Strongly interacting quantum fluids:

Transport theory

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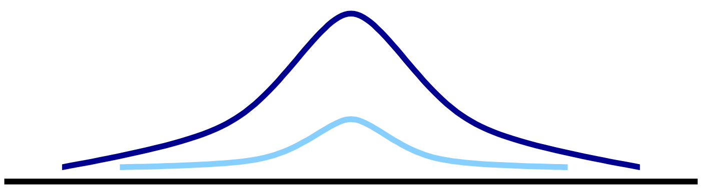
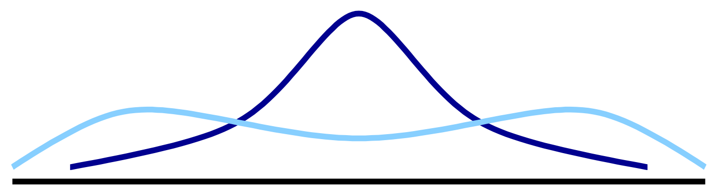
Fluids: Gases, Liquids, Plasmas, . . .

Hydrodynamics: Long-wavelength, low-frequency dynamics of conserved or spontaneously broken symmetry variables.



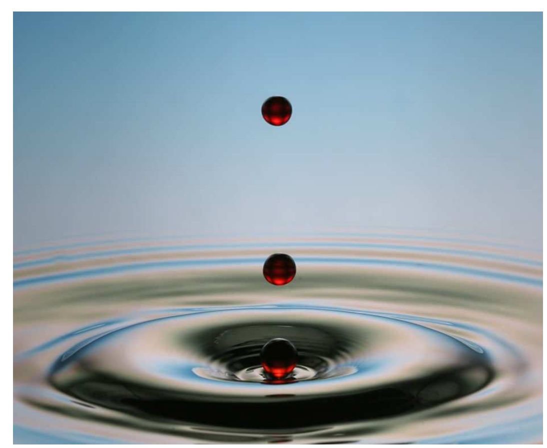
$\tau \sim \tau_{micro}$



$\tau \sim \lambda$

Historically: Water

$(\rho, \epsilon, \vec{\pi})$



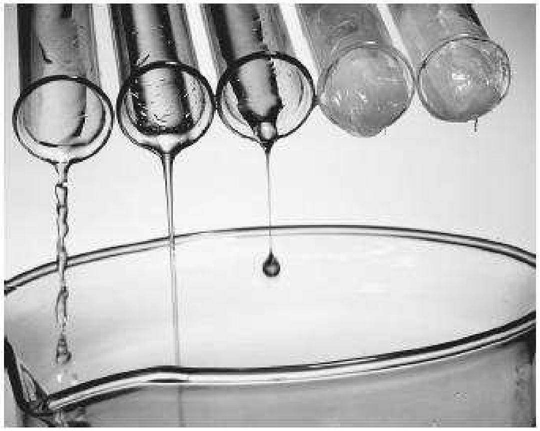
Simple non-relativistic fluid

Simple fluid: Conservation laws for mass, energy, momentum

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0$$

$$\frac{\partial \epsilon}{\partial t} + \vec{\nabla}\vec{j}^{\,\epsilon} = 0$$

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}\Pi_{ij} = 0$$



Constitutive relations: Energy momentum tensor

$$\Pi_{ij} = P\delta_{ij} + \rho v_i v_j + \eta\left(\partial_i v_j + \partial_j v_i - \frac{2}{3}\delta_{ij}\partial_k v_k\right) + O(\partial^2)$$

reactive dissipative 2nd order

Expansion $\Pi^0_{ij} \gg \delta\Pi^1_{ij} \gg \delta\Pi^2_{ij}$

Regime of applicability

Expansion parameter $Re^{-1} = \dfrac{\eta(\partial v)}{\rho v^2} = \dfrac{\eta}{\rho L v} \ll 1$

$$Re \quad = \quad \underset{\substack{\text{fluid}\\ \text{property}}}{\frac{\hbar n}{\eta}} \quad \times \quad \underset{\substack{\text{flow}\\ \text{property}}}{\frac{m v L}{\hbar}}$$

Kinetic theory estimate: $\eta \sim n p l_{mfp}$

$$Re^{-1} = \frac{v}{c_s} Kn \qquad\qquad Kn = \frac{l_{mfp}}{L}$$

expansion parameter $Kn \ll 1$

Relativistic hydrodynamics

Energy momentum tensor of an ideal fluid

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P\eta^{\mu\nu} \ ,$$

Energy-momentum conservation: $\partial_\mu T^{\mu\nu} = 0$

$$\partial_\mu(su^\mu) = 0 \qquad\qquad Du_\mu = -\frac{1}{\epsilon + P}\nabla_\mu^\perp P$$

$$D = u \cdot \partial \qquad\qquad \nabla_\mu^\perp = \Delta_{\mu\nu}\partial^\nu \qquad\qquad \Delta_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu$$

Viscous contribution $\delta^{(1)}T^{\mu\nu} = -\eta\sigma^{\mu\nu} - \zeta\Delta^{\mu\nu}\partial \cdot u$

$$\sigma^{\mu\nu} = \Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3}\eta_{\alpha\beta}\partial \cdot u\right) \ .$$

Relativistic hydrodynamics: causality

Linearized hydro: consider small fluctuations $g^i = \delta T^{0i}$

$$S_{gg}^L = 2sT\,\frac{\Gamma_s \omega^2 \mathbf{k}^2}{(\omega^2 - c_s^2\mathbf{k}^2)^2 + (\Gamma_s \omega \mathbf{k}^2)^2} \qquad \Gamma_s = \frac{\frac{4}{3}\eta + \zeta}{sT}$$

$$S_{gg}^T = \frac{2\eta \mathbf{k}^2}{\omega^2 + (\frac{\eta}{sT}\mathbf{k}^2)^2}$$

L/T channel: propagating sound mode, diffusive shear mode.

Consider "speed" of shear wave

$$v_{diff} = \frac{\partial|\omega|}{\partial k} = \frac{\eta}{sT}\,k$$

Find acausal* behavior $v_{diff} > c$ for $k > k_{cr}$.

* Occurs outside regime of validity of hydro. But: Causes numerical difficulties.

