An introduction to turbulence in fluids, and modelling aspects

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This chapter is primarily intended as an introductory text, a pedagogical platform, on the phenomenon of turbulence in fluids. For the sake of simplicity, the discussion is mostly limited to the case of an incompressible (constant-property) newtonian fluid in simple three-dimensional turbulent flows. Additional complexities related to thermal convection, magnetic forces, nuclear reactions and so forth are ignored on purpose. The main motivation is to exhibit the general problematic of turbulence in an as simple as possible physical setting. Modelling prospecting, which aims at elaborating numerically tractable mathematical models of turbulence, is also brought up.

I. INTRODUCTION

A. A few words on turbulence

The word *turbulence* is employed to label many different physical phenomena, which exhibit the common characteristics of disorder and complexity. It is the ubiquitous presence of spontaneous (intrinsic) fluctuations, distributed over a wide range of length and time scales, that makes turbulence a worthwhile research topic. The very nature of the turbulent fluctuations is extremely peculiar. Turbulence has to do with non-linearity; there is no hint of the non-linear solutions in the linearized approximations, and strong departure from absolute statistical equilibrium. With this respect, turbulence is of prime interest from the viewpoint of both (non-linear) dynamical systems (Takens & Ruelle [1971]) and irreversible statistical mechanics (Monin & Yaglom [1975]).

Most natural and industrial flows are turbulent; turbulence lies at the core of so much of what we observe. Turbulence generally eclipses the laminar (steady and regular) behavior of the flow, and contributes to a largely enhanced energy dissipation, mixing, heat and mass transfer, etc. In astrophysics, interstellar turbulence is said to be one of the key ingredients of modern theories of star formation (Kritsuk & Norman [2004]). Turbulence is expected to play a significant role in solar eruptions, or in regular pulsations of cepheids.

Non-equilibrium distributions which correspond to turbulent fluctuations depend on the detailed form of the interactions involved in the dynamics; in that sense, turbulence is foremost a problem of fluid mechanics. However, it seems quite clear that the statistics of turbulence can not be made determinate, unless the dynamical equations are supplemented by an additional information about the turbulent state; turbulence is therefore, also, a problem of statistical physics. Since pioneering Reynolds' experiments (Reynolds [1883]), there has been a continuous effort aiming at elucidating this additional information, and elaborating a self-consistent theory of turbulence.

The concept of *energy cascade*, which owes its origin to Richardson (Richardson [1922]), has been particularly

FIG. 1: Turbulence in a soap film behind a one-dimensional vertical grid (courtesy Hamid Kellay, Université de Bordeaux I, France).

important. In 1941, Kolmogorov envisaged a self-similar cascade of kinetic energy from the large (spatial) scales to the small scales of motion. Kolmogorov's cascade is local in scale and all statistical information concerning the large scales is lost, except for the mean energy cascade rate $\bar{\varepsilon}$ itself. Kolmogorov's theory (Kolmogorov [1941]) postulates universal, homogeneous and isotropic distributions for the small-scale velocity fluctuations, and predicts a universal law for the spectrum (in wavenumber k) of kinetic energy:

$$E(k) = C\ \bar{\varepsilon}^{2/3} k^{-5/3},$$

where C is termed the Kolmogorov's constant. A large body of experimental and numerical measurements corroborate the Kolmogorov's law but higher-order statistics are not universal in the sense of Kolmogorov's hypotheses. These discrepancies are rooted in the spatio-temporal fluctuations of the (local) cascade rate: The energy cascade is an highly non-uniform process in space and time. This feature is usually referred to as *intermittency*. From the viewpoint of statistical mechanics, intermittency implies that the macroscopic parameter $\bar{\varepsilon}$ is not sufficient to describe the *energy-cascade state of turbulence*.

Once "Kolmogorov's mean field theory" is abandoned, a pandora box of possibilities is opened, and a specific

contact with the dynamics must be achieved. Current models have not yet managed to establish this contact. They rely on (a priori) plausible hypotheses, but fail to relate themselves to the actual dynamics (see Frisch [1995], for a modern review). More recent works attempt to correlate turbulent velocity fluctuations with the presence of highly coherent vortical structures (She & Lévêque [1994], for instance).

The Kolmogorov's theory is sixty year old, but fundamental questions about turbulence remain mainly unsolved.

