Bonus Material:

Chiral Effective Field Theory

Low energy effective theory for the Goldstone modes

Step 1: Parameterize G/H =pseudoscalar GB's

$$U(x): \qquad U \to LUR^\dagger \qquad\qquad (L,R) \in SU(3)_L \times SU(3)_R$$

Vacuum $U^{fg} = \delta^{fg}$. Massless fluctuations (G/H)

$$U(x) = \exp(i\phi^a \lambda^a / f_\pi) \qquad\qquad \phi^a = (\pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \eta)$$

Step 2: Write most general G invariant effective lagrangian

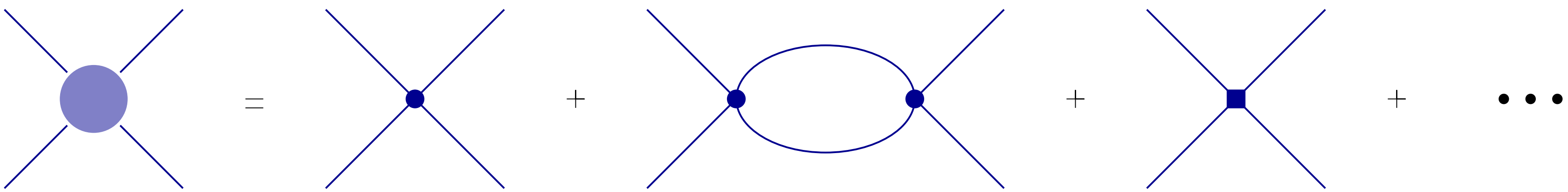
$$\mathcal{L} = \frac{f_\pi^2}{4} \mathrm{Tr}[\partial_\mu U \partial^\mu U^\dagger] + \ldots$$

Non-linear sigma model

Expand lagrangian ($SU(2)$ sector)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^a)^2 + \frac{1}{6f_\pi^2}\left[(\phi^a \partial_\mu \phi^a)^2 - (\phi^a)^2(\partial_\mu \phi^b)^2\right] + O\left(\frac{\partial^4}{f_\pi^4}\right)$$

Step 3: Low energy expansion (power counting)

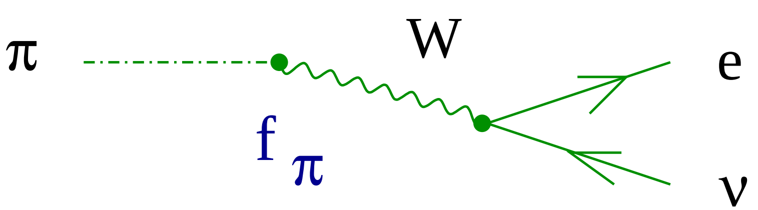


$$T_{\pi\pi} \;=\; O\left(k^2/f_\pi^2\right) \;+\; O\left((k^2/f_\pi^2)^2\right)$$

Relation to f_π: Couple weak gauge fields

$$\partial_\mu U \rightarrow (\partial_\mu + igW_\mu^\pm \tau^\mp)U$$

$$\mathcal{L} = g f_\pi W_\mu^\pm \partial^\mu \pi^\mp$$



Quark Masses

Non-zero quark masses: $\mathcal{L} = \bar{\psi}_L M \psi_R + \bar{\psi}_R M^\dagger \psi_R$

$$M \to L M R^\dagger \qquad\qquad \text{spurion field } M$$

Chiral lagrangian at leading order in M

$$\mathcal{L} = B\mathrm{Tr}[MU] + h.c.$$

Mass matrix $M = \mathrm{diag}(m_u, m_d m_s)$. Minimize effective potential

$$U_{vac} = 1, \qquad E_{vac} = -B\mathrm{Tr}[M] \qquad \langle\bar{\psi}\psi\rangle = -B$$

Expand around U_{vac}: pion mass

$$m_\pi^2 f_\pi^2 = (m_u + m_d)\langle\bar{\psi}\psi\rangle$$

Chiral expansion

$$\mathcal{L} = f_\pi^4 \left(\frac{\partial U}{\Lambda_\chi}\right)^m \left(\frac{m_\pi}{\Lambda_\chi}\right)^n \qquad \Lambda_\chi = 4\pi f_\pi$$

Bonus Material:

Remarks about χSB and Confinement

Notes

QCD with general N_f, N_c (with or without SUSY)

find theories without confinement and/or chiral symmetry breaking

QCD with $N_f = N_c = 3$

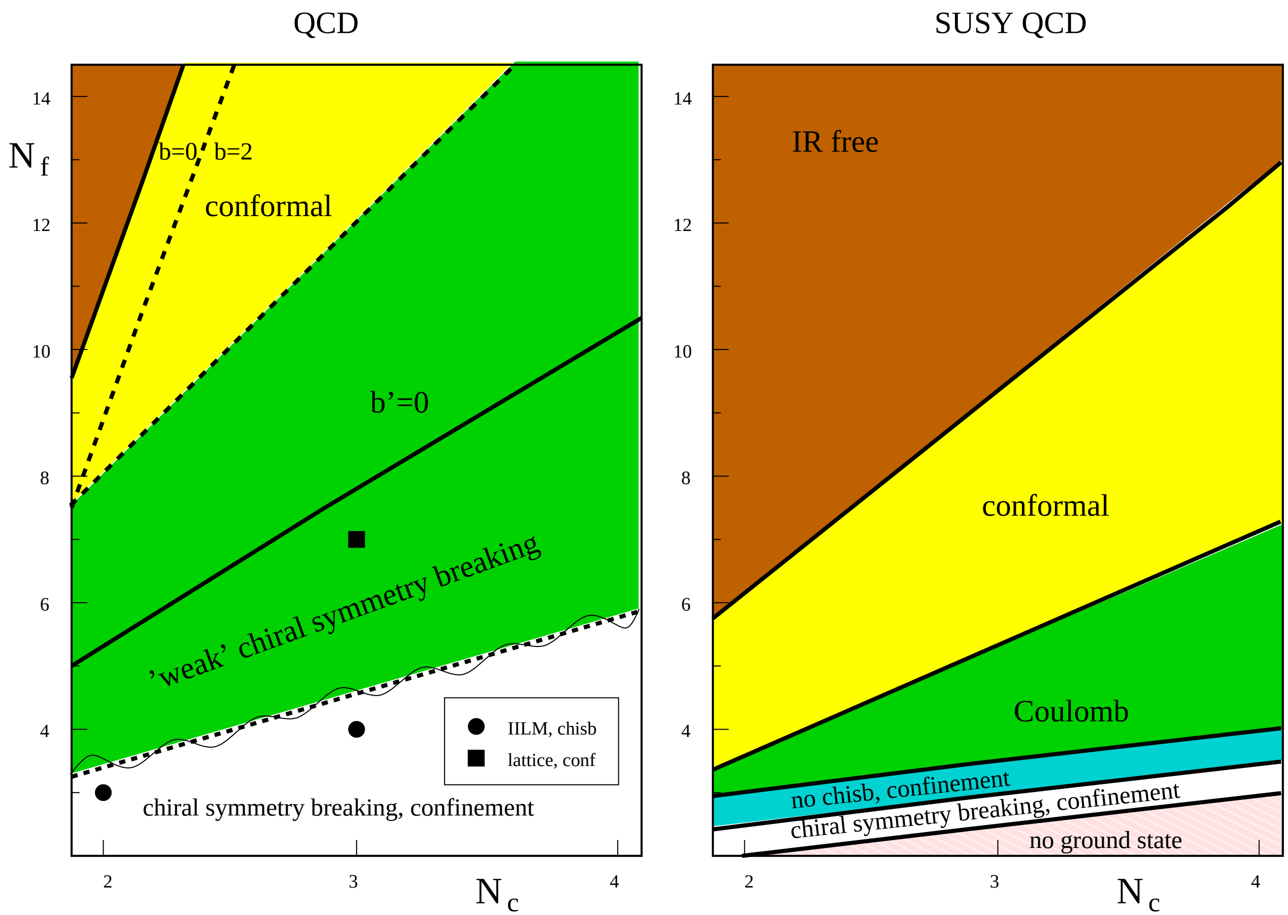
1. Confinement implies chiral symmetry breaking
2. Symmetry breaking pattern $SU(3)_L \times SU(3)_R \to SU(3)_V$ unique
3. Order parameter $\langle \bar{\psi}\psi \rangle \neq 0$