Second order hydrodynamics

Causality can be restored by introducing a finite relaxation time

$$\eta(\omega) \simeq \frac{\eta}{1 + i\omega\tau_\pi}$$

More formal approach: Second order hydrodynamics (BRSSS)

$$\begin{aligned} \delta^{(2)} T^{\mu\nu} \;&=\; \eta\tau_{II}\left[{}^{\langle} D\sigma^{\mu\nu\rangle} + \frac{1}{3}\sigma^{\mu\nu}(\partial\cdot u)\right] \\ &\quad + \lambda_1 \sigma^{\langle\mu}{}_{\lambda}\sigma^{\nu\rangle\lambda} + \lambda_2 \sigma^{\langle\mu}{}_{\lambda}\Omega^{\nu\rangle\lambda} + \lambda_3 \Omega^{\langle\mu}{}_{\lambda}\Omega^{\nu\rangle\lambda} \end{aligned}$$

$$A^{\langle\mu\nu\rangle} = \tfrac{1}{2}\Delta^{\mu\alpha}\Delta^{\nu\beta}\left(A_{\alpha\beta} + A_{\beta\alpha} - \tfrac{2}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}A_{\alpha\beta}\right) \quad \Omega^{\mu\nu} = \tfrac{1}{2}\Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\partial_\alpha u_\beta - \partial_\beta u_\alpha\right)$$

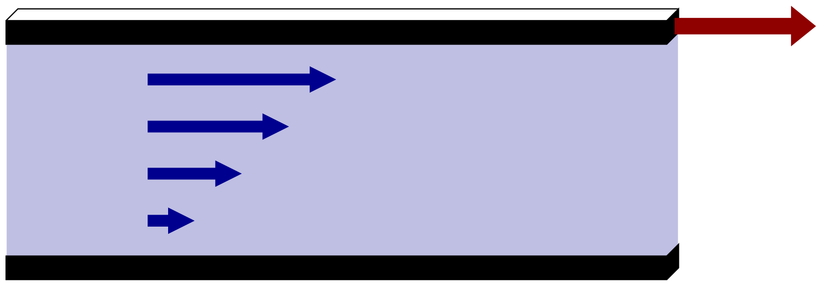
Contains four new transport coefficients τ_{II}, λ_i

Can be written as a relaxation equation for $\pi^{\mu\nu} \equiv \delta T^{\mu\nu}$

$$\pi^{\mu\nu} = -\eta\sigma^{\mu\nu} - \tau_{II}\, {}^{\langle} D\pi^{\mu\nu\rangle} + \ldots$$

Shear viscosity

Viscosity determines shear stress ("friction") in fluid flow

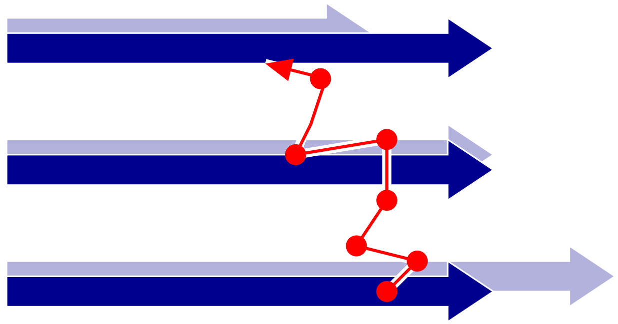


$$F = A\,\eta\,\frac{\partial v_x}{\partial y}$$

Kinetic theory: conserved quantities carried by quasi-particles

$$\frac{\partial f_p}{\partial t} + \vec{v}\cdot\vec{\nabla}_x f_p + \vec{F}\cdot\vec{\nabla}_p f_p = C[f_p]$$

$$\eta \sim \frac{1}{3} n\,\bar{p}\,l_{mfp}$$



Dilute, weakly interacting gas: $l_{mfp} \sim 1/(n\sigma)$

$$\eta \sim \frac{1}{3}\frac{\bar{p}}{\sigma}$$

independent of density!

Shear viscosity

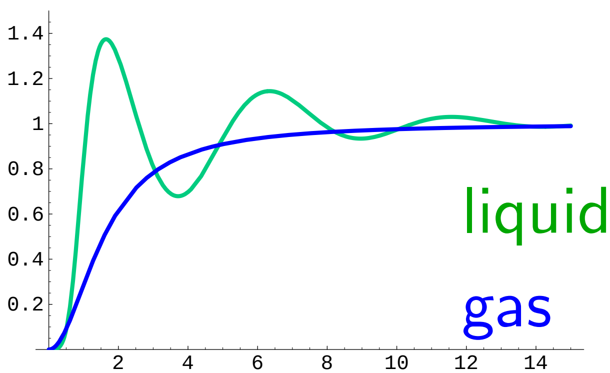
non-interacting gas ($\sigma \to 0$): $\eta \to \infty$

non-interacting and hydro limit ($T \to \infty$) limit do not commute

strongly interacting gas: $$\frac{\eta}{n} \sim \bar{p} l_{mfp} \geq \hbar$$

but: kinetic theory not reliable!

what happens if the gas condenses into a liquid?



Pair correlation function

Life at low Reynolds number

E. M. Purcell
Lyman Laboratory, Harvard University, Cambridge, Massachusetts 02138
(Received 12 June 1976)

Viscosity of a liquid is a dominant theme here, and you know Vicki's program of explaining everything in terms of fundamental constants. The viscosity of a liquid is a tough nut to crack ... because when the stuff is cooled by 40^o its viscosity can change by a factor of 10^6.

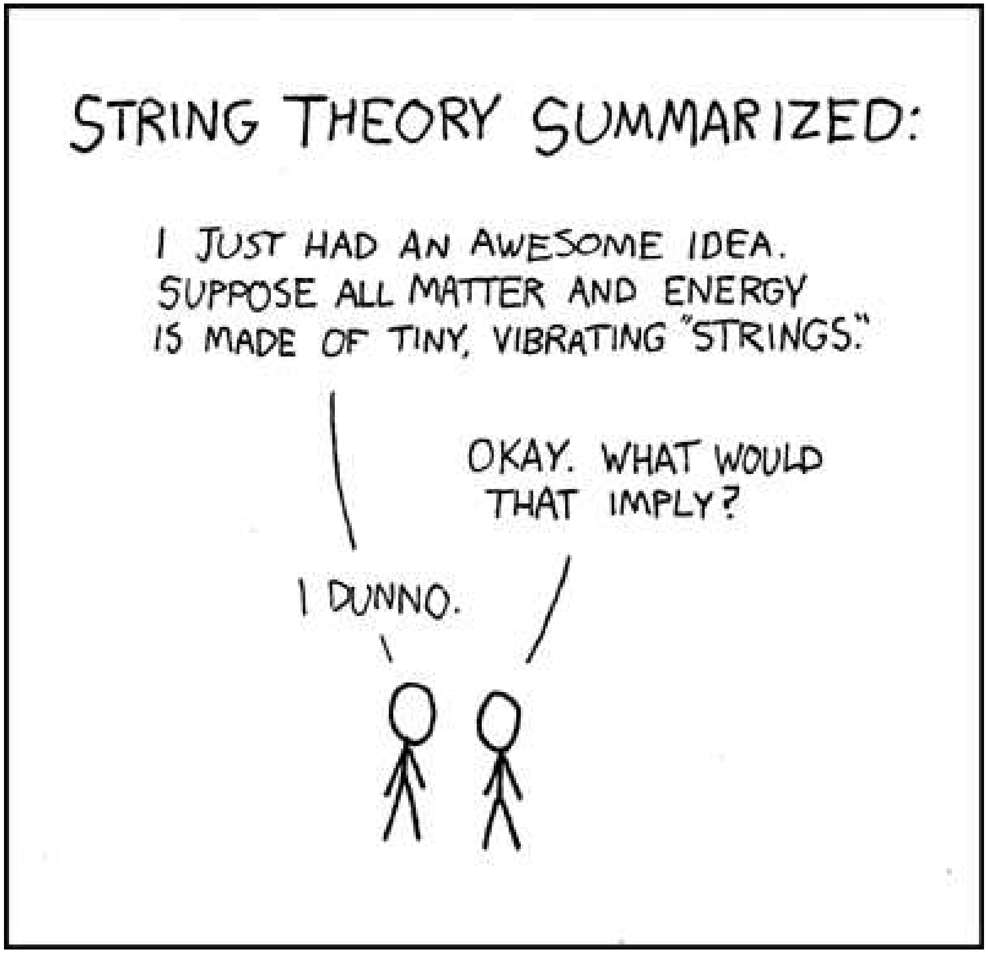
Purcell anticipates
Vicki's theory

$$\eta \sim \exp(E/T)$$

But it's more mysterious than that: The viscosities have a big range, but they all stop in the same place. I don't understand that.