B. Suggestions for further readings on turbulence

The following articles on turbulence, accessible to a general audience, may help to situate the topic of turbulence in the field of modern classical physics:

- "Turbulence: Challenges for Theory and Experiment"
  by Uriel Frisch & Stephen Orszag, Physics Today, p. 24 (January, 1990)
- "Some comments on Turbulence"
  by J. L. Lumley, Phys. Fluids A 4(2), p. 203 (1992)
- "Turbulence near a final answer"
  by Uriel Frisch, Physics World, vol. 12, p. 53 (December, 1999)

The following books (among many others) may be found enlightening on the topics broached in this chapter:

- *A first course in turbulence* (a reference book on turbulence)
  by H. Tennekes & J. L. Lumley
  ed. Mit Press, Cambridge, USA (1972)
- *Turbulent flows* (textbook of a course taught at Cornell University)
  by S. Pope
  ed. Cambridge University Press, Cambridge, United Kingdom (2000)

C. Content of the chapter

Turbulence is introduced from the standpoint of fluid mechanics in section II; the need for a statistical treatment is put forward. Turbulent-viscosity modelling of turbulence is briefly discussed in section III. The statistical mechanics of turbulence are outlined in section IV; the Kolmogorov's theory and its shortcomings, related to the phenomenon of intermittency, are presented. Finally, section V is devoted to a rapid introduction to the so-called large-eddy simulation of turbulent flows.

II. TURBULENCE AS A PROBLEM OF FLUID MECHANICS

A. An historical example: The Poiseuille's flow

Let us consider the internal flow of an incompressible newtonian fluid through a long, straight (slightly tilted) pipe (Fig. 2). This flow is known as the *Poiseuille's flow*; it has played an historical role in the development of our understanding of turbulent flows.

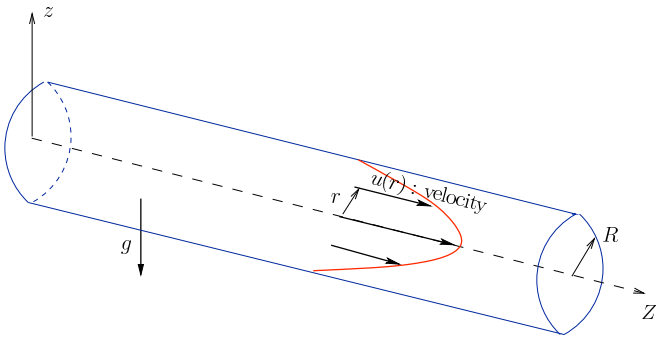

FIG. 2: Sketch of the Poiseuille's flow (in a slightly tilted pipe); g is the gravitational acceleration; $u(r)$ denotes the streamwise velocity at a distance r from the axis.

A *laminar regime* is achieved for low flow rates. The fluid motion is direct; the streamlines are parallel to the axis of the pipe. In the stationary regime, the drop in pressure between the entrance and the exit, here supplemented by gravity, drives the flow against the (internal) friction forces, resulting from collisions between the molecules of the fluid.

The (constant-property) newtonian-fluid hypothesis (see Batchelor [1967]) states that the tangential resistance per unit area, or shearing stress, writes

$$\tau(r) = -\mu \frac{du(r)}{dr}, \tag{1}$$

where μ is the *dynamic viscosity*. μ depends on the microscopic properties of the fluid. By balancing the forces along the axis, one obtains

$$\tau(r) = \frac{1}{2} G r, \tag{2}$$

where $G \equiv -dq(Z)/dZ > 0$ is the (constant) downward gradient of the modified pressure $q = p + gz$. Combining Eqs. (1) and (2) finally leads to the parabolic velocity profile

$$u(r) = \frac{G}{4\mu}(R^2 - r^2), \tag{3}$$

which satisfies the no-slip condition at the boundary.

Once the velocity profile is known, the flow rate Q (defined as the volume of fluid which flows through a section of the pipe per unit time) can be determined:

$$Q \equiv \int_0^{2\pi} 2\pi r u(r) dr = \frac{\pi G R^4}{8\mu}. \qquad (4)$$

This is the celebrated *Poiseuille's law*, which relates the flow rate, the radius of the pipe and the (modified) pressure drop (Poiseuille [1841]).

The overall dissipation rate of kinetic energy (per unit mass) expresses as

$$\varepsilon_{\text{diss.}} = \frac{GU}{\rho}, \qquad (5)$$

where $U \equiv Q/\pi R^2$ is the mean velocity over a section. Using Eq. (4), one gets

$$G = \frac{8\mu U}{R^2} \quad \text{and therefore} \quad \varepsilon_{\text{diss.}} = 8\nu \frac{U^2}{R^2}. \qquad (6)$$

$\nu \equiv \mu/\rho$ is termed the *kinematic viscosity*, since its dimensions $(length)^2 \times (time)^{-1}$ do not include mass. ν is about 0.01 cm^2/s for water at standard temperature. $\varepsilon_{\text{diss.}}$ corresponds to the (external) energy input required to maintain the flow stationary, and compensate the loss of kinetic energy due to viscous friction. In the laminar regime, one finds that $\varepsilon_{\text{diss.}}$ is proportional to the kinematic viscosity of the fluid and to the square of the mean velocity of the flow.