1. Follows from 't Hooft matching conditions

2. Proved in large N_C limit by Coleman and Witten

3. Kovner and Shifman showed that $\langle \bar{\psi}\psi \rangle = 0, \ \langle (\bar{\psi}\psi)^2 \rangle \neq 0$ violates Weingarten inequalities.

QCD Phase Diagram: N_c and N_f



Bonus Material:

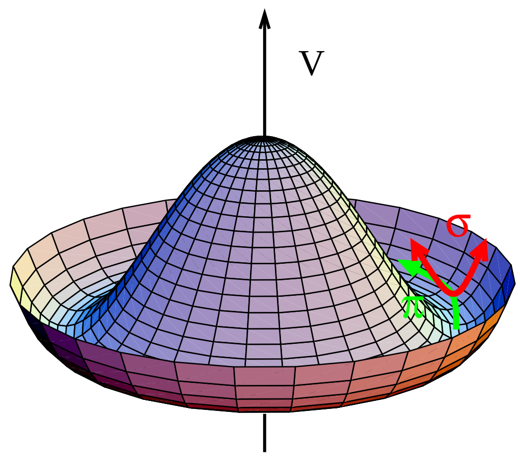
Sigma Model

Second Approach: Sigma Model

Simple model based on linear representation of $SU(2)_L \times SU(2)_R$

$$\phi^a = (\sigma, \vec{\pi}) \qquad\qquad O(4) = SU(2)_L \times SU(2)_R$$

Chirally symmetric lagrangian



$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^a)^2 + V(\phi^a \phi^a)$$

$$V(\phi^a \phi^a) = -\frac{\mu^2}{2}(\phi^a \phi^a) + \frac{\lambda}{4}(\phi^a \phi^a)^2$$

Minimum of potential

$$\partial V/\partial \phi^a = \phi^a(-\mu^2 + \lambda \phi^a \phi^a) = 0 \qquad \phi_0^a = (\sigma_0, \vec{0}) \;\; \sigma_0^2 = \mu^2/\lambda$$

Direction fixed by explicit breaking $\mathcal{L}_{SB} = -c\sigma$

σ_0 related to pion decay constant

$$\vec{A}_\mu = \sigma \partial_\mu \vec{\pi} + \vec{\pi} \partial_\mu \sigma \simeq \sigma_0 \partial_\mu \vec{\pi} \qquad \sigma_0 = f_\pi = 93 \text{ MeV}$$

Consider small oscillations. Equation of motion

$$\delta \mathcal{L} / \delta \phi^a = -\Box \phi^a - \partial V / \partial \phi^a = 0$$

Write $\phi^a = \phi_0^a + \delta \phi^a$

$$\begin{aligned} \Box(\delta \phi^a) &= (\phi_0^a + \delta \phi^a) \left(-\mu^2 + \lambda (\phi_0^a + \delta \phi^a)^2 \right) \\ &= (-\mu^2 + \lambda \phi_0^a \phi_0^a) \phi_0^a + \left(-\mu^2 + 2\lambda \phi_0^a \phi_0^b + \lambda \delta^{ab} \phi_0^c \phi_0^c \right) \delta \phi^b + \ldots \end{aligned}$$

Split in (σ, π) components

$$\begin{aligned} \Box(\delta \sigma) &= \left(-\mu^2 + 3\lambda \sigma_0^2 \right) \delta \sigma & \qquad m_\sigma^2 = 2\mu^2 \\ \Box(\delta \vec{\pi}) &= \left(-\mu^2 + \lambda \sigma_0^2 \right) \delta \vec{\pi} & \qquad m_\pi^2 = 0 \end{aligned}$$

Thermal Fluctuations

Write $\phi^a = \langle\phi^a\rangle + \tilde{\phi}^a$ where $\tilde{\phi}^a$ is a thermal fluctuation. Use

$$\begin{aligned}
\langle\tilde{\phi}^a\rangle &= 0 \\
\langle\tilde{\phi}^a\tilde{\phi}^b\rangle &= (\delta^{ab}/4)\langle\tilde{\phi}^a\tilde{\phi}^a\rangle \\
\langle\tilde{\phi}^a\tilde{\phi}^b\tilde{\phi}^c\rangle &= 0
\end{aligned}$$

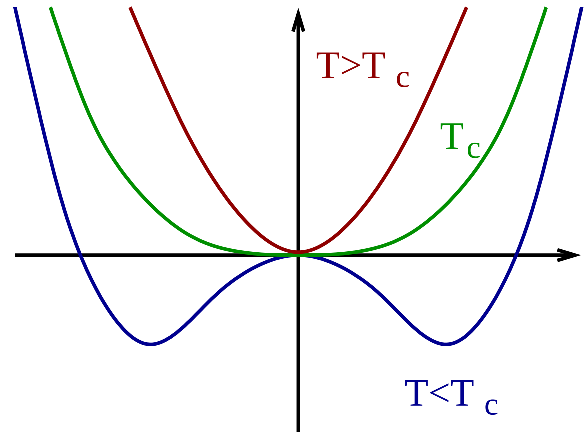
Equation of motion for $\langle\phi^a\rangle$ (use $1/N$)

$$\begin{aligned}
\Box\langle\phi^a\rangle &= -\mu^2\langle\phi^a\rangle + \lambda\left\langle\left(\langle\phi^a\rangle + \tilde{\phi}^a\right)\left(\langle\phi^b\rangle + \tilde{\phi}^b\right)^2\right\rangle \\
&= -\mu^2\langle\phi^a\rangle + \lambda\langle\phi^a\rangle\left[\langle\phi^b\rangle^2 + \langle\tilde{\phi}^b\tilde{\phi}^b\rangle\right]
\end{aligned}$$

