

Strongly interacting quantum fluids:

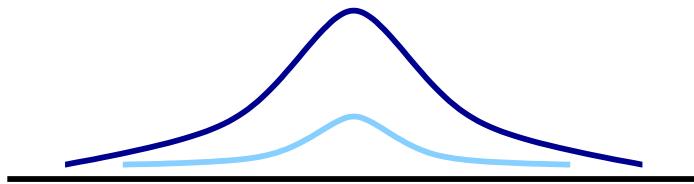
Transport theory

Thomas Schaefer

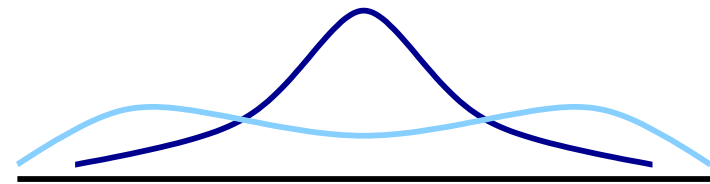
North Carolina State University

Fluids: Gases, Liquids, Plasmas, ...

Hydrodynamics: Long-wavelength, low-frequency dynamics of conserved or spontaneously broken symmetry variables.



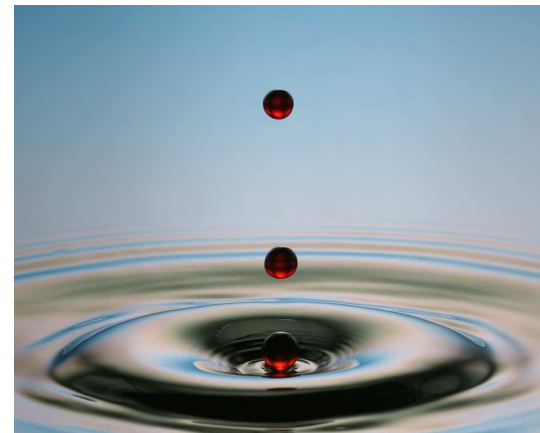
$$\tau \sim \tau_{micro}$$



$$\tau \sim \lambda$$

Historically: Water

$$(\rho, \epsilon, \vec{\pi})$$



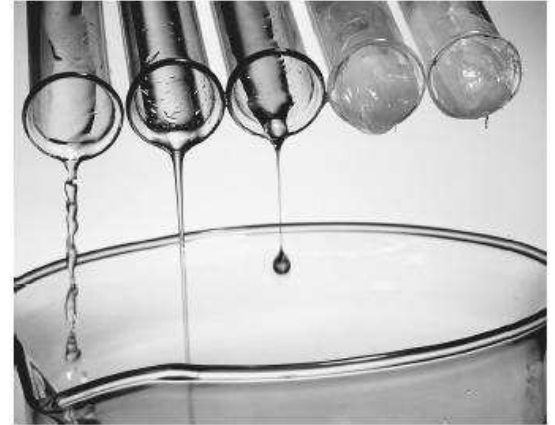
Simple non-relativistic fluid

Simple fluid: Conservation laws for mass, energy, momentum

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

$$\frac{\partial \epsilon}{\partial t} + \vec{\nabla} \cdot \vec{j}^\epsilon = 0$$

$$\frac{\partial}{\partial t} (\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0$$



Constitutive relations: Energy momentum tensor

$$\Pi_{ij} = P\delta_{ij} + \rho v_i v_j + \eta \left(\partial_i v_j + \partial_j v_i - \frac{2}{3} \delta_{ij} \partial_k v_k \right) + O(\partial^2)$$

reactive

dissipative

2nd order

$$\text{Expansion } \Pi_{ij}^0 \gg \delta \Pi_{ij}^1 \gg \delta \Pi_{ij}^2$$

Regime of applicability

Expansion parameter $Re^{-1} = \frac{\eta(\partial v)}{\rho v^2} = \frac{\eta}{\rho L v} \ll 1$

$$Re = \underbrace{\frac{\hbar n}{\eta}}_{\text{fluid property}} \times \underbrace{\frac{m v L}{\hbar}}_{\text{flow property}}$$

Kinetic theory estimate: $\eta \sim n p l_{mfp}$

$$Re^{-1} = \frac{v}{c_s} Kn \quad Kn = \frac{l_{mfp}}{L}$$

expansion parameter $Kn \ll 1$

Relativistic hydrodynamics

Energy momentum tensor of an ideal fluid

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P\eta^{\mu\nu},$$

Energy-momentum conservation: $\partial_\mu T^{\mu\nu} = 0$

$$\partial_\mu (su^\mu) = 0 \quad Du_\mu = -\frac{1}{\epsilon + P} \nabla_\mu^\perp P$$

$$D = u \cdot \partial \quad \nabla_\mu^\perp = \Delta_{\mu\nu} \partial^\nu \quad \Delta_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu$$

Viscous contribution $\delta^{(1)}T^{\mu\nu} = -\eta\sigma^{\mu\nu} - \zeta\Delta^{\mu\nu}\partial \cdot u$

$$\sigma^{\mu\nu} = \Delta^{\mu\alpha} \Delta^{\nu\beta} \left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3} \eta_{\alpha\beta} \partial \cdot u \right).$$

Relativistic hydrodynamics: causality

Linearized hydro: consider small fluctuations $g^i = \delta T^{0i}$

$$S_{gg}^L = 2sT \frac{\Gamma_s \omega^2 \mathbf{k}^2}{(\omega^2 - c_s^2 \mathbf{k}^2)^2 + (\Gamma_s \omega \mathbf{k}^2)^2} \quad \Gamma_s = \frac{\frac{4}{3}\eta + \zeta}{sT}$$

$$S_{gg}^T = \frac{2\eta \mathbf{k}^2}{\omega^2 + \left(\frac{\eta}{sT} \mathbf{k}^2\right)^2}$$

L/T channel: propagating sound mode, diffusive shear mode.

Consider “speed” of shear wave

$$v_{diff} = \frac{\partial |\omega|}{\partial k} = \frac{\eta}{sT} k$$

Find acausal* behavior $v_{diff} > c$ for $k > k_{cr}$.

* Occurs outside regime of validity of hydro. But: Causes numerical difficulties.