The previous computation is relevant only for low flow rates. Indeed, let us consider a pipe with a radius 1 cm, titled with a slope 0.1 %. According to the poiseuille's law (4), water takes under gravity (assuming the same pressure at the entrance and the exit) a speed 10 cm/s. This prediction is correct. However, for a 1 m radius pipe, the Poiseuille's law yields (in a similar situation) a speed 1 km/s. In practice, speeds are obviously much smaller; the actual flow dissipates much more energy than predicted by the laminar dissipation law (6). The mean dissipation rate is no longer proportional to U^2, but to U^3 (considering rugous walls). Furthermore, the (mean) velocity profile is no longer parabolic. It is almost constant around the centerline and decreases rapidly to zero near the boundary. What is the origin of such dissimilarities?

B. The transition to turbulence

As the velocity of the fluid exceeds some critical value, the stationarity and the regularity of the flow break off. Small (velocity) disturbances are no longer damped by the laminar flow, but grow by extracting kinetic energy from the mean flow. Disordered swirling motions, in which fluid particles follow complicated (non-brownian) trajectories, take place (Fig. 3). The flow is *turbulent*. In this situation, velocity gradients are much larger than

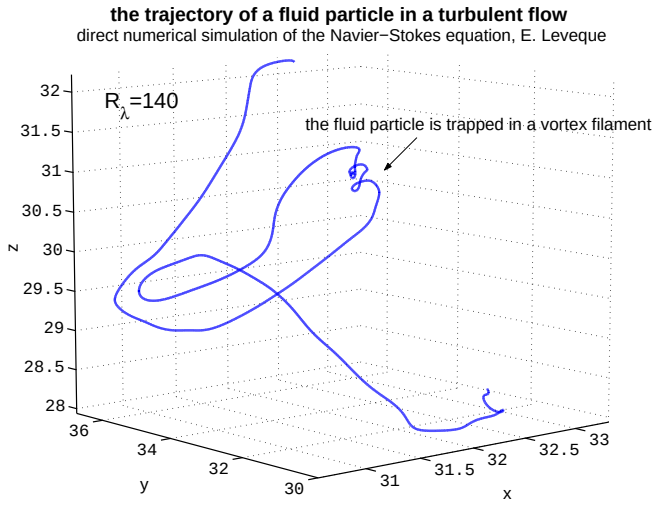

FIG. 3: The trajectory of a fluid particle (an infinitesimal material element of fluid) transported by a statistically homogeneous and isotropic turbulent flow, does not result from a sequence of independent random steps (brownian motion). Occasionally, the fluid particle is trapped in a vortex filament, which gives rise to *anomalous* velocity fluctuations.

in the laminar case, and consequently, viscous friction is strongly enhanced.

In 1883, Osborne Reynolds evidenced this transition by steadily injecting dye on the centerline of the pipe (Fig. 4). In the laminar regime, the dye forms a streak and does not mix with the surrounding fluid, except for molecular diffusion. Above a certain speed, the streak becomes unstable and the dye rapidly disperses across the whole pipe (turbulence is indeed very efficient for mixing fluids).

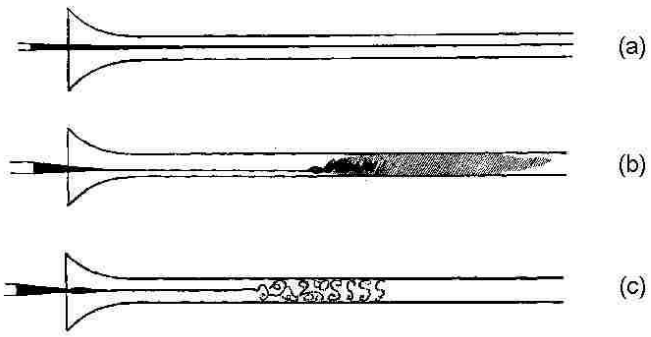

FIG. 4: (a): The laminar regime — (b): "The colour band would all at once mix up with the surrounding water, and fill the rest of the tube with a mass of coloured water" — (c): "On viewing the tube by the light of an electric spark, the mass of colour resolved itself into a mass of more or less distinct curls, showing eddies" (Reynolds [1883])