Fluctuations tend to restore symmetry

Thermal averages

$$\vec{\pi}_T = 0$$
$$\sigma_T^2 = f_\pi^2 \left(1 - \frac{1}{f_\pi^2}\langle \tilde{\phi}^a \tilde{\phi}^a \rangle\right)$$



Gaussian fluctuations ($m = 0$)

$$\langle \tilde{\phi}^a \tilde{\phi}^a \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_k} \frac{1}{e^{\beta\omega_k} - 1} = \frac{T^2}{12}$$

Critical temperature (3 light d.o.f.)

$$\sigma_T^2 = f_\pi^2 \left(1 - \frac{T^2}{3f_\pi^2}\right) \qquad T_c = \sqrt{3} f_\pi \simeq 150 \text{ MeV}$$

Bonus Material:

Universality

Universality

Chiral phase transition might be continuous (2nd order)

Near T_c masses go to zero and correlation length diverges

Physics independent of microscopic details

Long distance behavior is universal

Only depends on symmetries of the order parameter

Landau-Ginzburg effective action

$$F = \int d^3x \left\{ \frac{1}{2}(\vec{\nabla}\phi^a)^2 + \frac{\mu^2}{2}(\phi^a\phi^a) + \frac{\lambda}{4}(\phi^a\phi^a)^2 + \ldots \right\}$$

Consider $\lambda > 0$, $\mu^2(T_c) = 0$

Universality

$SU(2)_L \times SU(2)_R$	QCD	$\equiv$	$O(4)$	magnet
$\langle\bar{\psi}\psi\rangle$	χ condensate		$\vec{M}$	magnetization
m_q	quark mass		H_3	magnetic field
$\vec{\pi}$	pions		$\vec{\phi}$	spin waves

Predictions

$$C \sim t^\alpha \qquad \alpha = -0.19$$

$$\langle\bar{\psi}\psi\rangle \sim t^\beta \qquad \beta = 0.38$$

$$m_\pi \sim t^\nu \qquad \nu = 0.73$$

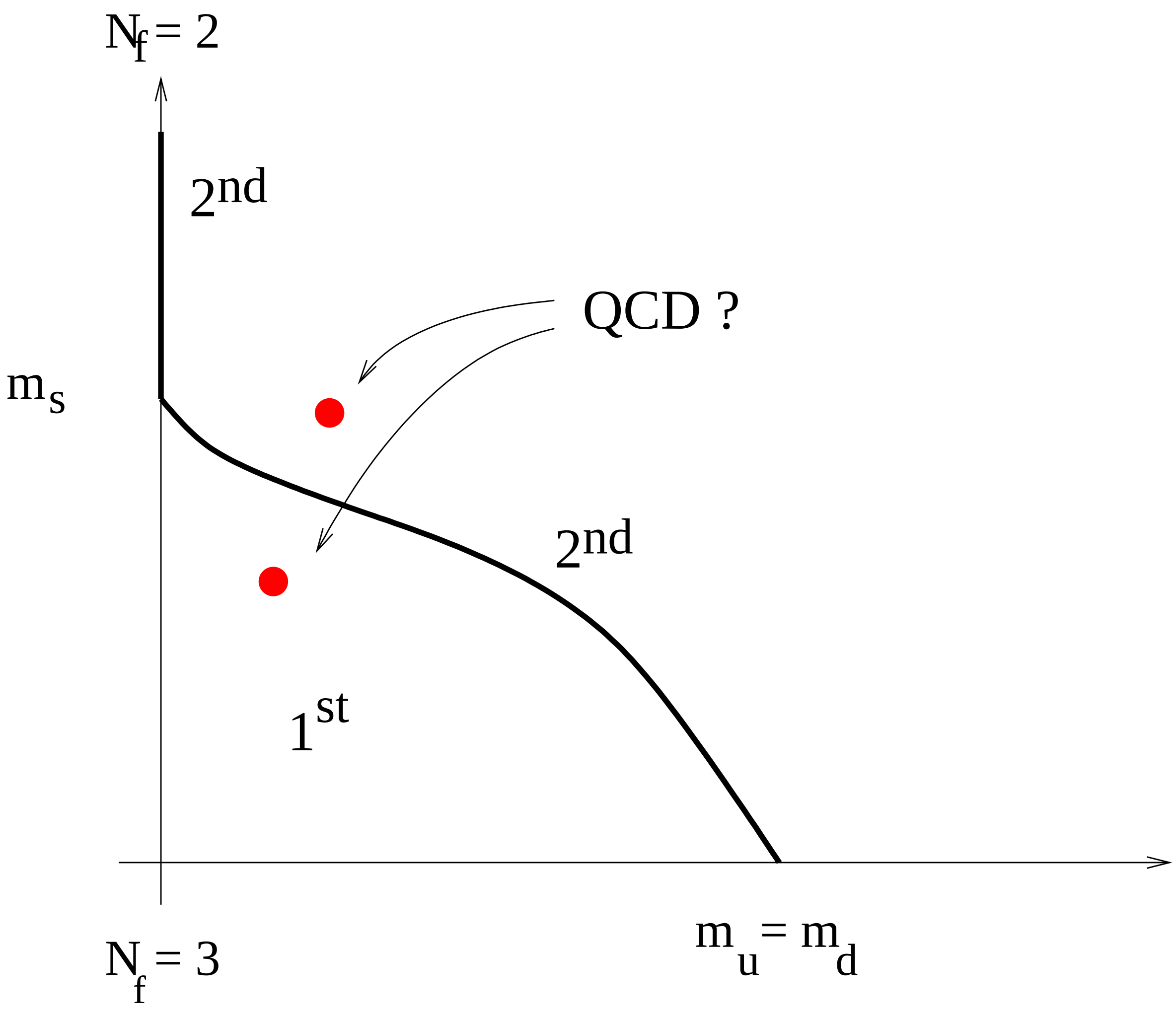
$t = (T - T_c)/T$

from ϵ expansion,

numerical simulations

$N_f = 3$: extra cubic invariant $\det(\phi)$, 2nd order transition unstable

$N_f = 3$ transition is 1st order

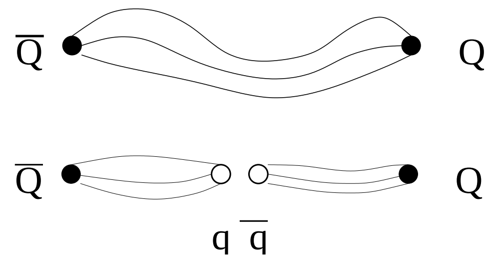


Universality: Confinement

Confinement characterized by heavy quark potential

$$V(r) \sim kr$$

$$k \sim 1 \text{ GeV/fm}$$

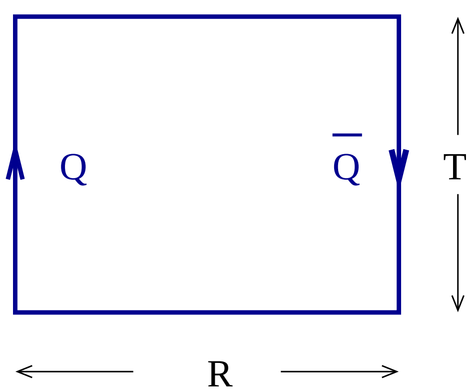


Propagator for heavy quark

$$\left(i\partial_0 + gA_0 + \vec{\alpha}(i\vec{\nabla} + g\vec{A}) + \gamma_0 M\right)\psi = 0$$

$$S(x, x') \simeq \exp\left(ig\int A_0 dt\right)\left(\frac{1+\gamma_0}{2}\right)e^{im(t-t')}\delta(\vec{x} - \vec{x}')$$