Eyring, Frenkel: $$\eta \simeq hn \exp(E/T)$$

And now for something completely different . . .

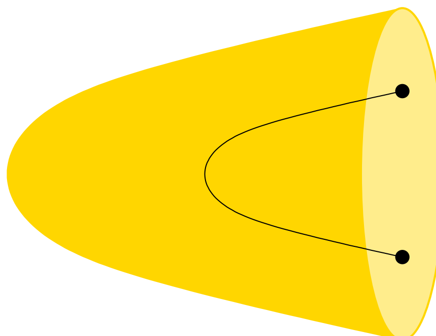


Gauge theory at strong coupling: Holographic duality

The AdS/CFT duality relates

large N_c (conformal) gauge theory in 4 dimensions	$\Leftrightarrow$	string theory on 5 dimensional Anti-de Sitter space $\times S_5$
correlation fcts of gauge invariant operators	$\Leftrightarrow$	boundary correlation fcts of AdS fields

$$\langle \exp \int dx \; \phi_0 \mathcal{O} \rangle = Z_{string}[\phi(\partial AdS) = \phi_0]$$



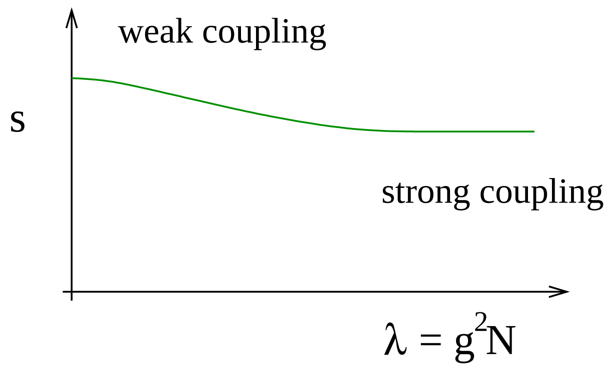
The correspondence is simplest at strong coupling $g^2 N_c$

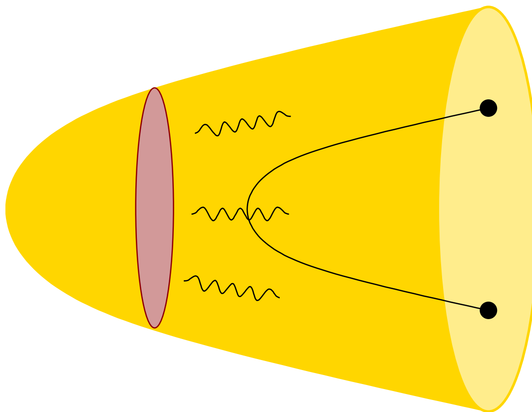
strongly coupled gauge theory $\Leftrightarrow$ classical string theory

Holographic duals at finite temperature

Thermal (conformal) field theory $\equiv AdS_5$ black hole

CFT temperature	$\Leftrightarrow$	Hawking temperature of black hole
CFT entropy	$\Leftrightarrow$	Hawking-Bekenstein entropy $\sim$ area of event horizon





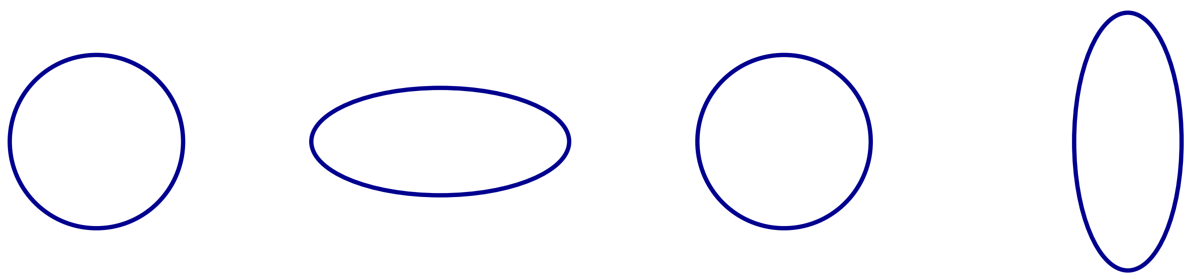
$$s(\lambda \to \infty) = \frac{\pi^2}{2} N_c^2 T^3 = \frac{3}{4} s(\lambda = 0)$$

Gubser and Klebanov

Holographic duals: Transport properties

Thermal (conformal) field theory $\equiv AdS_5$ black hole

$$T_{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} \qquad\qquad g_{\mu\nu} = g^{0}_{\mu\nu} + \gamma_{\mu\nu}$$



CFT entropy $\Leftrightarrow$ Hawking-Bekenstein entropy

$\sim$ area of event horizon

shear viscosity $\Leftrightarrow$ Graviton absorption cross section

$\sim$ area of event horizon

Holographic duals: Transport properties

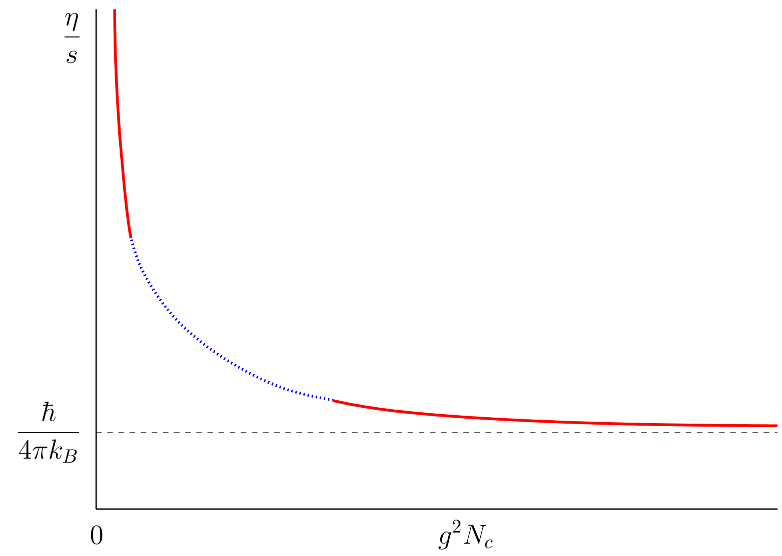
Thermal (conformal) field theory $\equiv AdS_5$ black hole