When the flow is turbulent, it is preferable to break the

instantaneous velocity field $u_i(\vec{x},t)$ into a mean (ensemble average is meant) value $\overline{u_i}(\vec{x},t)$, which varies slowly as a function of $\vec{x}$ and t, and a rapidly fluctuating component

$$u_i'(\vec{x},t) \equiv u_i(\vec{x},t) - \overline{u_i}(\vec{x},t).$$

This decomposition is called the *Reynolds' decomposition.*

Obviously, it is not realistic to obtain an ensemble average in an experiment or in a numerical simulation, since one can not carry out an infinite number of independent realizations. When the turbulent flow is (statistically) stationary, ergodicity is invoked (Eckmann & Ruelle [1985]); it is assumed that statistical properties obtained by averaging over a set of realizations (ensemble averages) coincide with those obtained by averaging a single realization for a sufficiently long interval of time (time average):

$$\begin{aligned}\overline{u_i}(\vec{x}) &\equiv \underbrace{\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} u_i^{(n)}(\vec{x},t)}_{\text{ensemble average}} \\ &\approx \frac{1}{T}\int_{t-T/2}^{t+T/2} u_i(\vec{x},t')dt', \qquad (7)\end{aligned}$$

where T should be taken (very) large compared to the correlation time of the turbulent velocity.

The turbulent velocity $u_i'(\vec{x},t)$ has zero mean but deeply influences the kinetics of the mean flow. Indeed, the mean flux of momentum writes

$$\rho\overline{u_i u_j} = \underbrace{\rho\overline{u_i}\;\overline{u_j}}_{\text{mean flow}} + \underbrace{\rho\overline{u_i' u_j'}}_{\text{turbulent agitation}}, \qquad (8)$$

which implies that the flux of momentum related to the mean flow is supplemented by the mean flux $\rho\overline{u_i' u_j'}$, related to the turbulent velocity. This latter may be viewed as an additional stress (acting on the mean flow) resulting from the underlying turbulent agitation. This stress is termed the *Reynolds stress.* In order to take into account turbulence in the mean flow dynamics, it is necessary to determine the Reynolds-stress tensor.

C. The kinetics of the mean flow

The equations governing the motion of a newtonian fluid have been known for long. It dates back to the works of Navier and Stokes (Navier [1823], Stokes [1843]). The *Navier-Stokes equation* stands for the conservation of the momentum of an infinitesimal material element of fluid, and satisfies the primary requirement of the second law of thermodynamics (the rate of energy dissipation be positive and the process irreversible).

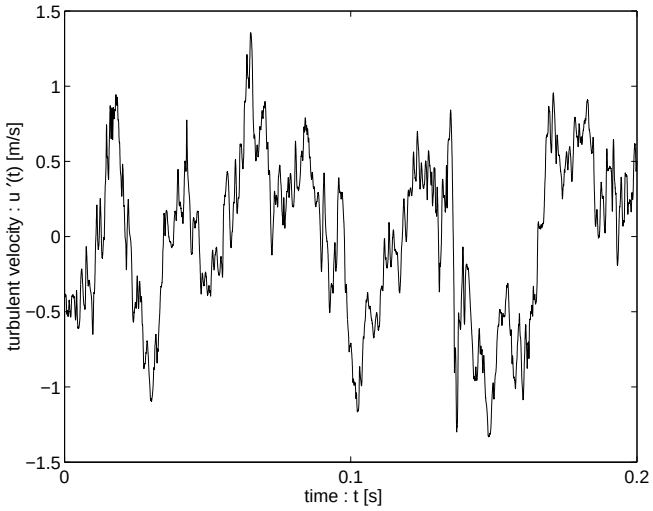

FIG. 5: Temporal fluctuations of the streamwise velocity in the turbulent wake behind a vertical cylinder; the fluid is air (courtesy C. Baudet, Ens-Lyon, France). The velocity has been measured by using a hot-wire anemometer (Comte-Bellot [1976]). Turbulent temporal fluctuations do not exhibit any characteristic temporal period; the signal is said *scale-invariant.*

For each component $u_i(\vec{x},t)$:

$$\frac{Du_i}{Dt} \equiv \left(\frac{\partial}{\partial t} + u_j\frac{\partial}{\partial x_j}\right)u_i = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_j \partial x_j}. \qquad (9)$$

Here and below, summation over repeated cartesian indices is implied.