Second order hydrodynamics

Causality can be restored by introducing a finite relaxation time

$$\eta(\omega) \simeq \frac{\eta}{1 + i\omega\tau_\pi}$$

More formal approach: Second order hydrodynamics (BRSSS)

$$\begin{aligned} \delta^{(2)}T^{\mu\nu} = & \eta\tau_{II} \left[\langle D\sigma^{\mu\nu} \rangle + \frac{1}{3} \sigma^{\mu\nu} (\partial \cdot u) \right] \\ & + \lambda_1 \sigma^{\langle\mu}{}_\lambda \sigma^{\nu\rangle\lambda} + \lambda_2 \sigma^{\langle\mu}{}_\lambda \Omega^{\nu\rangle\lambda} + \lambda_3 \Omega^{\langle\mu}{}_\lambda \Omega^{\nu\rangle\lambda} \end{aligned}$$

$$A^{\langle\mu\nu\rangle} = \frac{1}{2} \Delta^{\mu\alpha} \Delta^{\nu\beta} (A_{\alpha\beta} + A_{\beta\alpha} - \frac{2}{3} \Delta^{\mu\nu} \Delta^{\alpha\beta} A_{\alpha\beta}) \quad \Omega^{\mu\nu} = \frac{1}{2} \Delta^{\mu\alpha} \Delta^{\nu\beta} (\partial_\alpha u_\beta - \partial_\beta u_\alpha)$$

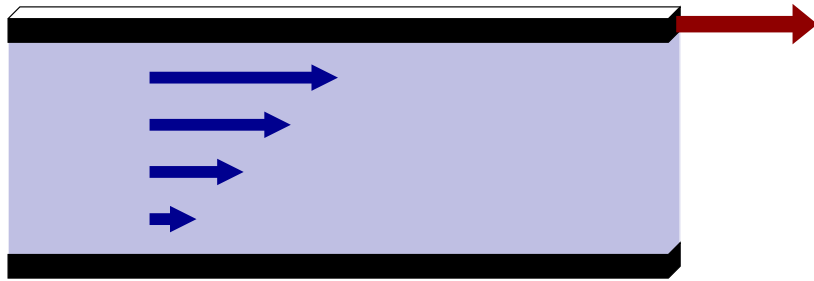
Contains four new transport coefficients τ_{II}, λ_i

Can be written as a relaxation equation for $\pi^{\mu\nu} \equiv \delta T^{\mu\nu}$

$$\pi^{\mu\nu} = -\eta\sigma^{\mu\nu} - \tau_{II} \langle D\pi^{\mu\nu} \rangle + \dots$$

Shear viscosity

Viscosity determines shear stress (“friction”) in fluid flow

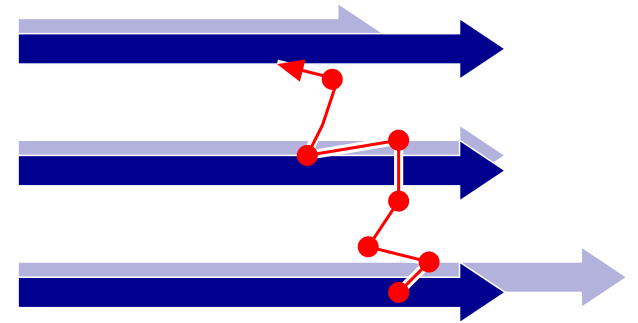


$$F = A \eta \frac{\partial v_x}{\partial y}$$

Kinetic theory: conserved quantities carried by quasi-particles

$$\frac{\partial f_p}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f_p + \vec{F} \cdot \vec{\nabla}_p f_p = C[f_p]$$

$$\eta \sim \frac{1}{3} n \bar{p} l_{mfp}$$



Dilute, weakly interacting gas: $l_{mfp} \sim 1/(n\sigma)$

$$\eta \sim \frac{1}{3} \bar{p} \sigma$$

independent of density!

Shear viscosity

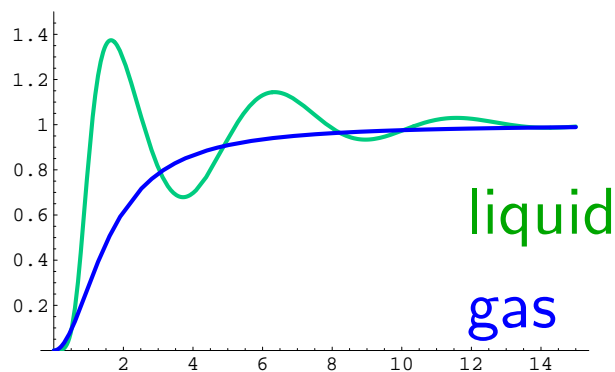
non-interacting gas ($\sigma \rightarrow 0$): $\eta \rightarrow \infty$

non-interacting and hydro limit ($T \rightarrow \infty$) limit do not commute

strongly interacting gas: $\frac{\eta}{n} \sim \bar{p} l_{mfp} \geq \hbar$

but: kinetic theory not reliable!

what happens if the gas condenses into a liquid?



Pair correlation
function

Life at low Reynolds number

E. M. Purcell

Lyman Laboratory, Harvard University, Cambridge, Massachusetts 02138

(Received 12 June 1976)

Viscosity of a liquid is a dominant theme here, and you know Vicki's program of explaining everything in terms of fundamental constants. The viscosity of a liquid is a tough nut to crack ... because when the stuff is cooled by 40° its viscosity can change by a factor of 10^6 .

Purcell anticipates

Vicki's theory

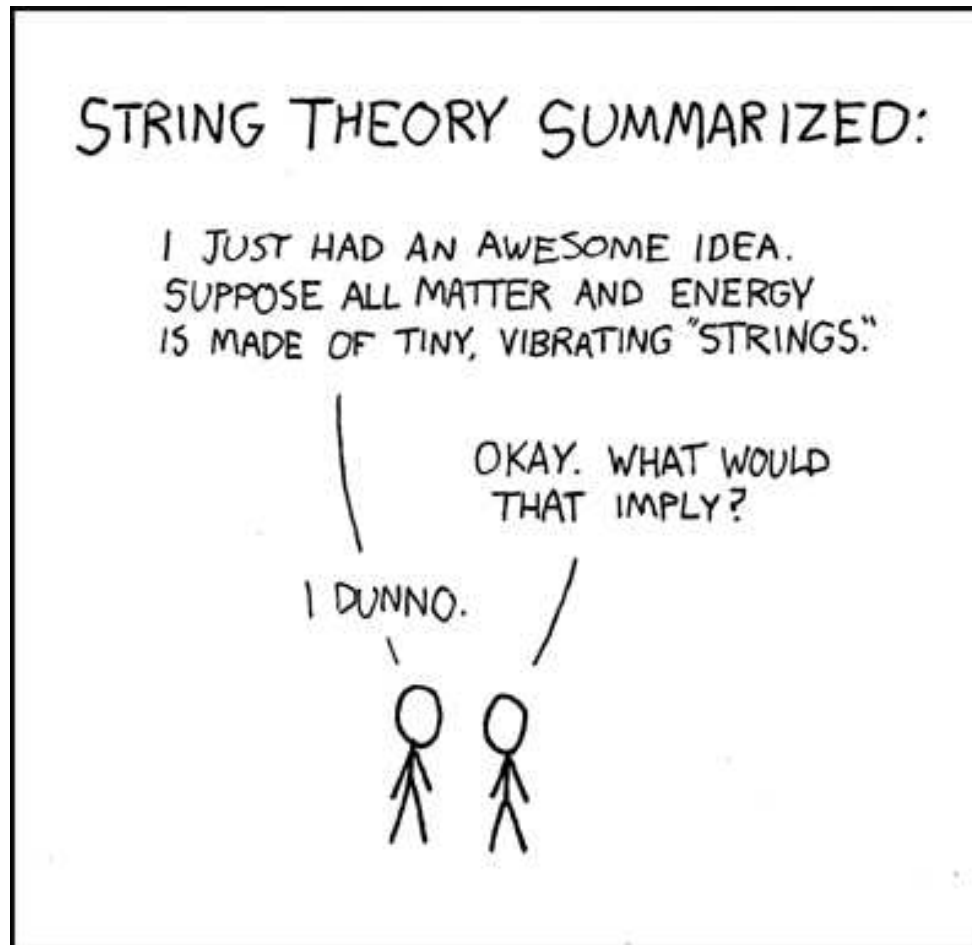
$$\eta \sim \exp(E/T)$$

But it's more mysterious than that: The viscosities have a big range, but they all stop in the same place. I don't understand that.