CFT entropy	$\Leftrightarrow$	Hawking-Bekenstein entropy $\sim$ area of event horizon
shear viscosity	$\Leftrightarrow$	Graviton absorption cross section $\sim$ area of event horizon

Strong coupling limit

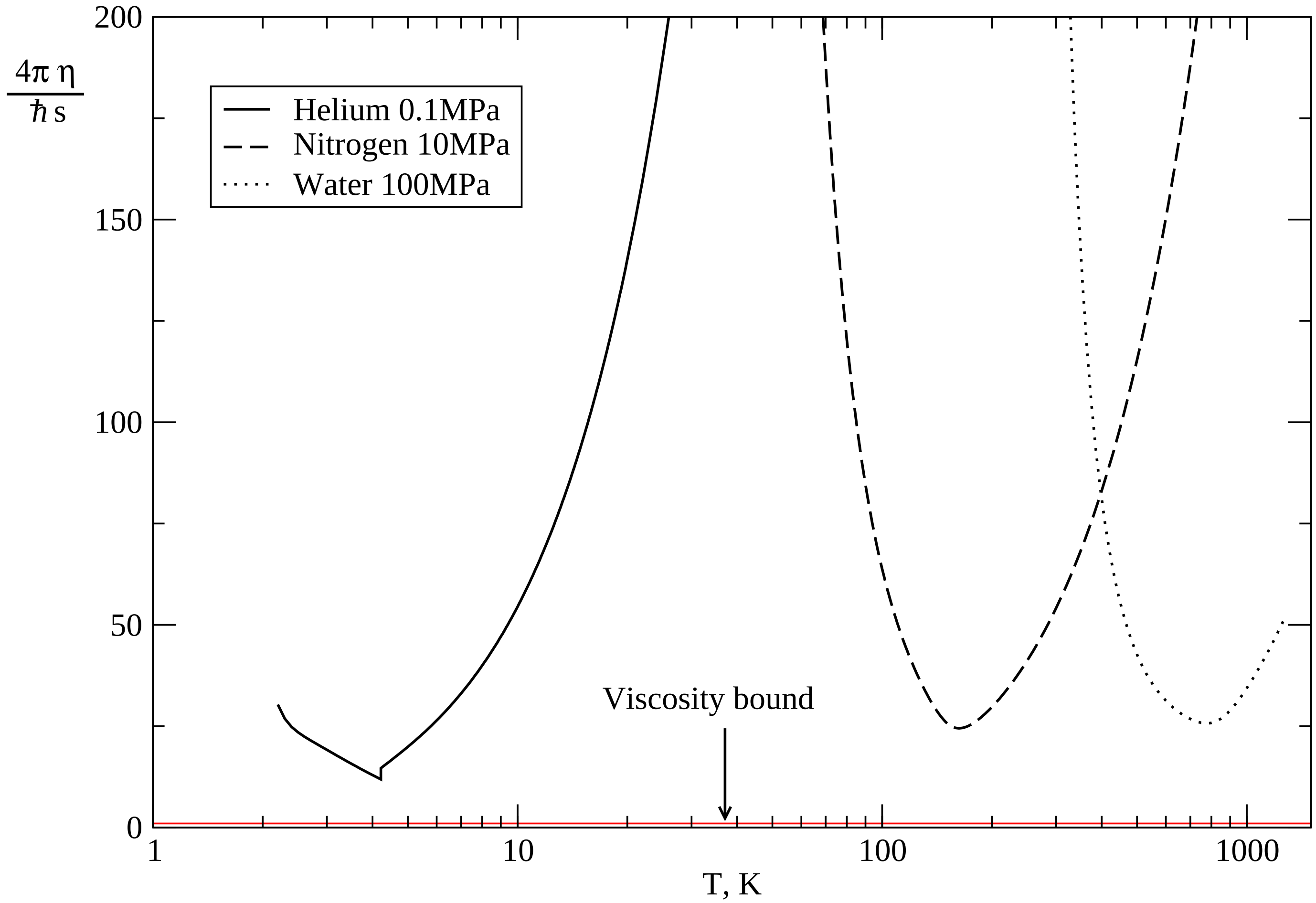
$$\frac{\eta}{s} = \frac{\hbar}{4\pi k_B}$$

Son and Starinets (2001)

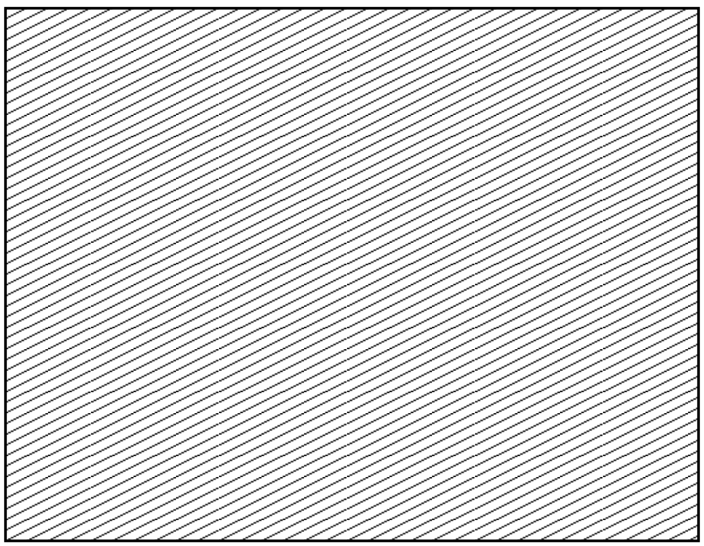


Strong coupling limit universal? Provides lower bound for all theories?

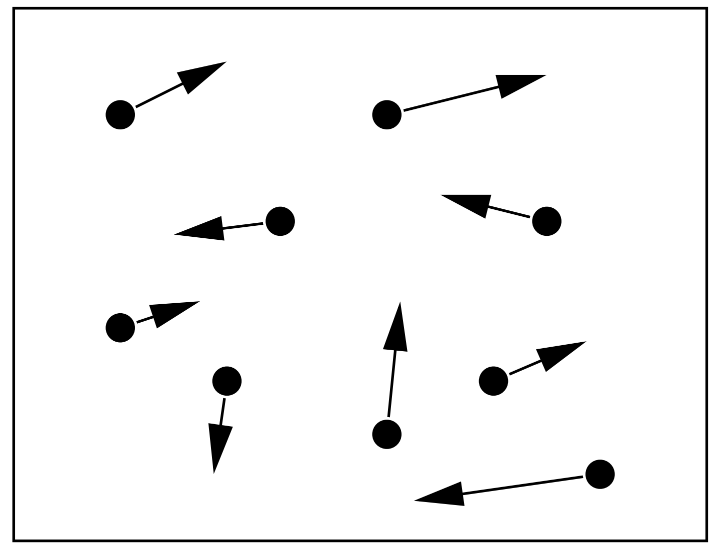
Viscosity bound: Common fluids



Kinetics vs No-Kinetics



AdS/CFT low viscosity goo



pQCD kinetic plasma

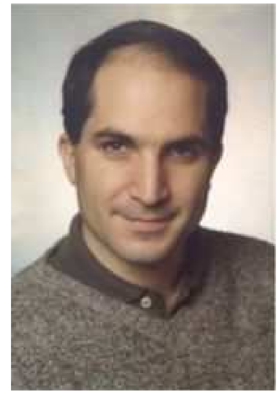
Effective theories for fluids (Here: Weak coupling QCD)

$$\mathcal{L} = \bar{q}_f (i\not{D} - m_f) q_f - \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu}$$

$$\frac{\partial f_p}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f_p = C[f_p] \qquad (\omega < T)$$