Furthermore, it is assumed that the density of the fluid does not vary; the fluid is considered as *incompressible.* The solenoidal condition

$$\frac{\partial u_i}{\partial x_i} = 0 \qquad (10)$$

follows from the mass conservation. This condition is also verified by the mean-flow velocity $\overline{u_i}$ and the turbulent velocity u_i'.

1. The mean-flow equations

By averaging the Navier-Stokes equations, one obtains for each component $\overline{u_i}(\vec{x},t)$:

$$(\frac{\partial}{\partial t} + \overline{u_j}\frac{\partial}{\partial x_j})\overline{u_i} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} - \frac{\partial \overline{u_i' u_j'}}{\partial x_j} + \nu\frac{\partial^2 \overline{u_i}}{\partial x_j \partial x_j}. \qquad (11)$$

For the kinetic energy (per unit mass) $\overline{k} \equiv \frac{1}{2}\overline{u_i}^2$:

$$\frac{\overline{D}\;\overline{k}}{\overline{D}t} = \frac{\partial}{\partial x_j}\left(-\frac{1}{\rho}\overline{u_j}\;\overline{p} + \left[2\nu\overline{S_{ij}} - \overline{u_i' u_j'}\right]\overline{u_i}\right) - \left[2\nu\overline{S_{ij}} - \overline{u_i' u_j'}\right]\overline{S_{ij}}, \qquad (12)$$

where

$$\overline{S_{ij}} \equiv \frac{1}{2}\left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i}\right) \tag{13}$$

is the (symmetric) mean strain-rate tensor.

2. The transition to turbulence and the Reynolds number

The transition to turbulence is related to the preponderance of the turbulent stress $\overline{u_i' u_j'}$ (turbulent transport of momentum) over the mean viscous stress $-2\nu\overline{S_{ij}}$ (molecular diffusion of momentum) in Eq.(12). Reynolds (Reynolds [1895]) proposed to quantify the relative efficiency of these two mechanisms by an order parameter of the flow, defined as

$$Re \equiv \frac{UL}{\nu} \quad : \quad \text{the so-called } \textit{Reynolds number}, \tag{14}$$

where U and L characterize respectively the velocity et the size of the flow.

From the dynamical equations, the Reynolds number is recovered by assuming that

$$\int_{\mathcal{V}} -\overline{u_i' u_j'}\ \overline{S_{ij}} \sim \frac{U^3}{L} \quad \text{and} \quad \int_{\mathcal{V}} |\overline{S}|^2 \sim \frac{U^2}{L^2}. \tag{15}$$

$\int_{\mathcal{V}}$ denotes an average over the flow and $\sim$ means "behaves as".

The turbulent regime corresponds to high Reynolds numbers, and indicates the supremacy of the turbulent stress over the mean-flow viscous stress. In that case, the kinematic viscosity is no longer a relevant parameter of the mean-flow dynamics; the budget equation for the kinetic energy becomes

$$\frac{\overline{D}\ \overline{k}}{\overline{D}t} = \frac{\partial}{\partial x_j}\overbrace{\left(-\frac{1}{\rho}\overline{u_j}\ \overline{p} - \overline{u_i' u_j'}\,\overline{u_i}\right)}^{\text{transport within the mean flow}} + \underbrace{\overline{u_i' u_j'}\ \overline{S_{ij}}}_{\text{transfert of energy to turbulence}}. \tag{16}$$

The term $\overline{u_i' u_j'}\ \overline{S_{ij}}$ is important; it stands for a transfert of kinetic energy from the mean flow to the turbulent agitation. As a rule of thumb, one may claim that *the mean-flow inhomogeneities trigger turbulence*. The kinetic energy extracted from the mean flow will be eventually transformed into heat by viscous dissipation acting on turbulent motions (at very small length scales).

According to Eq. (15), the overall mean dissipation rate read

$$\varepsilon_{\text{diss.}} = \int_{\mathcal{V}} -\overline{u_i' u_j'}\ \overline{S_{ij}} \sim \frac{U^3}{L}. \tag{17}$$

$\varepsilon_{\text{diss.}}$ does not depend on the viscosity of the fluid: *Turbulence is a property of the flow, not of the fluid.*

In order to illustrate these results, let us come back to the Poiseuille's flow. In the turbulent regime, the pressure drop per unit length expresses as $G(\rho, U, R)$ because ν is no longer a relevant parameter. A dimensional argument leads to

$$G \sim \frac{\rho U^2}{R}. \tag{18}$$

Concerning the overall mean dissipation rate, one obtains

$$\varepsilon_{\text{diss.}} = \frac{GU}{\rho} \sim \frac{U^3}{R}, \tag{19}$$

in agreement with experimental observations.