Potential related to Wilson loop



$$W(R, T) = \exp\left(ig\oint A_\mu dz_\mu\right)$$

Have $W(R,T) = \exp(-E \cdot T) = \exp(-V(R)T)$

$$W(R,T) \sim \exp(-kA) \qquad \text{Confinement} \;\equiv\; \text{AreaLaw}$$

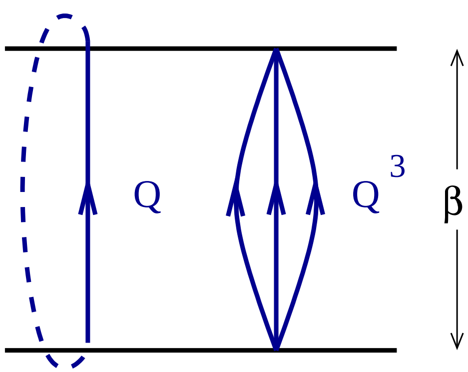
Local order parameter? Polyakov line

$$P(\vec{x}) = \frac{1}{N_c}\mathrm{Tr}[L(\vec{x})] = \frac{1}{N_c} P\mathrm{Tr}\left[\exp\left(ig\int_0^\beta A_0 dt\right)\right]$$

Naive Interpretation: $\langle P \rangle \sim \exp(-m_Q \beta)$

$$\langle P \rangle = 0 \qquad \text{confined} \qquad \langle P \rangle \neq 0 \qquad \text{deconfined}$$

Symmetry: Consider $L \to zL \quad z = \exp(2\pi k i/N_c) \in Z_{N_c}$



$$\mathrm{Tr}[L(\vec{x})] \;\to\; z\mathrm{Tr}[L(\vec{x})]$$
$$\mathrm{Tr}[L(\vec{x})^3] \;\to\; \mathrm{Tr}[L(\vec{x})^3]$$

Polyakov line: $P \to zP$

$$\langle P \rangle = 0 \qquad Z_{N_c} \text{ unbroken} \qquad T < T_c$$

$$\langle P \rangle \neq 0 \qquad Z_{N_c} \text{ broken} \qquad T > T_c$$

Landau-Ginzburg Theory (cubic invariant: $SU(3)$ only)

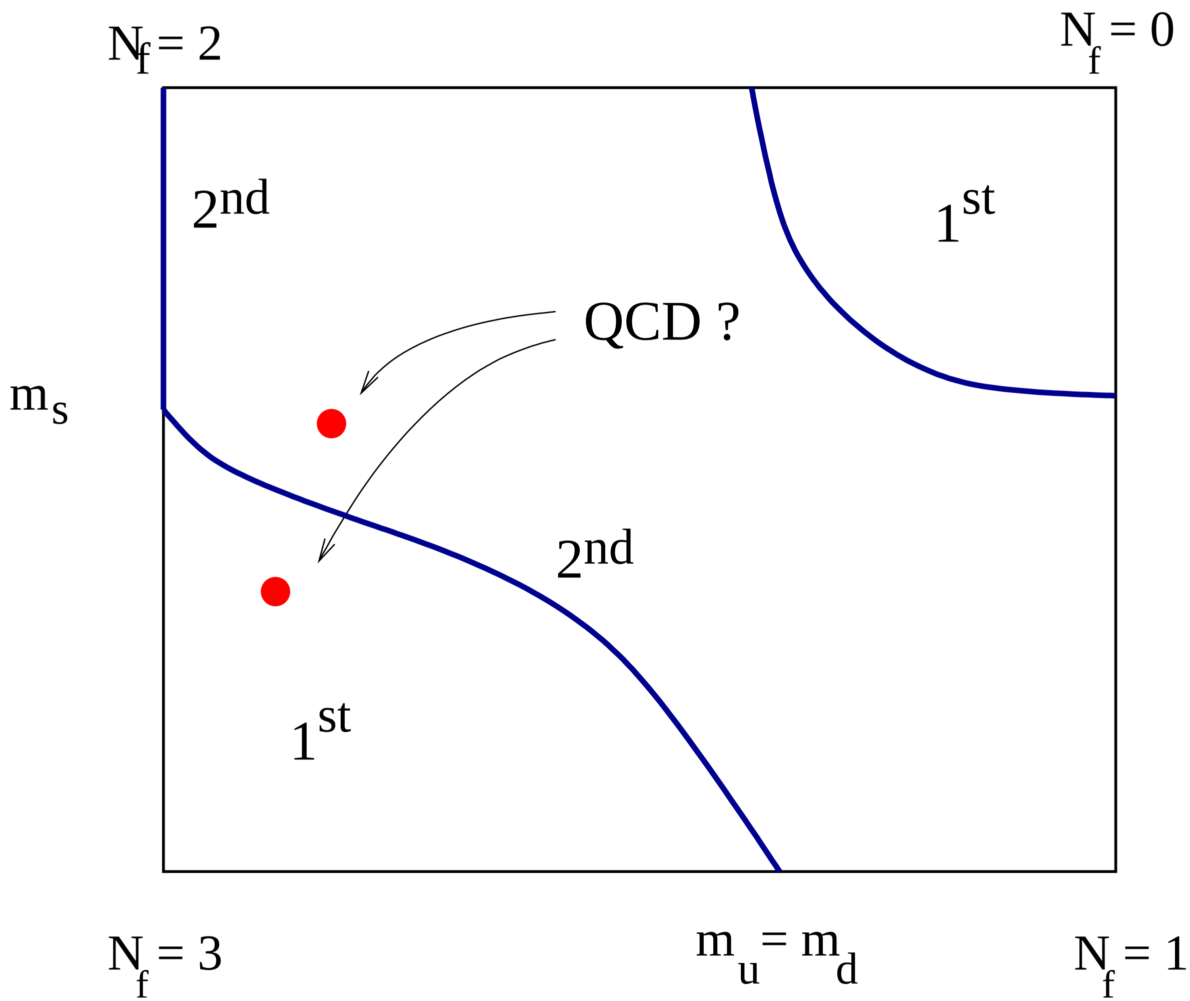
$$F = \int d^3x \left\{ \frac{1}{2} |\vec{\nabla} P|^2 + \mu^2 |P|^2 + g \mathrm{Re}(P^3) + \lambda |P|^4 + \ldots \right\}$$