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0 \qquad (\omega < g^4 T)$$

Effective theories (Strong coupling)

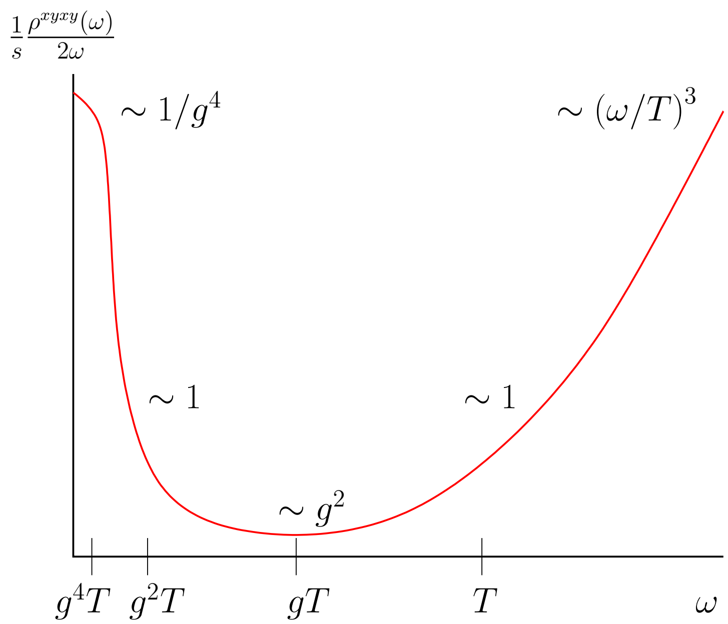


$$\mathcal{L} = \bar{\lambda}(i\sigma \cdot D)\lambda - \frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} + \ldots \Leftrightarrow S = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}\mathcal{R} + \ldots$$

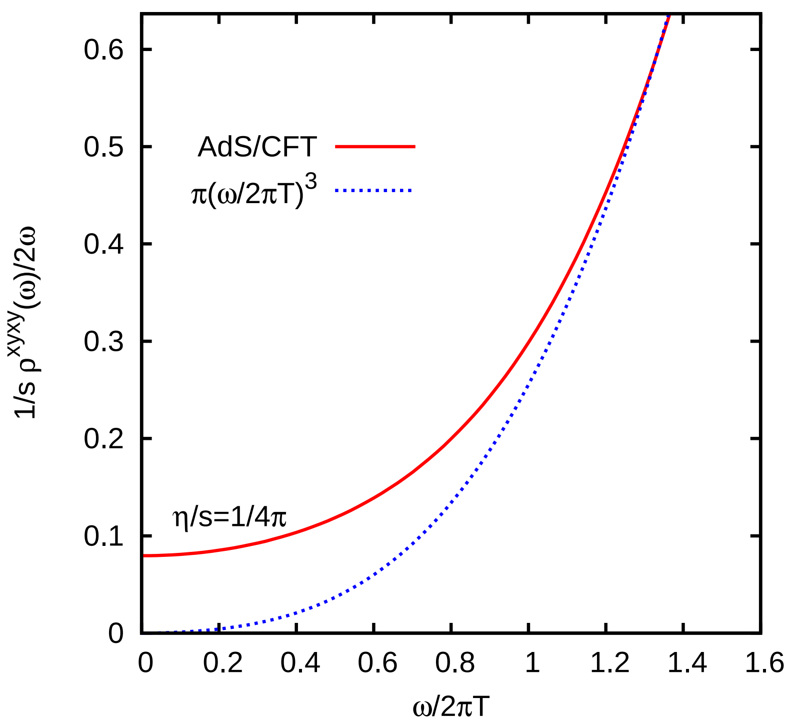
$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}\Pi_{ij} = 0 \quad (\omega < T)$$

Kinetics vs No-Kinetics

Spectral function $\rho(\omega) = \mathrm{Im}G_R(\omega, 0)$ associated with T_{xy}



weak coupling QCD



strong coupling AdS/CFT

transport peak vs no transport peak

Transport coefficients, theory

1. Kinetic theory
2. Kubo formula, lattice
3. Dynamic universality
4. Holography

Kinetic theory

Quasi-Particles ($\gamma \ll \omega$): introduce distribution function $f_p(x,t)$

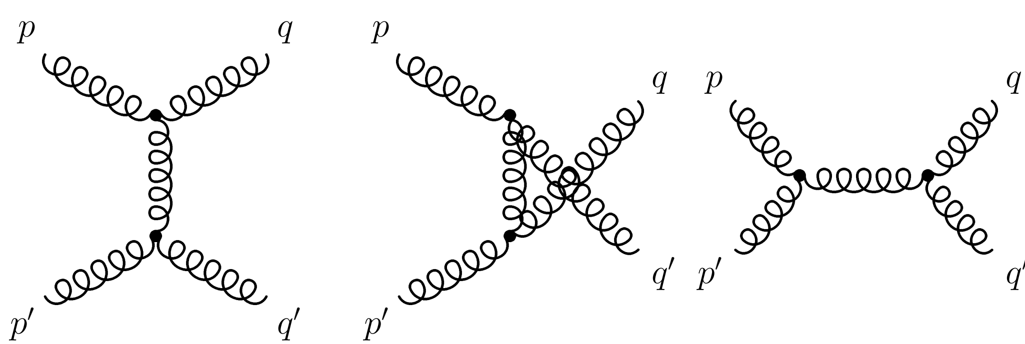
$$N = \int d^3p\, f_p \qquad T_{ij} = \int d^3p\, p_i p_j f_p,$$

Boltzmann equation

$$\frac{\partial f_p}{\partial t} + \vec{v}\cdot\vec{\nabla}_x f_p + \vec{F}\cdot\vec{\nabla}_p f_p = C[f_p]$$

Collision term $C[f_p] = C_{gain} - C_{loss}$

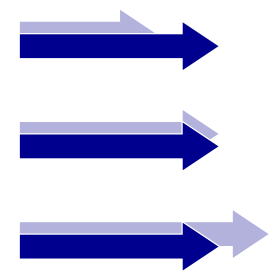
$$C_{loss} = \int dp'dqdq'\; f_p f_{p'} w(p,p';q,q') \qquad C_{gain} = \ldots$$



Linearized theory (Chapman-Enskog): $f_p = f_p^0(1 + \chi_p/T)$

$$RHS = C[f_p] \equiv \frac{f_p^0}{T} C_p \chi_p \qquad \text{linear collision operator}$$