Eyring, Frenkel:

$$\eta \simeq hn \exp(E/T)$$

And now for something completely different ...



Gauge theory at strong coupling: Holographic duality

The AdS/CFT duality relates

large N_c (conformal) gauge
theory in 4 dimensions

correlation fcts of gauge
invariant operators



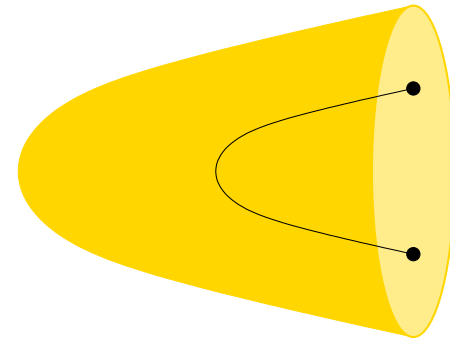
string theory on 5 dimensional
Anti-de Sitter space $\times S^5$



boundary correlation fcts
of AdS fields

$$\langle \exp \int dx \phi_0 \mathcal{O} \rangle =$$

$$Z_{string}[\phi(\partial AdS) = \phi_0]$$



The correspondence is simplest at strong coupling $g^2 N_c$

strongly coupled gauge theory \Leftrightarrow

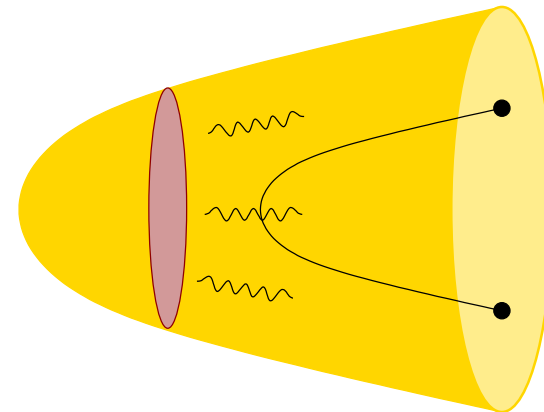
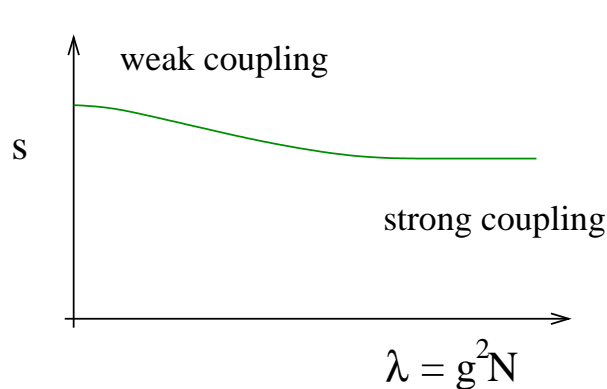
classical string theory

Holographic duals at finite temperature

Thermal (conformal) field theory $\equiv AdS_5$ black hole

CFT temperature \Leftrightarrow Hawking temperature of black hole

CFT entropy \Leftrightarrow Hawking-Bekenstein entropy
 \sim area of event horizon

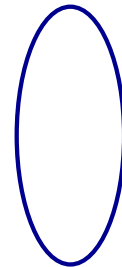
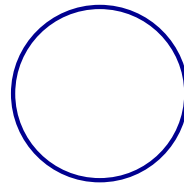
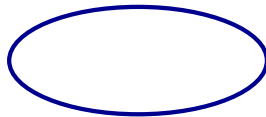
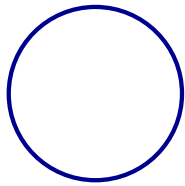


$$s(\lambda \rightarrow \infty) = \frac{\pi^2}{2} N_c^2 T^3 = \frac{3}{4} s(\lambda = 0)$$

Holographic duals: Transport properties

Thermal (conformal) field theory \equiv AdS_5 black hole

$$T_{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} \quad g_{\mu\nu} = g_{\mu\nu}^0 + \gamma_{\mu\nu}$$