Predictions

$SU(2)$-color: 2nd order

$SU(3)$-color: 1st order

Summary: Universality



Bonus Material:

Partition Function of Free Gas

Example: Free energy of non-interacting bosons

Partition function: $Z = [\det(p^2 + m^2)]^{-1/2}$

$$\log Z = -\frac{1}{2}\sum_n \log(\omega_n^2 + \omega^2) \qquad\qquad \omega^2 = \vec{p}^2 + m^2$$

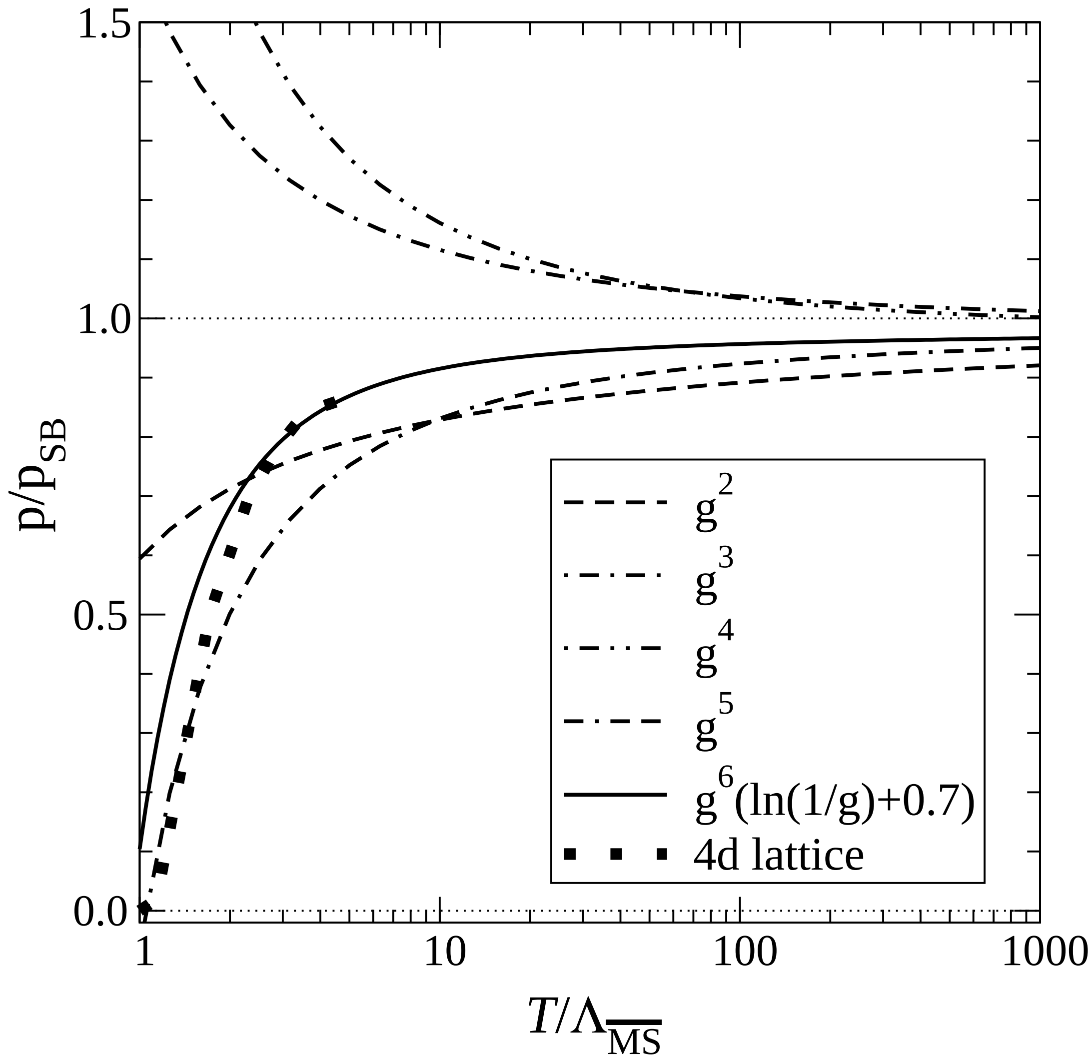
Consider derivative with respect to ω^2

$$\frac{d\log Z}{d\omega^2} = -\frac{1}{2}\sum_n \frac{1}{\omega_n^2 + \omega^2}$$

Use bosonic Matsubara sum and integrate back

$$-T\log Z = \frac{\omega}{2} + \frac{1}{\beta}\log\left(1 - e^{-\beta\omega}\right)$$

Weak Coupling Thermodynamics



Bonus Material:

Kinetic Theory and Shear Viscosity

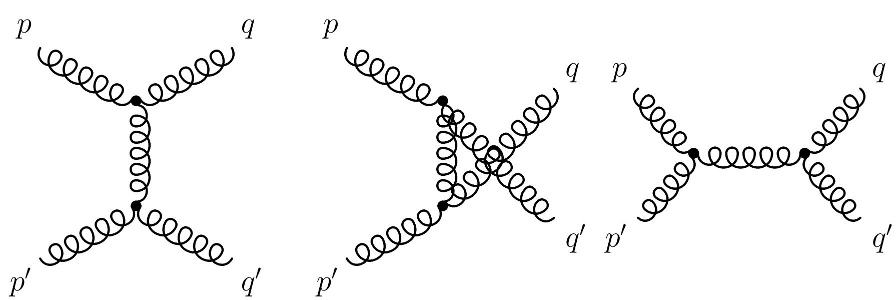
Kinetic Theory

Boltzmann equation

$$\frac{\partial f_p}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f_p + \vec{F} \cdot \vec{\nabla}_p f_p = C[f_p]$$

Collision term $C[f_p] = C_{gain} - C_{loss}$

$$C_{loss} = \int dp' dq dq' \; f_p f_{p'} w(p, p'; q, q') \qquad C_{gain} = \ldots$$



$$C[f_p^0] = 0 \;\; (equ.)$$

Linearized theory (Chapman-Enskog): $f_p = f_p^0(1 + \chi_p/T)$

$$C[f_p] \equiv C_p \chi_p \qquad \text{linear collision operator}$$

Linear response to flow gradient

$$f_p = \exp(-(E_p - \vec{p} \cdot \vec{v})/(kT))$$

Drift term proportional to "driving term" $(v_{ij} = \partial_i v_j + \partial_j v_i - trace)$

$$\frac{\partial f_p}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f_p \equiv X \qquad X \equiv p_i p_j v_{ij}$$