Linear response to flow gradient



$$f_p = \exp(-(E_p - \vec{p} \cdot \vec{v}(x))/(kT))$$

Drift term proportional to "driving term" $(v_{ij} = \partial_i v_j + \partial_j v_i - trace)$

$$LHS = \frac{\partial f_p}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f_p \equiv \frac{f_p^0}{T} X \qquad X \equiv p_i p_j v_{ij}$$

Boltzmann equation

$$C_p \chi_p = X \qquad \chi_p \equiv g_p p_i p_j v_{ij} \equiv (\chi_p)_{ij} v_{ij}$$

compute $T_{ij}[f_p^0 + \delta f_p] \equiv T_{ij}^0 + \eta v_{ij}$

$$\eta \sim \langle X|\chi\rangle \qquad \langle X|\chi\rangle = \int d^3p\, f_p^0\, (p_i p_j \chi_p^{ij})$$

Use Boltzmann equation $C_p \chi_p = X$: $\quad \eta \sim \langle \chi|C_p|\chi\rangle$

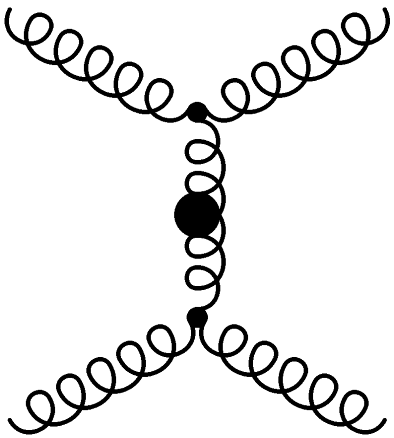
Variational principle

$$\langle \chi_{var}|C_p|\chi_{var}\rangle\langle \chi|C_p|\chi\rangle \geq \langle \chi_{var}|C_p|\chi\rangle^2 = \langle \chi_{var}|X\rangle^2$$

$$\eta \geq \frac{\langle \chi_{var}|X\rangle^2}{\langle \chi_{var}|C|\chi_{var}\rangle}$$

Best bound for $g_p \sim p^\alpha$ $(\alpha \simeq 0.1)$

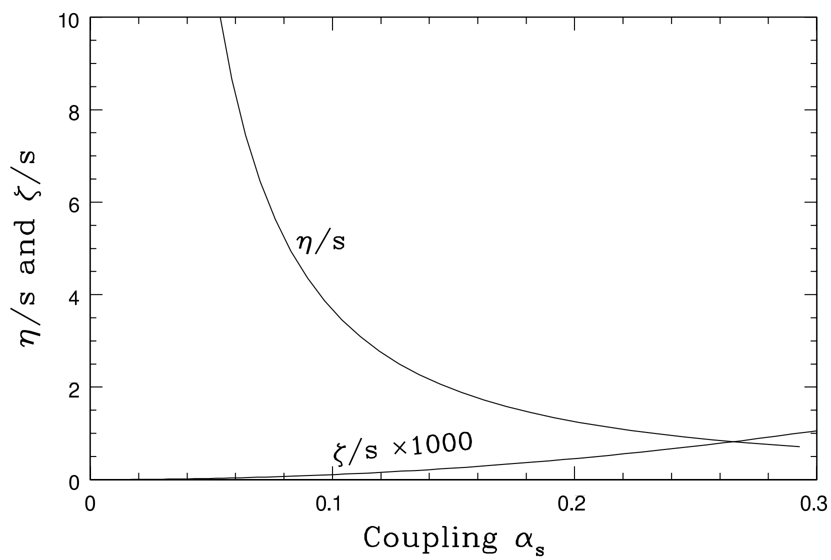
$$\eta = \frac{0.34T^3}{\alpha_s^2 \log(1/\alpha_s)}$$



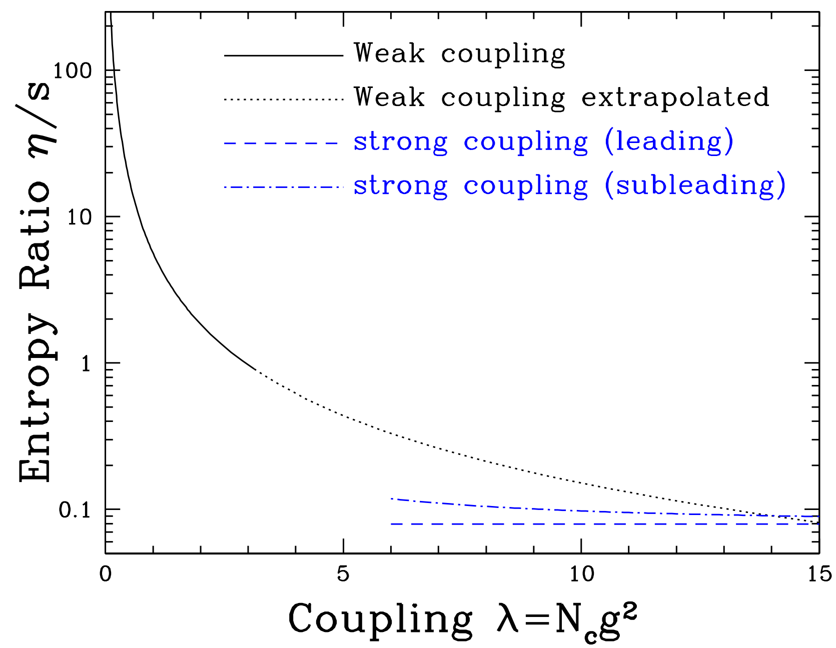
$\log(\alpha_s)$ from dynamic screening

Baym et al. (1990)

pQCD and pSYM: weak versus strong coupling

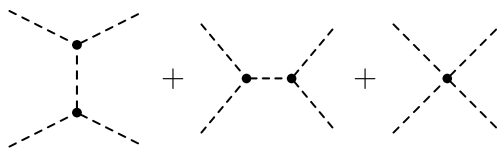
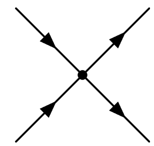
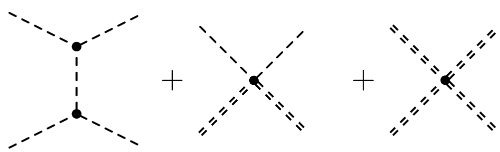
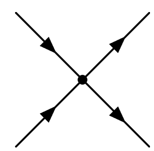
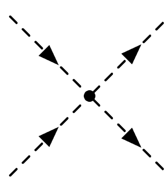
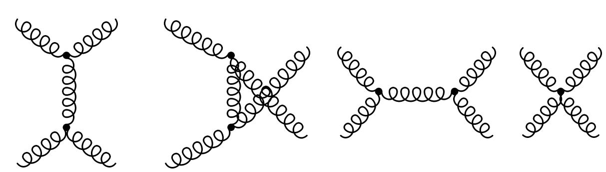


Arnold, Dogan, Moore (2006)

