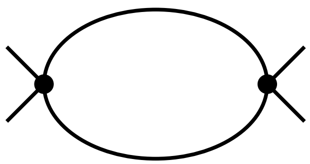
$$A_\mu(\vec{x},\tau) = \sum_n \int d^3k A_\mu^n(\vec{k}) e^{i(\vec{k}\vec{x}+\omega_n\tau)}$$

Matsubara frequencies

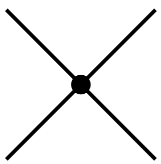
$$\omega_n = 2\pi n T \qquad \text{bosons}$$

$$\omega_n = (2n+1)\pi T \qquad \text{fermions}$$

Feynman rules: Euclidean QCD with discrete energies



$$T\sum_n \int \frac{d^3p}{(2\pi)^3}$$



$$(2\pi)^3\delta^3\left(\sum \vec{p}_i\right)\delta_{\sum n_i}$$

Typical Matsubara Sums

$$\sum_k \frac{1}{x^2+k^2} = \frac{2\pi}{x}\left(\frac{1}{2}+\frac{1}{e^{2\pi x}-1}\right) \qquad \text{bosons}$$

$$\sum_k \frac{1}{x^2+(2k+1)^2} = \frac{\pi}{x}\left(\frac{1}{2}-\frac{1}{e^{\pi x}+1}\right) \qquad \text{fermions}$$

Example: Free energy of non-interacting bosons

Partition function: $Z = [\det(p^2 + m^2)]^{-1/2}$

$$\log Z = -\frac{1}{2} \sum_n \log(\omega_n^2 + \omega^2) \qquad \omega^2 = \vec{p}^2 + m^2$$

Consider derivative with respect to ω^2

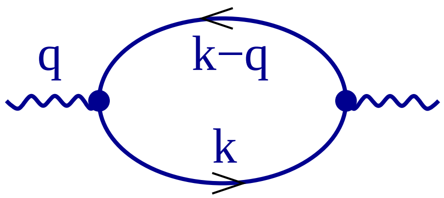
$$\frac{d \log Z}{d\omega^2} = -\frac{1}{2} \sum_n \frac{1}{\omega_n^2 + \omega^2}$$

Use bosonic Matsubara sum and integrate back

$$-T \log Z = \frac{\omega}{2} + \frac{1}{\beta} \log\left(1 - e^{-\beta\omega}\right)$$

Gluon Polarization Tensor

Warmup: Photon polarization function $\Pi_{\mu\nu}$



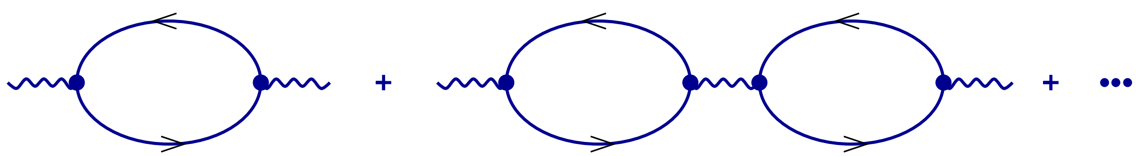
$$= e^2 T \sum_n \int \frac{d^3k}{(2\pi)^3} \mathrm{tr}[\gamma_\mu \not{k} \gamma_\nu (\not{k} - \not{q})] \Delta(k) \Delta(k-q)$$

Hard Thermal Loop (HTL) limit ($q \ll k \sim T$)

$$\Pi_{\mu\nu} = 2m^2 \int \frac{d\Omega}{4\pi} \left(\frac{i\omega \hat{K}_\mu \hat{K}_\nu}{q \cdot \hat{K}} + \delta_{\mu 4} \delta_{\nu 4} \right) \qquad \hat{K} = (-i, \hat{k})$$

$2m^2 = \frac{1}{3} e^2 T^2$ Debye mass

Significance of $\Pi_{\mu\nu}$

$D_{\mu\nu} =$  $= \dfrac{1}{(D^0_{\mu\nu})^{-1} + \Pi_{\mu\nu}}$

$D_{00}(\omega = 0, \vec{q})$ determines static potential

$$V(r) = e \int \frac{d^3q}{(2\pi)^3} \frac{e^{iqr}}{\vec{q}^2 + \Pi_{00}} \simeq -\frac{e}{r} \exp(-m_D r)$$

screened Coulomb potential

D_{ij} determines magnetic interaction

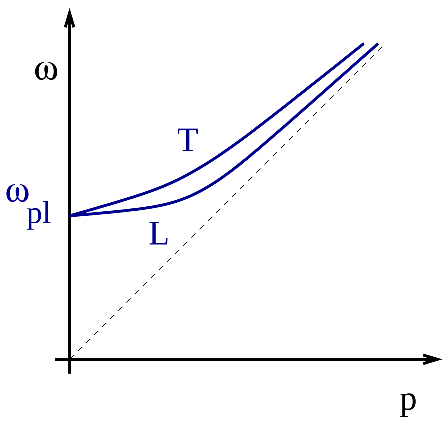
$$\Pi_{ii}(\omega \to 0, 0) = 0$$

no magnetic screening

$$\mathrm{Im}\Pi_{ii}(\omega, q) \sim \frac{\omega}{q} m_D^2 \Theta(q - \omega)$$