D. The need for a statistical treatment

In order to determine the turbulent-stress tensor

$$\rho\ \overline{u_i'(\vec{x},t) u_j'(\vec{x},t)},$$

it is needed to describe the statistics of the turbulent velocity field. This has been the subject of many studies based on statistical considerations, where $u_i'(\vec{x},t)$ is viewed as a random function of $(\vec{x},t)$.

From the Navier-Stokes equation, a dynamical equation for $\overline{u_i' u_j'}$ can be derived by considering

$$\frac{\partial \overline{u_i' u_j'}}{\partial t} = \overline{u_i' \frac{\partial u_j'}{\partial t}} + \overline{u_j' \frac{\partial u_i'}{\partial t}} \tag{20}$$

and by substituting the equations for $\partial u_j'/\partial t$ and $\partial u_i'/\partial t$. This computation leads to the so-called *Reynolds-stress equation*, which has been the primary vehicle for much of the turbulence-modelling efforts.

$$\begin{aligned}\frac{\overline{D}\ \rho\overline{u_i' u_j'}}{\overline{D}t} = &-\overbrace{2\ \overline{p' S_{ij}'}}^{\text{pressure strain-rate tensor}} \\ &- \frac{\partial}{\partial x_k}\overbrace{\left(\overline{p' u_i'}\delta_{jk} + \overline{p' u_j'}\delta_{ik} + \rho\overline{u_i' u_j' u_k'} - \nu\frac{\partial\ \rho\overline{u_i' u_j'}}{\partial x_k}\right)}^{\text{(turbulent+diffusive) transport tensor}} \\ &- \overbrace{\left(\rho\overline{u_i' u_k'}\frac{\partial \overline{u_j}}{\partial x_k} + \rho\overline{u_j' u_k'}\frac{\partial \overline{u_i}}{\partial x_k}\right)}^{\text{production tensor}} \\ &- \overbrace{2\mu\left(\overline{\frac{\partial u_i'}{\partial x_k}\ \frac{\partial u_j'}{\partial x_k}}\right)}^{\text{dissipation tensor}},\end{aligned} \tag{21}$$

where S_{ij}' denotes the turbulent strain-rate tensor.

The Reynolds-stress equation (21) involves the third-order tensor $\overline{u_i' u_j' u_k'}$. Similarly, the equation for $\overline{u_i' u_j' u_k'}$ would involve the fourth-order tensor $\overline{u_i' u_j' u_k' u_l'}$ as an additional unknown, and so forth. This difficulty arises from the non-linearity of the Navier-Stokes equation. It means that the Navier-Stokes equation is equivalent to an infinite hierarchy of statistical equations coupling all the moments of the velocity field. Any finite subset of this hierarchy is not closed and possesses more unknowns than are set by the equations of the subset. In this situation, a (statistical) *closure condition* must be posed in order to provide a description with a reduced set of averaged (macroscopic) quantities. Closure attempts must connect with the dynamics of turbulence. By understanding how turbulence behaves, one may hope to guess an appropriate reduced set of modelled constitutive equations, usually referred to as the *Reynolds Averaged Navier-Stokes* equations (see Wilcox [1993], for a comprehensive review).

III. THE TURBULENT-VISCOSITY MODELLING OF TURBULENCE

A. The concept of turbulent viscosity and related ideas

Important features of turbulence can be described in terms of *turbulent viscosity*. Despite its numerous flaws, the concept of turbulent viscosity — originally introduced by Boussinesq in 1877 — provides a valuable, simple contact with the dynamics of turbulence.

The idea behind the turbulent viscosity is to treat the deviatoric (traceless) part of the turbulent stress $\rho \overline{u_i' u_j'}$ like the viscous stress in a newtonian fluid:

$$-\rho \overline{u_i' u_j'} + \underbrace{\frac{2}{3}\, \rho \overline{k'}\, \delta_{ij}}_{\text{isotropic turbulent stress}} = 2\rho \nu_{\text{turb.}} \overline{S_{ij}}, \qquad (22)$$

where the (kinematic) turbulent viscosity $\nu_{\text{turb.}}(\vec{x}, t)$ can a priori depend on the position $\vec{x}$ and the time t. This approach relies on the hypothesis of a turbulent friction responsible for the diffusive transport of momentum from the rapid to the slow mean-flow regions. In this context, the mean-flow equation writes

$$\frac{\overline{D}\,\overline{u_j}}{\overline{D}t} = -\frac{1}{\rho}\frac{\partial}{\partial x_j} \overbrace{\left(\bar{p} + \frac{2}{3}\rho \overline{k'}\right)}^{\text{generalized press.}} + \frac{\partial}{\partial x_i}\left(2\nu_{\text{eff.}} \overline{S_{ij}}\right), \qquad (23)$$

with the effective viscosity $\nu_{\text{eff.}}(\vec{x}, t) = \nu + \nu_{\text{turb.}}(\vec{x}, t) \approx \nu_{\text{turb.}}(\vec{x}, t)$. An appropriate specification of the turbulent viscosity is still required.