CFT entropy



Hawking-Bekenstein entropy

\sim area of event horizon

shear viscosity



Graviton absorption cross section

\sim area of event horizon

Holographic duals: Transport properties

Thermal (conformal) field theory $\equiv AdS_5$ black hole

CFT entropy \Leftrightarrow

Hawking-Bekenstein entropy

\sim area of event horizon

shear viscosity \Leftrightarrow

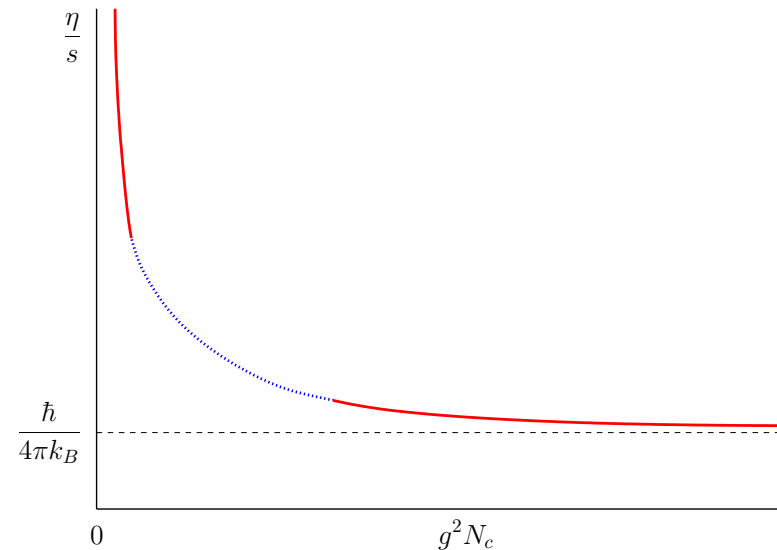
Graviton absorption cross section

\sim area of event horizon

Strong coupling limit

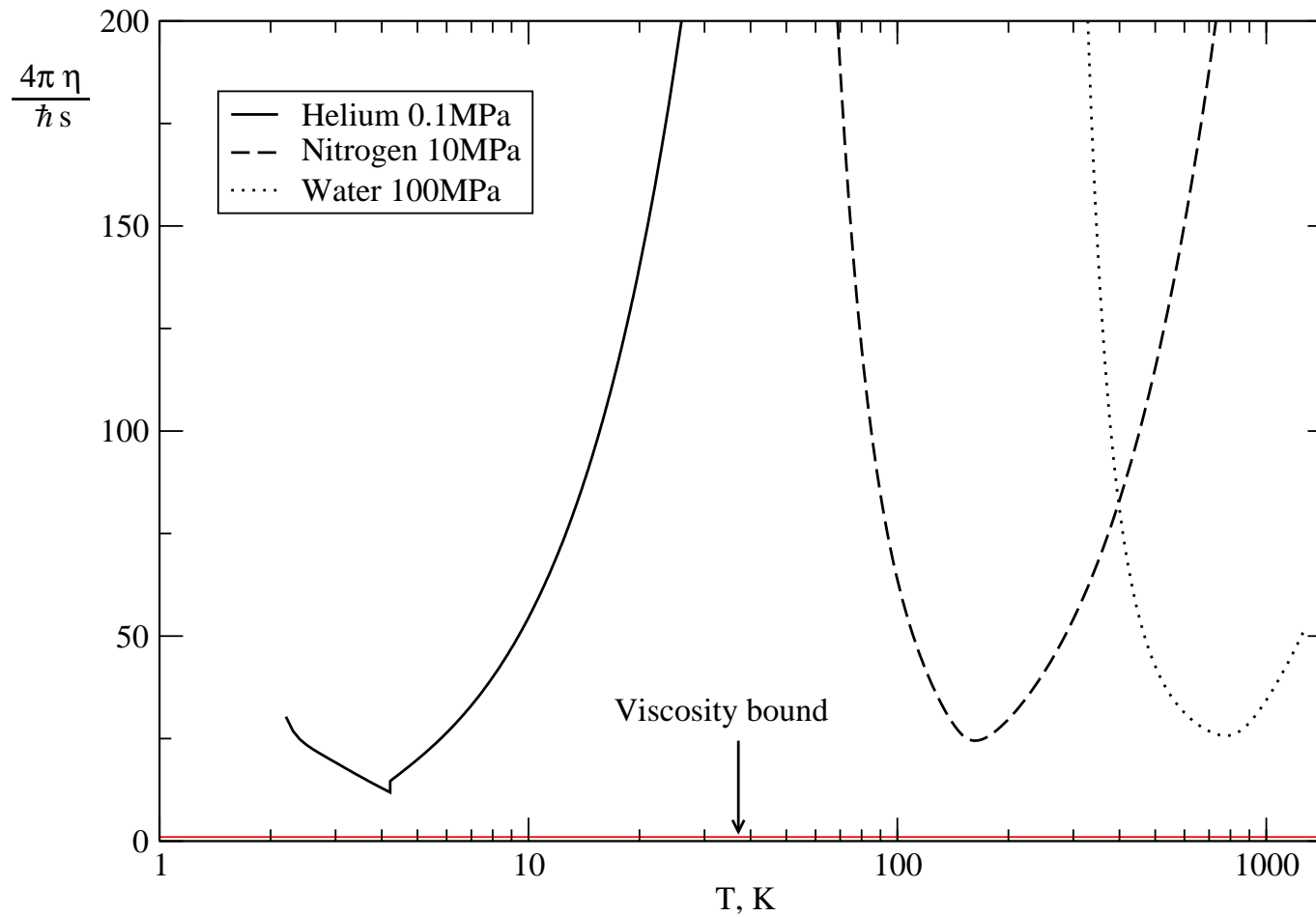
$$\frac{\eta}{s} = \frac{\hbar}{4\pi k_B}$$

Son and Starinets (2001)

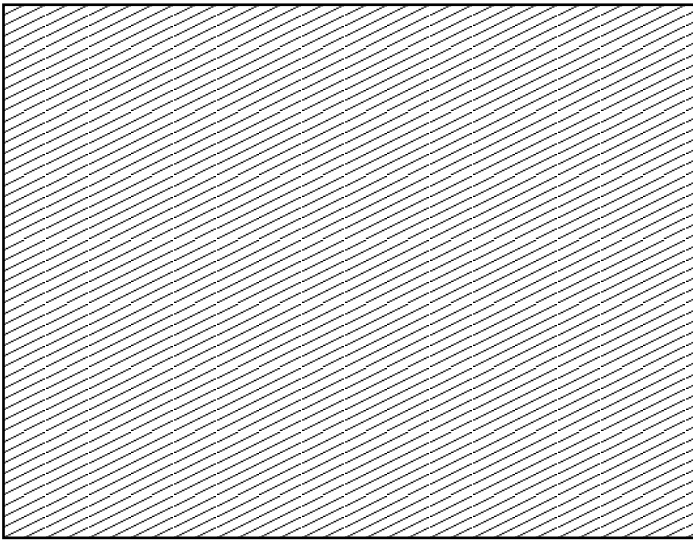


Strong coupling limit universal? Provides lower bound for all theories?

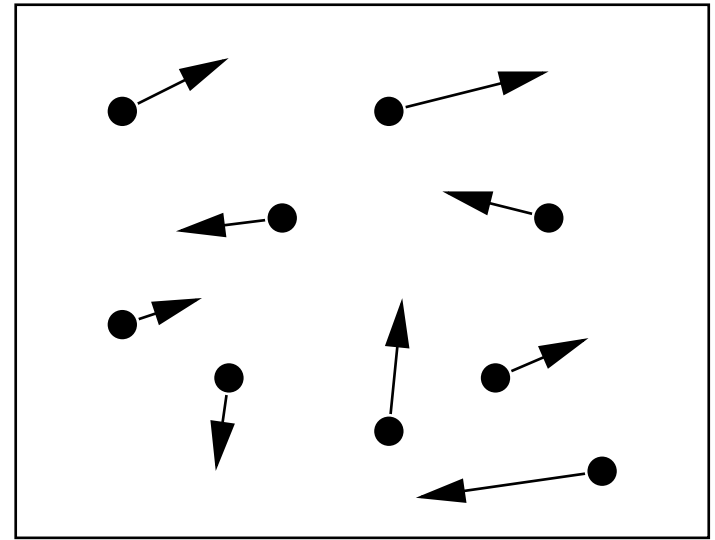
Viscosity bound: Common fluids



Kinetics vs No-Kinetics



AdS/CFT low viscosity goo



pQCD kinetic plasma

Effective theories for fluids (Here: Weak coupling QCD)



$$\mathcal{L} = \bar{q}_f (i\not{D} - m_f) q_f - \frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a$$

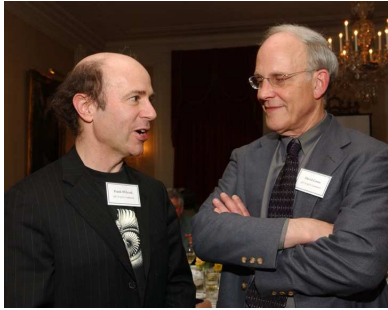


$$\frac{\partial f_p}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f_p = C[f_p] \quad (\omega < T)$$



$$\frac{\partial}{\partial t} (\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0 \quad (\omega < g^4 T)$$