Landau damping

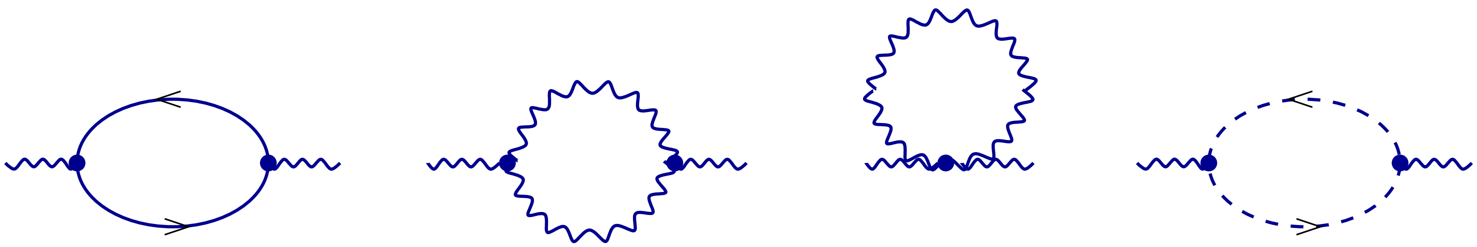
Poles of propagator: Plasmon dispersion relation



$$D(\omega, q \to 0) = 0$$

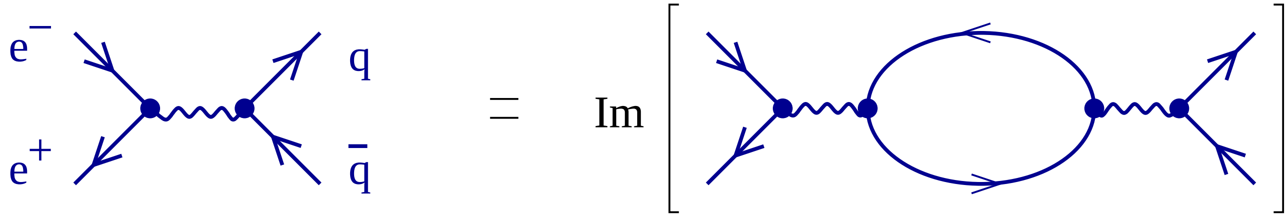
$$\omega_L^2 = \omega_T^2 = \frac{1}{3} m_D^2$$

QCD looks more complicated



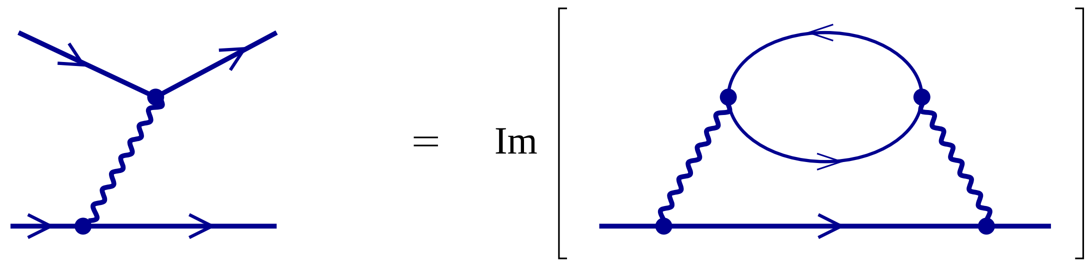
same result as QED with $m_D^2 = g^2T^2(1 + N_f/6)$

Dilepton production



$$\frac{dR}{d^4q} = \frac{\alpha^2}{48\pi^2}\left(12\sum_q e_q^2\right)e^{-E/T}$$

Collisional energy loss



$$\frac{dE}{dx} = \frac{8\pi}{3}\alpha_s^2 T^2\left(1+\frac{N_f}{6}\right)\log\left(c\frac{\sqrt{ET}}{m_D}\right) \qquad E \gg M^2/T$$

$E = 20$ GeV: $dE/dx \simeq 0.3\ \mathrm{GeV/fm}$ for c, b quarks

note: for light quarks radiative energy loss dominates

Weak Coupling Thermodynamics