Similarly, the *gradient-diffusion hypothesis* states that the turbulent transport of a scalar field $\phi(\vec{x}, t)$ is down the mean scalar gradient:

$$\overline{u_i' \phi'} = -\kappa_{\text{turb.}} \frac{\partial \overline{\phi}}{\partial x_i}, \qquad (24)$$

where $\kappa_{\text{turb.}}(\vec{x}, t)$ is the turbulent diffusivity associated to the scalar field ϕ. This is analogous to the Fourier's law for heat conduction, or, to the Fick's law for molecular diffusion.

1. The magnitude of the turbulent viscosity

The magnitude of the turbulent viscosity can be estimated by considering the local mean energy-dissipation rate (per unit mass)

$$\bar{\varepsilon} = -\overline{u_i' u_j'}\ \overline{S_{ij}}. \qquad (25)$$

On the other hand, the turbulent-viscosity hypothesis yields

$$\bar{\varepsilon} = 2\nu_{\text{turb.}}\ |\overline{S}|^2, \qquad (26)$$

where $|\overline{S}|^2 = \sum_{ij} \overline{S_{ij}}^2$. By comparing the Eqs. (25) and (26), one finally gets

$$\frac{\nu_{\text{turb.}}}{\nu} = \frac{-\overline{u_i' u_j'}\ \overline{S_{ij}}}{2\nu\ |\overline{S}|^2} \sim Re_{\text{local}}. \qquad (27)$$

Eq.(27) means that the molecular viscosity is multiplied by a factor of the order of the *local Reynolds number* (see next section) to give the turbulent viscosity. This explains the highly dissipative nature of turbulent flows.

Dimensionally, $\nu_{\text{turb.}}$ is equivalent to the product of a velocity and a length scale. This suggests to write, by analogy to the kinetic theory for gases,

$$\nu_{\text{turb.}} = u'\ \ell', \qquad (28)$$

where u' and ℓ' would represent respectively the velocity and length scales of the turbulent motion.

In the *mixing-length model*, u' and ℓ' are specified on the basis of the geometry of the flow. In two-equation models — the so-called *(k-ε) model* being the prime example — u' and ℓ' are related to the turbulent kinetic energy and the turbulent dissipation, for which modelled constitutive equations are explicated.

2. The mixing-length model (zero-equation model)

The so-called *mixing-length model* has been proposed by Prandtl (Prandtl [1925]) for two-dimensional boundary flows. The turbulent viscosity reads

$$\nu_{\text{turb.}} = \ell_{\text{m}}^2 \left| \frac{\partial\, \overline{u_x}}{\partial y} \right| \qquad (29)$$

where y is the coordinate perpendicular to the wall (along the x axis).

The turbulent length scale is the mixing length and the velocity scale is locally determined by the mean velocity gradient:

$$\ell' = \ell_{\rm m} \quad \text{and} \quad u' = \ell_{\rm m} \left| \frac{\partial\, \overline{u_x}}{\partial y} \right|. \tag{30}$$

Note that u' is zero wherever the mean velocity gradient is zero. This may be problematic for bounded flows in some particular circumstances (on the centerline of a channel flow).

Prandtl's mixing length may be viewed as the turbulent analog of the mean free path of molecules in the kinetic theory of gases. The mixing length $\ell_{\rm m}$ must be specified accordingly to the geometry of the flow. This specification is usually empirical; it is derived from experiment and observation rather than theory.

A possible generalization of the mixing-length hypothesis read

$$\nu_{\rm turb.} = \ell_{\rm m}^2 |\overline{S}|. \tag{31}$$

In its generalized form, this model is arguably the simplest modelling of turbulence.

The mixing-length model is valuable in simple flows such as shear-layer or pressure-driven flows, but cannot account for the transport effects of turbulence. Indeed, it implies that the local level of turbulence depends upon the local generation and dissipation rates. In reality, turbulence may be carried or diffused to locations where no turbulence is actually being generated at all.