Effective theories (Strong coupling)



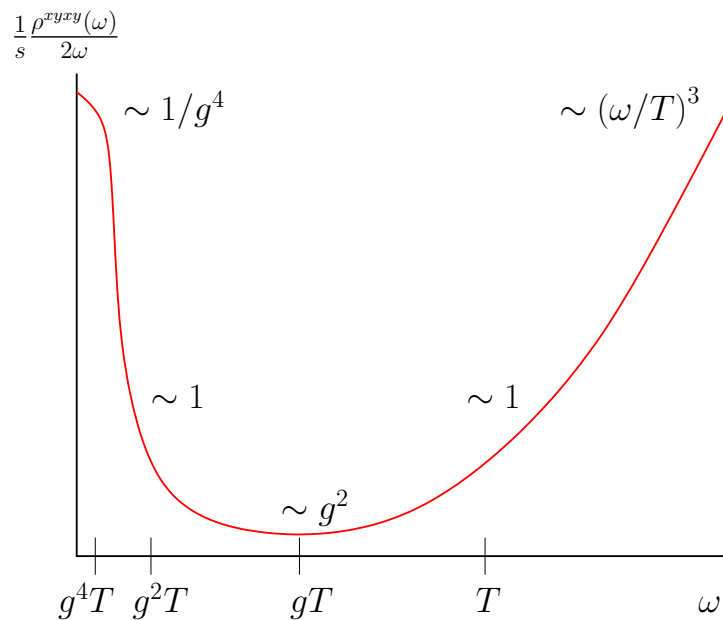
$$\mathcal{L} = \bar{\lambda}(i\sigma \cdot D)\lambda - \frac{1}{4}G_{\mu\nu}^a G_{\mu\nu}^a + \dots \Leftrightarrow S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \mathcal{R} + \dots$$



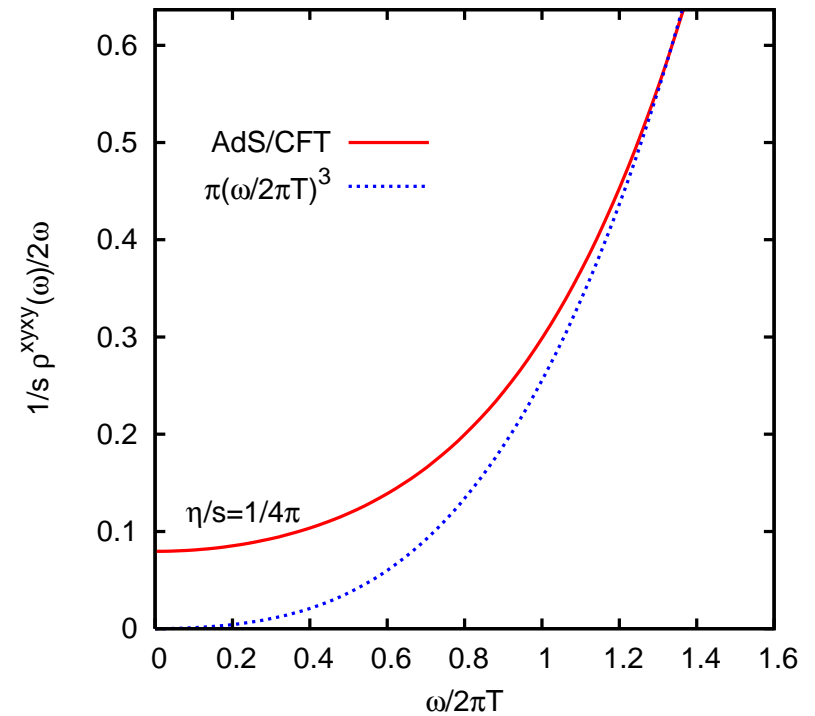
$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0 \quad (\omega < T)$$

Kinetics vs No-Kinetics

Spectral function $\rho(\omega) = \text{Im}G_R(\omega, 0)$ associated with T_{xy}



weak coupling QCD



strong coupling AdS/CFT

transport peak vs no transport peak

Transport coefficients, theory

1. Kinetic theory
2. Kubo formula, lattice
3. Dynamic universality
4. Holography

Kinetic theory

Quasi-Particles ($\gamma \ll \omega$): introduce distribution function $f_p(x, t)$

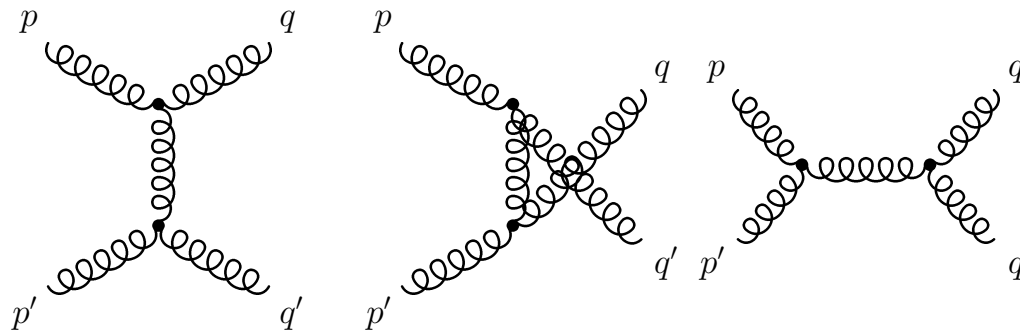
$$N = \int d^3p f_p \quad T_{ij} = \int d^3p p_i p_j f_p,$$

Boltzmann equation

$$\frac{\partial f_p}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f_p + \vec{F} \cdot \vec{\nabla}_p f_p = C[f_p]$$

Collision term $C[f_p] = C_{gain} - C_{loss}$

$$C_{loss} = \int dp' dq dq' f_p f_{p'} w(p, p'; q, q') \quad C_{gain} = \dots$$

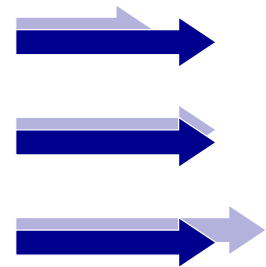


Linearized theory (Chapman-Enskog): $f_p = f_p^0(1 + \chi_p/T)$

$$RHS = C[f_p] \equiv \frac{f_p^0}{T} C_p[\chi_p] \quad \text{linear collision operator}$$

Linear response to flow gradient

$$f_p = \exp(-(E_p - \vec{p} \cdot \vec{v}(x))/(kT))$$



Drift term proportional to “driving term” ($v_{ij} = \partial_i v_j + \partial_j v_i - \text{trace}$)

$$LHS = \frac{\partial f_p}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f_p \equiv \frac{f_p^0}{T} X \quad X \equiv p_i p_j v_{ij}$$