Boltzmann equation

$$C_p \chi_p = X \qquad \chi_p \equiv g_p p_i p_j v_{ij}$$

Viscosity $T_{ij} = T_{ij}^0 + \eta v_{ij}$

$$\eta \sim \langle X | \chi \rangle \qquad \langle \chi | X \rangle = \int d^3p\, f_p^0\, (\chi_p \cdot p_i p_j v_{ij})$$

$$\eta \sim \langle \chi | C_p | \chi \rangle$$

Variational principle

$$\langle \chi_{var}|C_p|\chi_{var}\rangle\langle\chi|C_p|\chi\rangle \geq \langle\chi_{var}|C_p|\chi\rangle^2 = \langle\chi_{var}|X\rangle^2$$

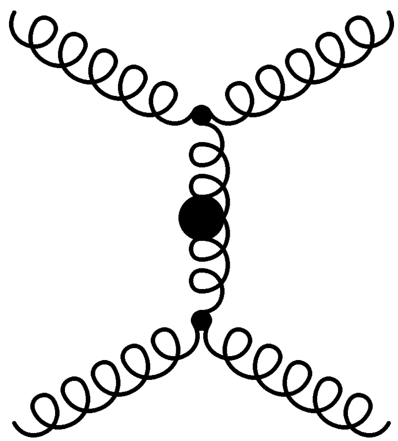
Variational bound

$$\eta \geq \frac{\langle\chi_{var}|X\rangle^2}{\langle\chi_{var}|C|\chi_{var}\rangle}$$

Best bound for $g_p \sim p^\alpha$ ($\alpha \simeq 0.1$)

$$\eta = \frac{0.34T^3}{\alpha_s^2 \log(1/\alpha_s)}$$

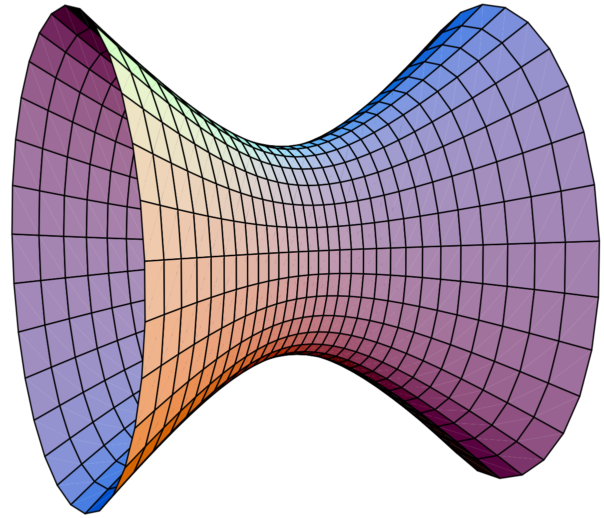
$\log(\alpha)$ from dynamic screening



Bonus Material:

AdS/CFT

Anti-DeSitter Space



Consider a hyperboloid embedded in 6-d euclidean space

$$-R^2 = \sum_{i=1,4} x_i^2 - x_0^2 - x_5^2$$

This is a space of constant negative curvature, and a solution of the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2} g_{\mu\nu} \Lambda$$

with negative cosmological constant. Isometries of AdS_5: $SO(4,2)$

Many possible choices of coordinates. Witten uses

$$ds^2 = \frac{1}{z^2} \left(-dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu \right)$$

$\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory

Fields: Gluons, Gluinos, Higgses; all in the adjoint of $SU(N_c)$

$$\mathcal{L} = \frac{1}{4}(F^a_{\mu\nu})^2 + \bar{\lambda}^a_A \sigma^\mu (D_\mu \lambda^A)^a + (D_\mu \Phi_{AB})^a (D_\mu \Phi^{AB})^a + \ldots$$

$$A^a_\mu \qquad \lambda^a_A \; (\bar{4}_R) \qquad \Phi^a_{AB} \; (6_R)$$

Global symmetries: Conformal and $SU(4)_R$

$$SO(4,2) \times SU(4)_R$$

Properties: Conformal $\beta(g) = 0$, extra scalars, no fundamental fermions, no chiral symmetry breaking, no confinement

Bonus Material:

Simple Model of Phase Diagram in $\mu - T$ Plane

QQ vs $\bar{Q}Q$ Condensation

Schematic interaction: $\mathcal{L} = G(\bar{\psi}\gamma_\mu\lambda^a\psi)^2$

$$\mathcal{L} = G_M(\bar{\psi}\psi)^2 + \ldots \qquad\qquad \mathcal{L} = G_D(\psi C\gamma_5\tau_2\lambda_2\psi)(h.c.) + \ldots$$

$\bar{Q}Q$ gap equation

$$M = m_0 + G_M\langle\bar{q}q\rangle \qquad \langle\bar{q}q\rangle = -\frac{3}{\pi^2}\int p^2 dp\, \frac{M}{E_p}(1 - n_F)$$

QQ gap equation

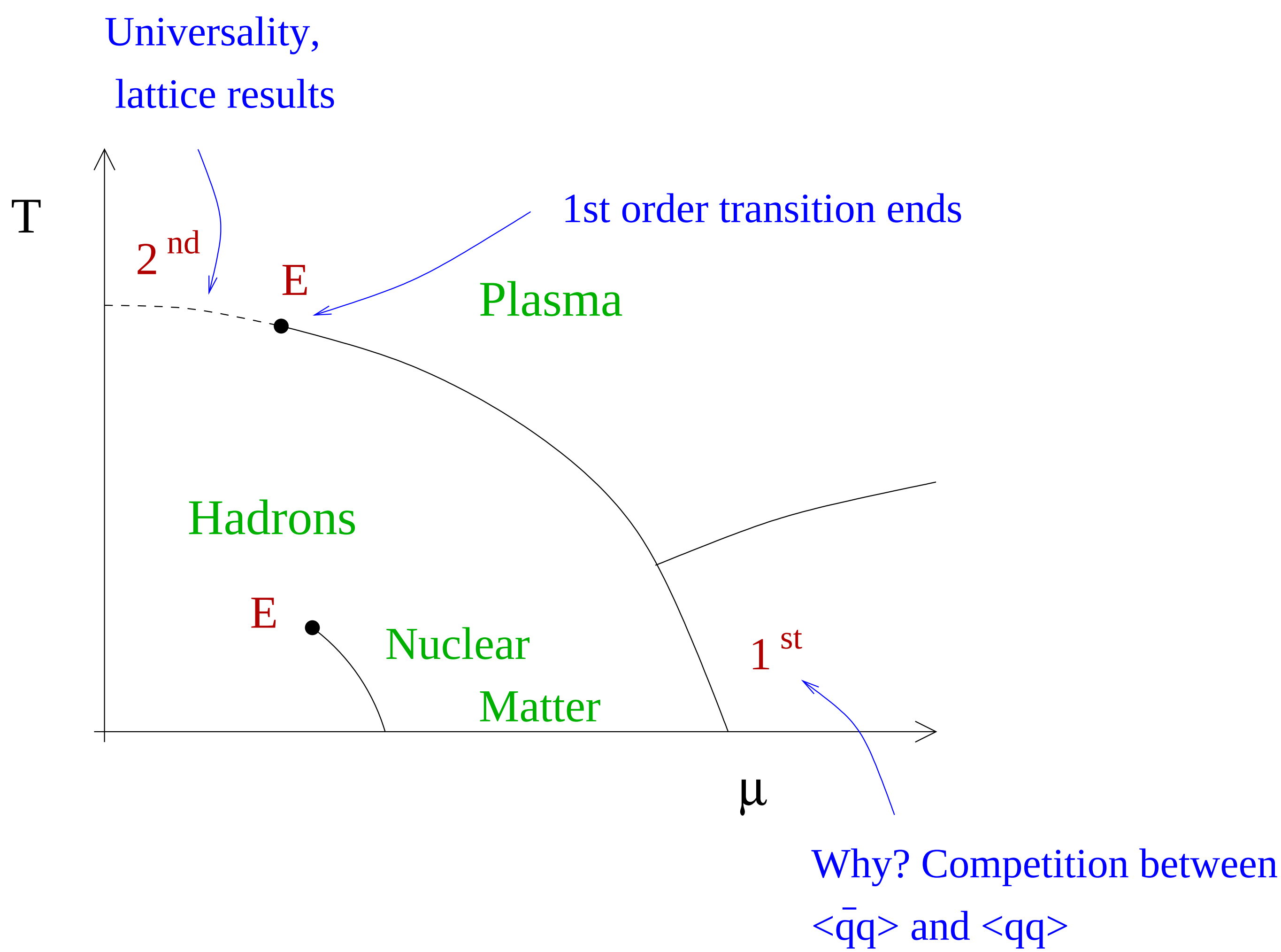
$$\Delta = G_D\langle qq\rangle \qquad \langle qq\rangle = \frac{1}{4\pi^2}\int p^2 dp\, \frac{\Delta}{[(E_p-\mu)^2 + \Delta^2]^{1/2}}$$