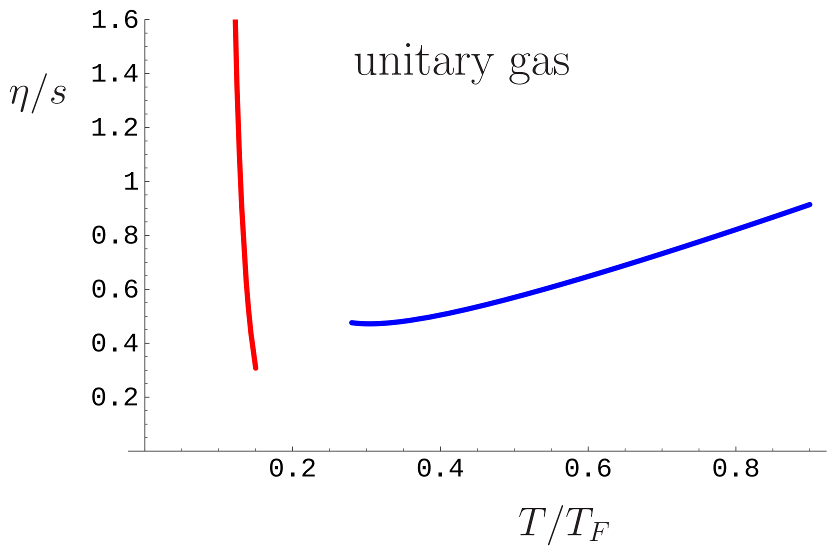


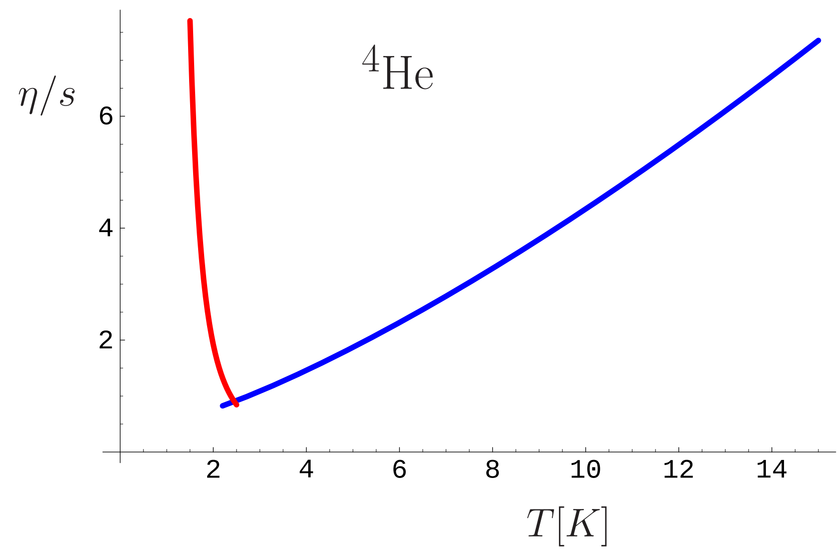
Huot, Jeon, Moore (2006)

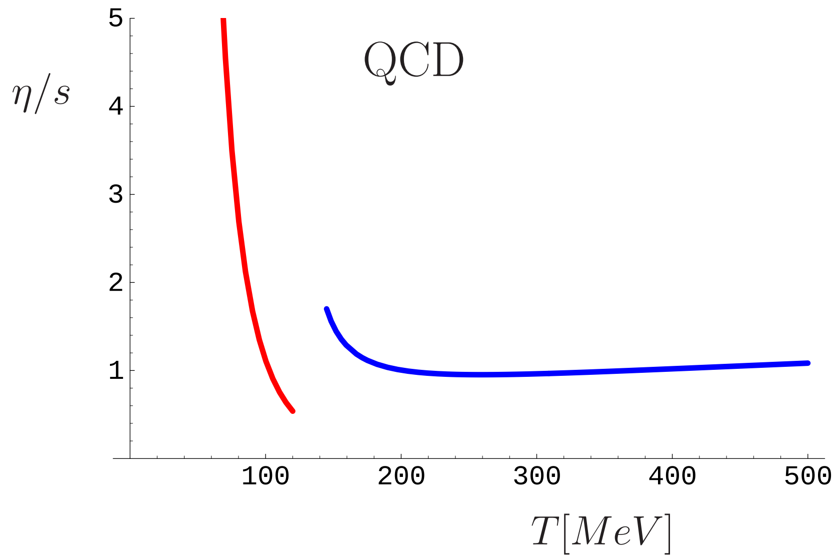
Kinetic Theory: Quasiparticles

	low temperature	high temperature
unitary gas	phonons	atoms
		
helium	phonons, rotons	atoms
		
QCD	pions	quarks, gluons
		

Theory Summary







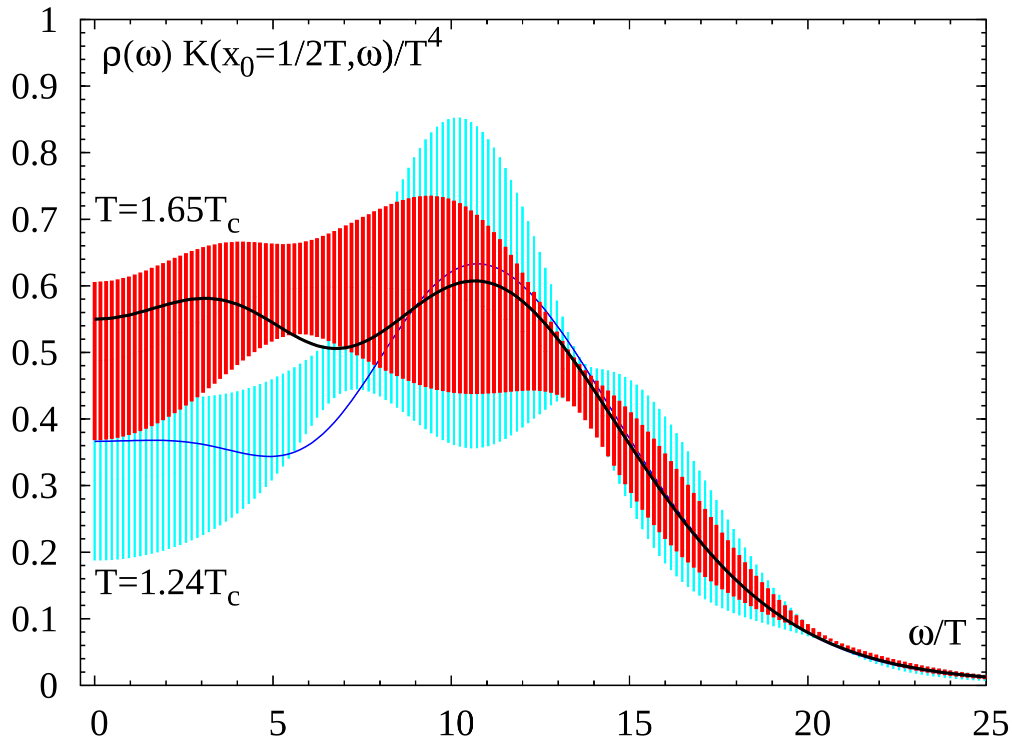
What if the coupling is strong? Kubo Formula

Linear response theory provides relation between transport coefficients and Green functions

$$G_R(\omega, 0) = \int dt\, d^3x\, e^{i\omega t}\Theta(t)\langle[T_{xy}(t,x), T_{xy}(0,0)]\rangle$$

$$\eta = -\lim_{\omega\to 0}\frac{1}{\omega}G_R(\omega, 0)$$

This result is hard to use for quantum fluids, but there are some heroic efforts by lattice QCD theorists, e.g. Meyer (2007).