Boltzmann equation

$$C_p[\chi_p] = X \quad \chi_p \equiv g_p p_i p_j v_{ij} \equiv (\chi_p)_{ij} v_{ij}$$

compute $T_{ij}[f_p^0 + \delta f_p] \equiv T_{ij}^0 + \eta v_{ij}$

$$\eta \sim \langle X | \chi \rangle \quad \langle X | \chi \rangle = \int d^3 p f_p^0 (p_i p_j \chi_p^{ij})$$

Use Boltzmann equation $C_p[\chi_p] = X$: $\eta \sim \langle \chi | C_p | \chi \rangle$

Variational principle

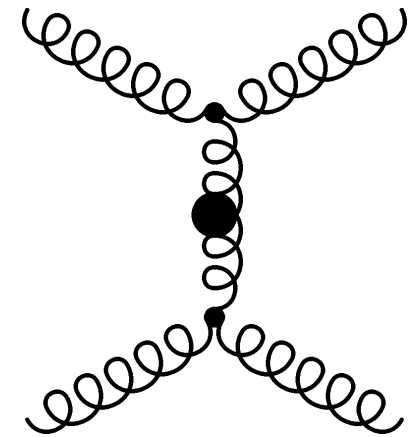
$$\langle \chi_{var} | C_p | \chi_{var} \rangle \langle \chi | C_p | \chi \rangle \geq \langle \chi_{var} | C_p | \chi \rangle^2 = \langle \chi_{var} | X \rangle^2$$

$$\eta \geq \frac{\langle \chi_{var} | X \rangle^2}{\langle \chi_{var} | C_p | \chi_{var} \rangle}$$

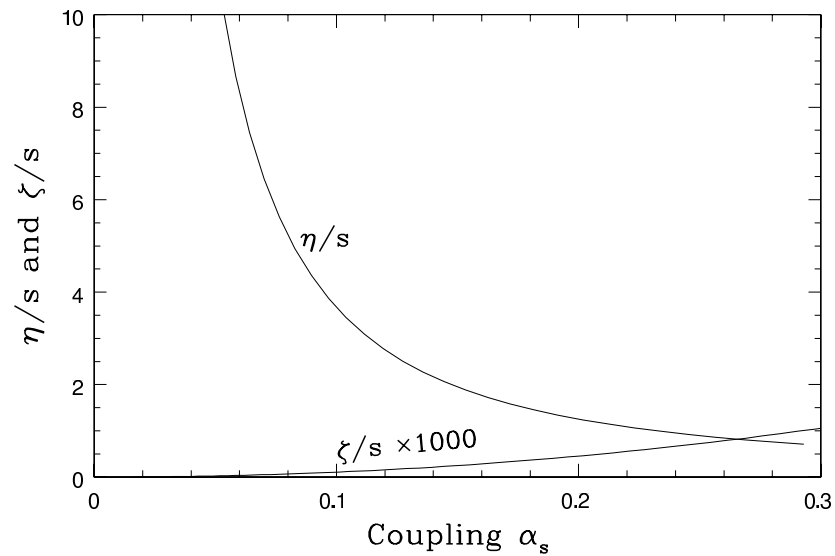
Best bound for $g_p \sim p^\alpha$ ($\alpha \simeq 0.1$)

$$\eta = \frac{0.34 T^3}{\alpha_s^2 \log(1/\alpha_s)}$$

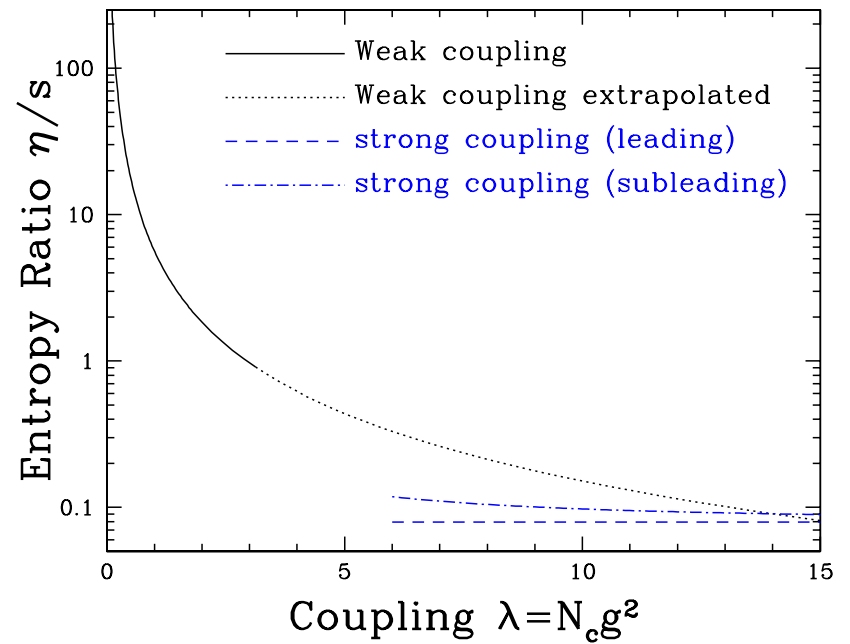
$\log(\alpha_s)$ from dynamic screening



pQCD and pSYM: weak versus strong coupling



Arnold, Dogan, Moore (2006)



Huot, Jeon, Moore (2006)

Kinetic Theory: Quasiparticles

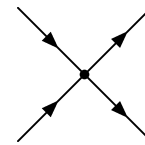
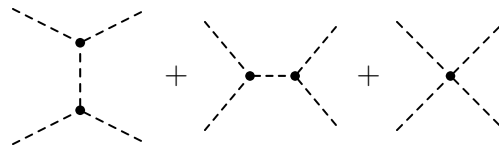
low temperature

high temperature

unitary gas

phonons

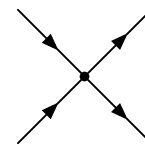
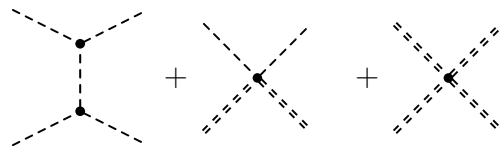
atoms



helium

phonons, rotons

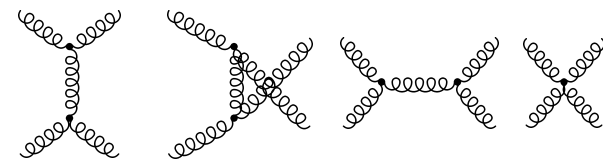
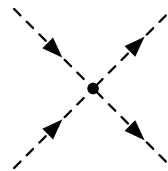
atoms