Condensation energy

$$E_{\bar{Q}Q} = -f_\pi^2 M^2 \qquad\qquad E_{QQ} = \frac{\mu^2}{2\pi^2}\Delta^2$$

$G_M > G_D$ favors $\bar{Q}Q$ $\qquad$ $\mu > 0$ favors QQ

Phase Diagram: Second Revision

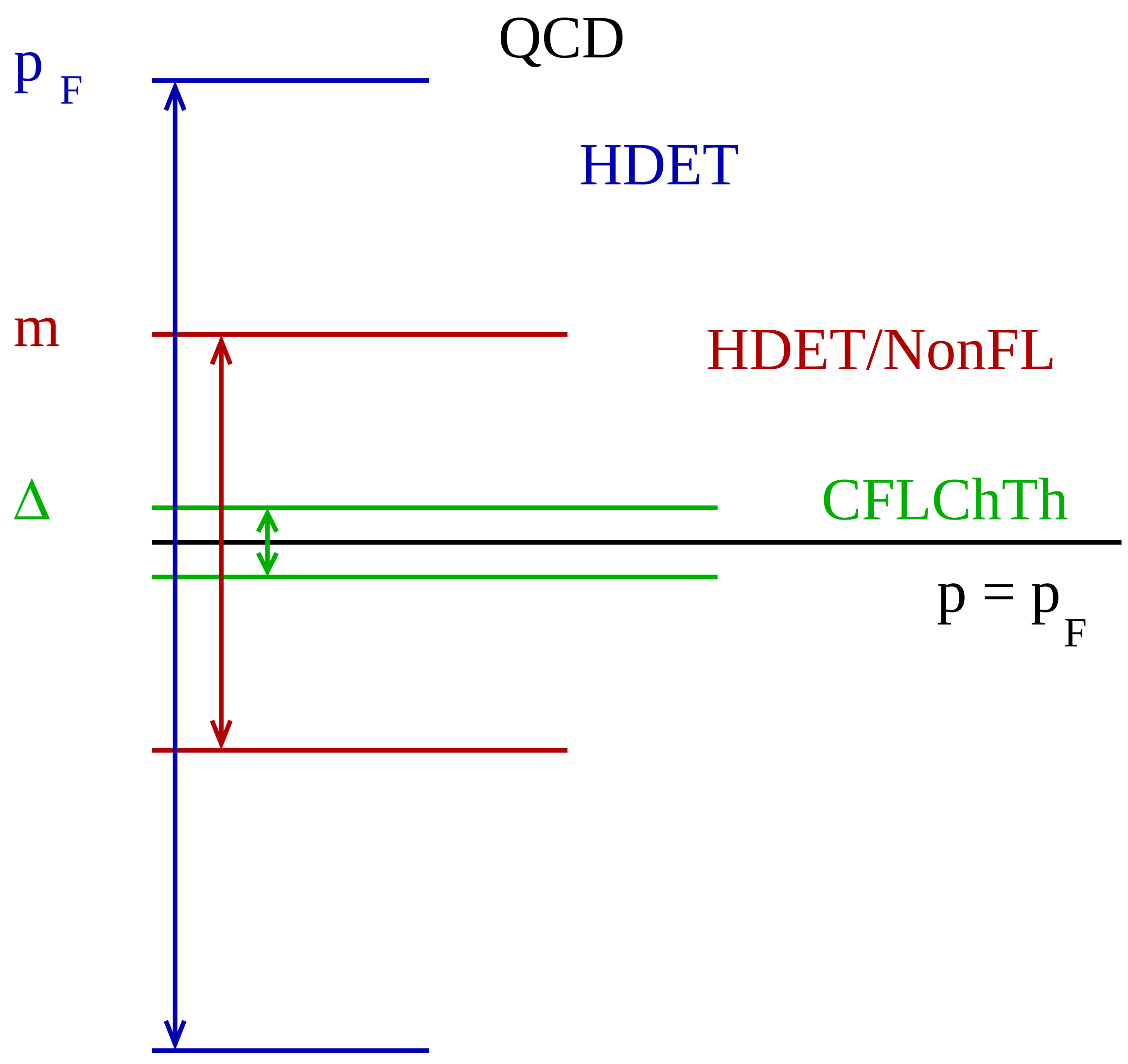


critical endpoint (E) persists even if $m \neq 0$

Bonus Material:

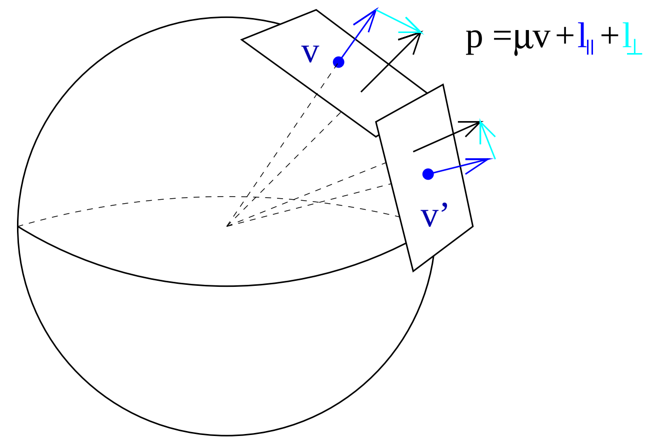
Effective Theories of the CFL Phase

Very Dense Matter: Effective Field Theories



High Density Effective Theory

Effective field theory on v-patches



$$\psi_{v\pm} = e^{-i\mu v\cdot x}\left(\frac{1 \pm \vec{\alpha}\cdot\vec{v}}{2}\right)\psi$$