T	$1.02\ T_c$	$1.24\ T_c$	$1.65\ T_c$
η/s		0.102(56)	0.134(33)
ζ/s	0.73(3)	0.065(17)	0.008(7)

Dynamic Universality

Continuous phase transition: Dynamics of low energy modes universal

Universality for transport coefficients

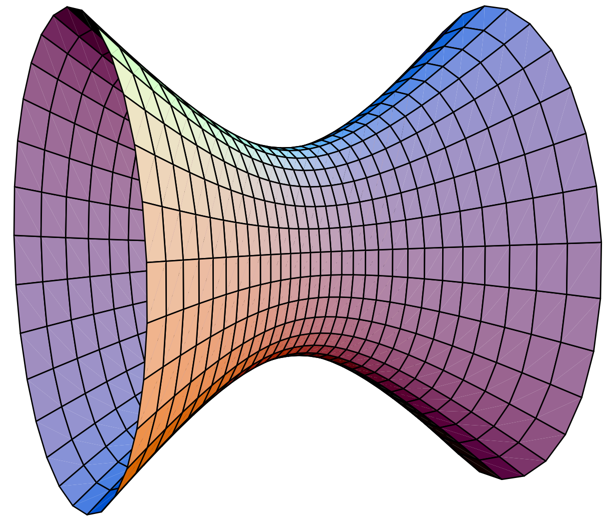
Universal theory: Hydro (diffusive modes), order parameters (time dependent LG), stochastic forces (Langevin)

$$\frac{\partial}{\partial t}(\rho v_i) = P_{ij}^{\perp}\left[\eta_0 \nabla^2 \frac{\delta \mathcal{H}}{\delta(\rho v_j)} + w_0 (\nabla_j \phi) \frac{\delta \mathcal{H}}{\delta \phi} + \zeta_j\right]$$

Model H of Hohenberg and Halperin

$$\eta \sim \xi^{x_\eta} \ (x_\eta \simeq 0.06) \qquad \zeta \sim \xi^{x_\zeta} \ (x_\zeta \simeq 2.8)$$

Anti-DeSitter Space



Consider a hyperboloid embedded in 6-d euclidean space

$$-R^2 = \sum_{i=1,4} x_i^2 - x_0^2 - x_5^2$$

This is a space of constant negative curvature, and a solution of the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2} g_{\mu\nu} \Lambda$$

with negative cosmological constant.

$$\text{Isometries of } AdS_5 : \; SO(4,2) \equiv \text{conformal group in } d = 3+1$$

metric of $\mathrm{AdS}_5 \times \mathrm{S}_5$ (note that $(L/\ell_s)^4 = g^2 N_c$)

$$ds^2 = \frac{r^2}{L^2}\left(-dt^2 + d\mathbf{x}^2\right) + \frac{L^2}{r^2}\,dr^2 + L^2 d\Omega_5^2$$

$r \to \infty$ "boundary" of AdS_5

Finite temperature: AdS_5 black hole solution

$$ds^2 = \frac{r^2}{L^2}\left(-f(r)dt^2 + d\mathbf{x}^2\right) + \frac{L^2}{f(r)r^2}\,dr^2\,,$$

where $f(r) = 1 - (r_0/r)^4$. Hawking temperature $T_H = r_0/\pi$.

Compute induced stress tensor on the boundary

$$\langle T_{\mu\nu}\rangle = \lim_{\epsilon\to 0}\frac{-2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}\bigg|_{bd}$$

Find ideal fluid with

$$\langle T_{\mu\nu}\rangle = \mathrm{diag}(\epsilon, P, P, P)\,, \qquad \frac{\epsilon}{3} = P = \frac{N_c^2}{8\pi^2}(\pi T)^4$$

Hydrodynamics from AdS/CFT

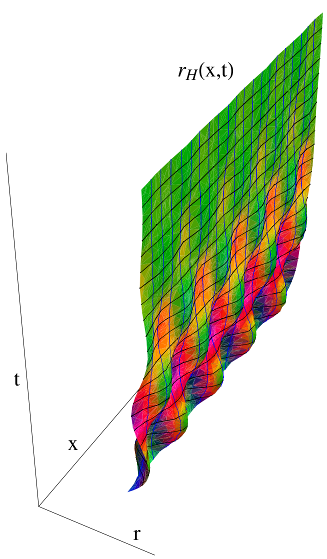
Eddington-Finkelstein coordinates

$$ds^2 = 2\,dv\,dr + \frac{r^2}{L^2}\left[-f(r)dv^2 + r^2 d\mathbf{x}^2\right]$$

Introduce local rest frame $u^\mu = (1, \mathbf{0})$, scale parameter b

$$ds^2 = -2u_\mu dx^\mu dr + \frac{r^2}{L^2}\left[-f(br)u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu\right]$$

$P_{\mu\nu} = u_\mu u_\nu + \eta_{\mu\nu}$



promote $u_\mu(x)$ and $b(x)$ to fields

determine metric order by order in gradients

compute induced stress

leading order: ideal fluid dynamics with $\epsilon = 3P$

$$T_0^{\mu\nu} = \frac{N_c^2}{8\pi^2}(\pi T)^4 \left(\eta^{\mu\nu} + 4u^\mu u^\nu\right)$$

next-to-leading order: Navier-Stokes with $\eta/s = 1/(4\pi)$

$$\delta^{(1)} T^{\mu\nu} = -\frac{N_c^2}{8\pi^2}(\pi T)^3 \sigma^{\mu\nu}$$

next-to-next-to-leading order: second order conformal hydro

$$\begin{aligned}\delta^{(2)} T^{\mu\nu} &= \eta\tau_{II}\left[{}^{\langle}D\sigma^{\mu\nu\rangle} + \frac{1}{3}\sigma^{\mu\nu}(\partial\cdot u)\right] \\ &\quad + \lambda_1 \sigma^{\langle\mu}{}_{\lambda}\sigma^{\nu\rangle\lambda} + \lambda_2 \sigma^{\langle\mu}{}_{\lambda}\Omega^{\nu\rangle\lambda} + \lambda_3 \Omega^{\langle\mu}{}_{\lambda}\Omega^{\nu\rangle\lambda}\end{aligned}$$

relaxation times

$$\tau_\Pi = \frac{2-\ln 2}{\pi T} \qquad \lambda_1 = \frac{2\eta}{\pi T} \qquad \lambda_2 = \frac{2\eta \ln 2}{\pi T} \qquad \lambda_3 = 0$$