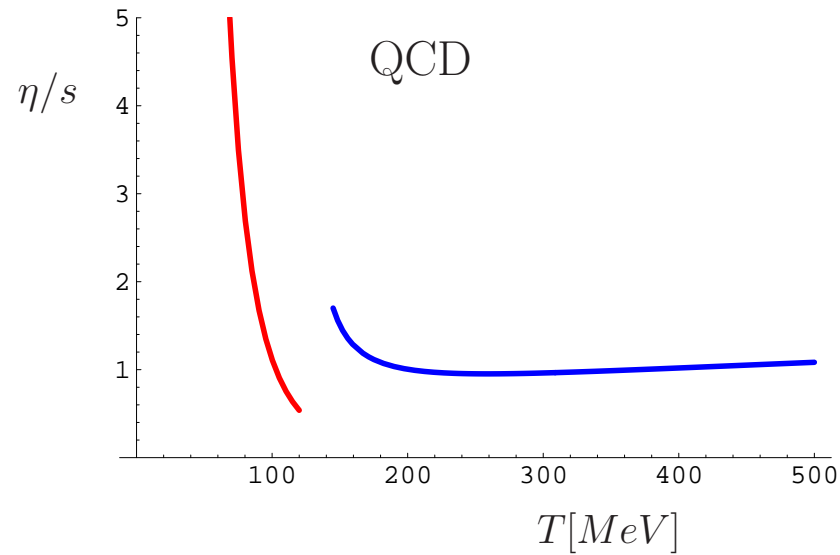
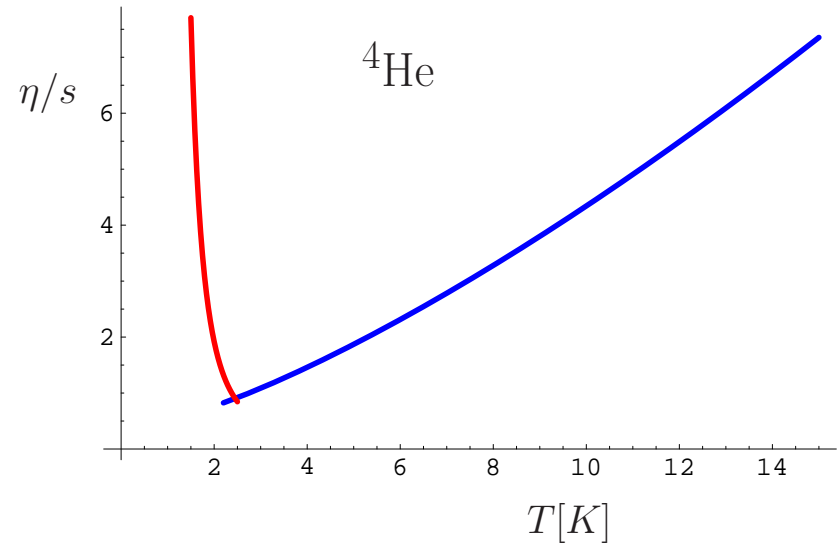
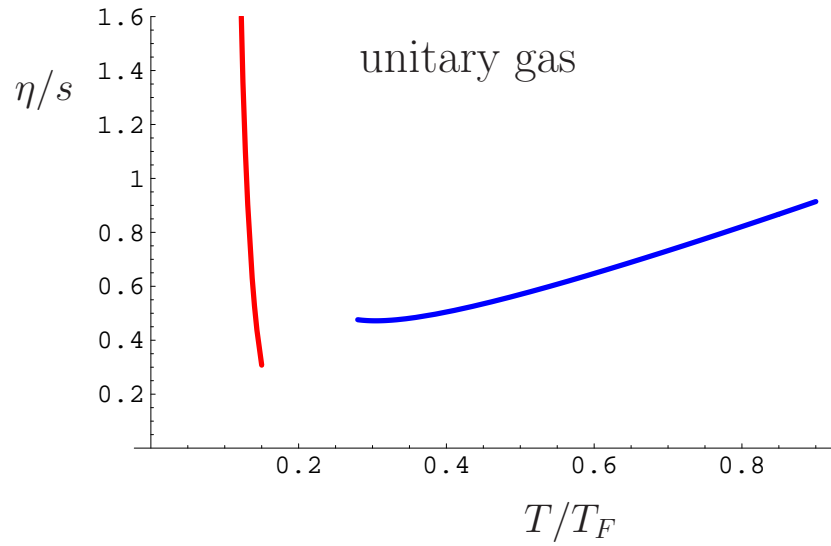
QCD

pions

quarks, gluons



Theory Summary



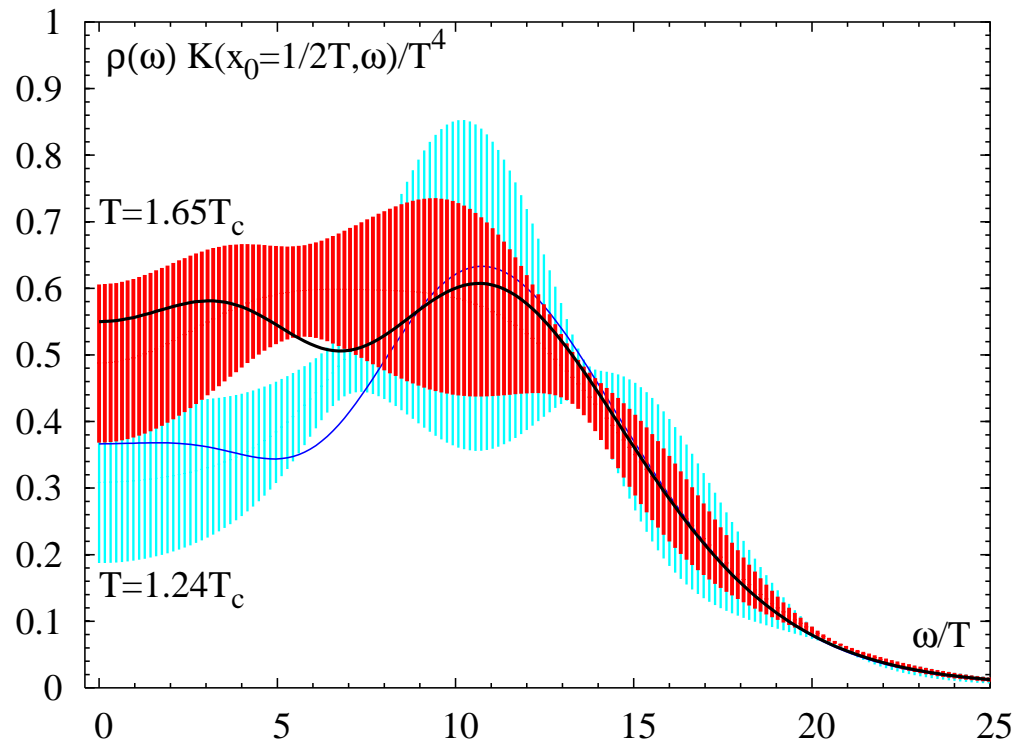
What if the coupling is strong? Kubo Formula

Linear response theory provides relation between transport coefficients and Green functions

$$G_R(\omega, 0) = \int dt d^3x e^{i\omega t} \Theta(t) \langle [T_{xy}(t, x), T_{xy}(0, 0)] \rangle$$

$$\eta = - \lim_{\omega \rightarrow 0} \frac{1}{\omega} G_R(\omega, 0)$$

This result is hard to use for quantum fluids, but there are some heroic efforts by lattice QCD theorists, e.g. Meyer (2007).