Effective lagrangian for $p_0 < m$

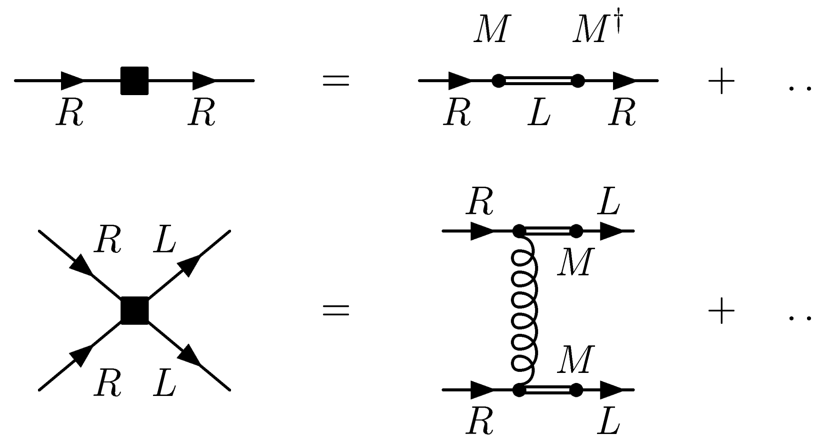
$$\mathcal{L} = \psi_v^\dagger\left(iv\cdot D - \frac{D_\perp^2}{2\mu}\right)\psi_v + \mathcal{L}_{4f} - \frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} + \mathcal{L}_{HDL}$$

$$\mathcal{L}_{HDL} = -\frac{m^2}{2}\sum_v G^a_{\mu\alpha}\frac{v^\alpha v^\beta}{(v\cdot D)^2}G^b_{\mu\beta}$$

Mass Terms: Match HDET to QCD

$$\mathcal{L} = \psi_R^\dagger \frac{MM^\dagger}{2\mu}\psi_R + \psi_L^\dagger \frac{M^\dagger M}{2\mu}\psi_L$$

$$+ \frac{V_M^0}{\mu^2}(\psi_R^\dagger M\lambda^a\psi_L)(\psi_R^\dagger M\lambda^a\psi_L)$$



mass corrections to FL parameters $\hat{\mu}_{L,R}$ and $V^0(RR \to LL)$

EFT in the CFL Phase

Consider HDET with a CFL gap term

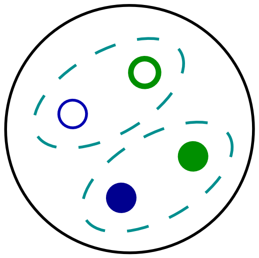
$$\mathcal{L} = \mathrm{Tr}\left(\psi_L^\dagger(iv\cdot D)\psi_L\right) + \frac{\Delta}{2}\left\{\mathrm{Tr}\left(X^\dagger\psi_L X^\dagger\psi_L\right) - \kappa\left[\mathrm{Tr}\left(X^\dagger\psi_L\right)\right]^2\right\} + (L\leftrightarrow R, X\leftrightarrow Y)$$

$$\psi_L \to L\psi_L C^T,\ X\to LXC^T, \qquad \langle X\rangle = \langle Y\rangle = \mathbb{1}$$

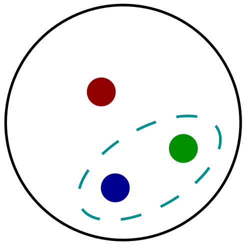
Quark loops generate a kinetic term for X, Y

Integrate out gluons, identify low energy fields ($\xi = \Sigma^{1/2}$)

$$\Sigma = XY^\dagger$$



$$N_L = \xi(\psi_L X^\dagger)\xi^\dagger$$



[8]+[1] GBs

[8]+[1] Baryons

Effective theory: (CFL) baryon chiral perturbation theory

$$
\begin{aligned}
\mathcal{L} \; = \; & \frac{f_\pi^2}{4}\left\{\mathrm{Tr}\left(\nabla_0\Sigma\nabla_0\Sigma^\dagger\right) - v_\pi^2\,\mathrm{Tr}\left(\nabla_i\Sigma\nabla_i\Sigma^\dagger\right)\right\} \\
& + A\left\{\left[\mathrm{Tr}\left(M\Sigma\right)\right]^2 - \mathrm{Tr}\left(M\Sigma M\Sigma\right) + h.c.\right\} \\
& + \mathrm{Tr}\left(N^\dagger i v^\mu D_\mu N\right) - D\,\mathrm{Tr}\left(N^\dagger v^\mu \gamma_5 \left\{\mathcal{A}_\mu, N\right\}\right) \\
& - F\,\mathrm{Tr}\left(N^\dagger v^\mu \gamma_5 \left[\mathcal{A}_\mu, N\right]\right) + \frac{\Delta}{2}\left\{\mathrm{Tr}\left(NN\right) - \left[\mathrm{Tr}\left(N\right)\right]^2\right\}
\end{aligned}
$$

$$\nabla_0\Sigma = \partial_0\Sigma + i\hat{\mu}_L\Sigma - i\Sigma\hat{\mu}_R$$

$$D_\mu N = \partial_\mu N + i[\mathcal{V}_\mu, N]$$

$$f_\pi^2 = \frac{21 - 8\log 2}{18}\,\frac{\mu^2}{2\pi^2} \quad v_\pi^2 = \frac{1}{3} \quad A = \frac{3\Delta^2}{4\pi^2} \quad D = F = \frac{1}{2}$$

Phase Structure and Spectrum

Phase structure determined by effective potential

$$V(\Sigma) = \frac{f_\pi^2}{2} \mathrm{Tr}\left(\hat{\mu}_L \Sigma \hat{\mu}_R \Sigma^\dagger\right) - A\mathrm{Tr}(M\Sigma^\dagger) - B_1 \left[\mathrm{Tr}(M\Sigma^\dagger)\right]^2 + \ldots$$

$$V(\Sigma_0) \equiv min$$

Fermion spectrum determined by

$$\mathcal{L} = \mathrm{Tr}\left(N^\dagger i v^\mu D_\mu N\right) + \mathrm{Tr}\left(N^\dagger \gamma_5 \rho_A N\right) + \frac{\Delta}{2}\left\{\mathrm{Tr}\left(NN\right) - \left[\mathrm{Tr}\left(N\right)\right]^2\right\},$$

$$\rho_{V,A} = \frac{1}{2}\left\{\xi \hat{\mu}_L \xi^\dagger \pm \xi^\dagger \hat{\mu}_R \xi\right\} \qquad \xi = \sqrt{\Sigma_0}$$