T	$1.02 T_c$	$1.24 T_c$	$1.65 T_c$
η/s		0.102(56)	0.134(33)
ζ/s	0.73(3)	0.065(17)	0.008(7)

Dynamic Universality

Continuous phase transition: Dynamics of low energy modes universal

Universality for transport coefficients

Universal theory: Hydro (diffusive modes), order parameters (time dependent LG), stochastic forces (Langevin)

$$\frac{\partial}{\partial t}(\rho v_i) = P_{ij}^\perp \left[\eta_0 \nabla^2 \frac{\delta \mathcal{H}}{\delta(\rho v_j)} + w_0 (\nabla_j \phi) \frac{\delta \mathcal{H}}{\delta \phi} + \zeta_j \right]$$

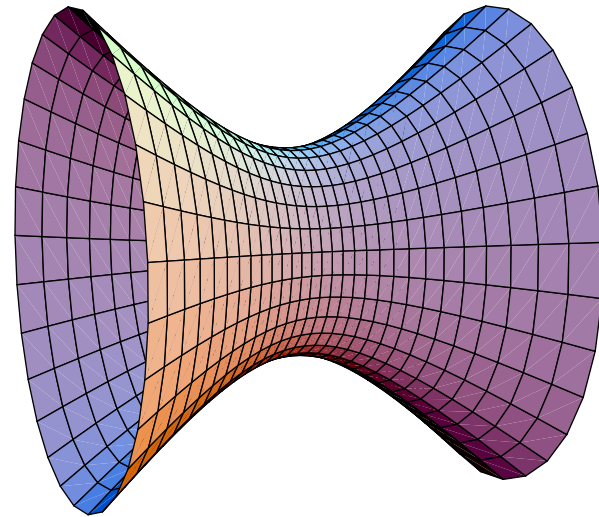
Model H of Hohenberg and Halperin

$$\eta \sim \xi^{x_\eta} \quad (x_\eta \simeq 0.06) \qquad \zeta \sim \xi^{x_\zeta} \quad (x_\zeta \simeq 2.8)$$

Anti-DeSitter Space

Consider a hyperboloid embedded in
6-d euclidean space

$$-R^2 = \sum_{i=1,4} x_i^2 - x_0^2 - x_5^2$$



This is a space of constant negative curvature, and a solution of the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{2}g_{\mu\nu}\Lambda$$

with negative cosmological constant.

Isometries of AdS_5 : $SO(4, 2) \equiv$ conformal group in $d = 3 + 1$

metric of $\text{AdS}_5 \times \text{S}_5$ (note that $(L/\ell_s)^4 = g^2 N_c$)

$$ds^2 = \frac{r^2}{L^2} (-dt^2 + d\mathbf{x}^2) + \frac{L^2}{r^2} dr^2 + L^2 d\Omega_5^2$$

$r \rightarrow \infty$ “boundary” of AdS_5

Finite temperature: AdS_5 black hole solution

$$ds^2 = \frac{r^2}{L^2} (-f(r)dt^2 + d\mathbf{x}^2) + \frac{L^2}{f(r)r^2} dr^2,$$

where $f(r) = 1 - (r_0/r)^4$. Hawking temperature $T_H = r_0/\pi$.

Compute induced stress tensor on the boundary

$$\langle T_{\mu\nu} \rangle = \lim_{\epsilon \rightarrow 0} \frac{-2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} \Big|_{bd}$$

Find ideal fluid with

$$\langle T_{\mu\nu} \rangle = \text{diag}(\epsilon, P, P, P), \quad \frac{\epsilon}{3} = P = \frac{N_c^2}{8\pi^2} (\pi T)^4$$

Hydrodynamics from AdS/CFT

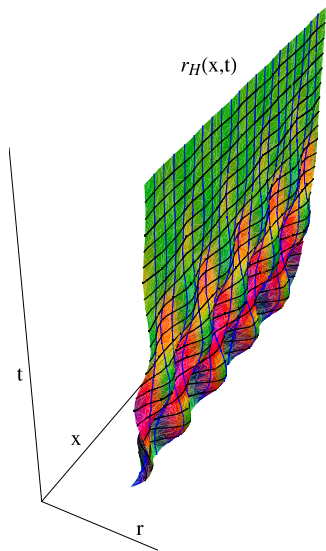
Eddington-Finkelstein coordinates

$$ds^2 = 2 dv dr + \frac{r^2}{L^2} [-f(r)dv^2 + r^2 d\mathbf{x}^2]$$

Introduce local rest frame $u^\mu = (1, \mathbf{0})$, scale parameter b

$$ds^2 = -2u_\mu dx^\mu dr + \frac{r^2}{L^2} [-f(br)u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu]$$

$$P_{\mu\nu} = u_\mu u_\nu + \eta_{\mu\nu}$$



promote $u_\mu(x)$ and $b(x)$ to fields
determine metric order by order in gradients
compute induced stress

leading order: ideal fluid dynamics with $\epsilon = 3P$

$$T_0^{\mu\nu} = \frac{N_c^2}{8\pi^2} (\pi T)^4 (\eta^{\mu\nu} + 4u^\mu u^\nu)$$

next-to-leading order: Navier-Stokes with $\eta/s = 1/(4\pi)$

$$\delta^{(1)} T^{\mu\nu} = -\frac{N_c^2}{8\pi^2} (\pi T)^3 \sigma^{\mu\nu}$$

next-to-next-to-leading order: second order conformal hydro

$$\begin{aligned} \delta^{(2)} T^{\mu\nu} = & \eta\tau_{II} \left[\langle D\sigma^{\mu\nu} \rangle + \frac{1}{3} \sigma^{\mu\nu} (\partial \cdot u) \right] \\ & + \lambda_1 \sigma^{\langle\mu}{}_\lambda \sigma^{\nu\rangle\lambda} + \lambda_2 \sigma^{\langle\mu}{}_\lambda \Omega^{\nu\rangle\lambda} + \lambda_3 \Omega^{\langle\mu}{}_\lambda \Omega^{\nu\rangle\lambda} \end{aligned}$$

relaxation times

$$\tau_{\Pi} = \frac{2 - \ln 2}{\pi T} \quad \lambda_1 = \frac{2\eta}{\pi T} \quad \lambda_2 = \frac{2\eta \ln 2}{\pi T} \quad \lambda_3 = 0$$

Summary

Hydrodynamics is a universal long distance theory (not just a theory of liquids)

Hydro is both more and less than kinetics: Hydro is more because it is universal, whereas kinetic theory is based on extra assumptions. Hydro is less because, if applicable, it contains extra information.

Kinetics is not (completely) wrong even if it is not applicable. This is because the long distance limit of kinetics is hydro.

AdS/CFT is an example of a theory with a hydro limit, but no kinetic description.