3. One-equation model: The k-model

Independently, Kolmogorov ([1942]) and Prandtl ([1945]) suggested to base the turbulent velocity scale on the mean turbulent kinetic energy: $u' \propto \overline{k'}^{1/2}$. If the turbulent length scale is taken as the mixing length, the turbulent viscosity becomes

$$\nu_{\rm turb.} = \ell_{\rm m} \overline{k'}^{1/2}. \tag{32}$$

A transport equation for $\overline{k'}$ is required.

The (exact) dynamical equation for $\overline{k'}$ writes

$$\frac{\bar{D}\, \overline{k'}}{\bar{D}t} = -\frac{\partial}{\partial x_j} \left(\underbrace{\frac{1}{2}\overline{u_i' u_i' u_j'} + \frac{1}{\rho}\overline{u_j' p'}}_{\text{turbulent transport}} + \nu \frac{\partial \overline{k'}}{\partial x_j} \right) \underbrace{-\overline{u_i' u_j'}\, \overline{S_{ij}}}_{\text{production: } \bar{P}} \underbrace{-\nu \overline{\left(\frac{\partial u_i'}{\partial x_j} \right)^2}}_{\text{dissipation: } \bar{\varepsilon}} \tag{33}$$

- $-\overline{u_i' u_j'} \approx 2\nu_{\rm turb.} \overline{S_{ij}} - 2/3\, \overline{k'} \delta_{ij}$ with $\nu_{\rm turb.} = \ell_{\rm m} \overline{k'}^{1/2}$ according to the turbulent-viscosity hypothesis.
- $-\frac{1}{2}\overline{u_i' u_i' u_j'} + \frac{1}{\rho}\overline{u_j' p'} \approx \frac{\nu_{\rm turb.}}{\sigma_k} \frac{\partial \overline{k'}}{\partial x_j}$. This stems from the gradient-diffusion hypothesis. The empirical coefficient σ_k may be viewed as a turbulent Prandtl number (of order unity) for the kinetic energy.
- from a dimensional argument, the mean dissipation rate $\bar{\varepsilon} = C \overline{k'}^{3/2} / \ell_{\rm m}$ where C is a model constant.

Finally, the transport equation for the mean turbulent kinetic energy writes

$$\frac{\bar{D}\, \overline{k'}}{\bar{D}t} = -\frac{\partial}{\partial x_j} \left(\left(\nu + \frac{\nu_{\rm turb.}}{\sigma_k} \right) \frac{\partial \overline{k'}}{\partial x_j} \right) + \bar{P} - \bar{\varepsilon} \tag{34}$$

with $\nu_{\rm turb.} = \ell_{\rm m} \overline{k'}^{1/2}$ and $\bar{\varepsilon} = C \overline{k'}^{3/2} / \ell_{\rm m}$ together with the turbulent-viscosity hypothesis for $\overline{u_i' u_j'}$ and the specification of the mixing length $\ell_{\rm m}$.

The k-model does allow for the transport of turbulence into regions where there is locally no generation. It is therefore inherently capable of simulating some phenomena more realistically than the mixing-length model. However the mixing length remains an empirical parameter, and knowledge is almost totally absent for recirculating and three-dimensional flows. This model is found useful in boundary-layer flows, where the mixing length is fairly well known.

4. Two-equation model: The (k-ε) model

The (k-ε) model belongs to the class of two-equation models, which are more efficient in the case of flows in complex geometry and higher Reynolds numbers. Its development is often credited to Jones and Launder ([1972]) but acknowledgement should also be addressed to Kolmogorov for his original insight.

The two relevant macroscopic parameters in the (k-ε) model are the turbulent mean kinetic energy and mean energy dissipation. From dimensional analysis, one gets that the turbulent length scale $\ell' \sim \overline{k'}^{3/2} / \bar{\varepsilon}$ and the turbulent velocity scale $u' \sim \overline{k'}^{1/2}$. It follows that

$$\nu_{\rm turb.} = C_\mu \frac{\overline{k'}^2}{\bar{\varepsilon}}, \tag{35}$$

where C_μ is an empirical constant.

The standard model equation for $\bar{\varepsilon}$ is viewed as

$$\begin{aligned} \frac{\bar{D}\, \overline{\varepsilon}}{\bar{D}t} &= \frac{\partial}{\partial x_j} \left(\left(\nu + \frac{\nu_{\rm turb.}}{\sigma_\varepsilon} \right) \frac{\partial \bar{\varepsilon}}{\partial x_j} \right) \\ &- 2c_1 \frac{\nu_{\rm turb.}\, \bar{\varepsilon}}{k}\, \overline{S_{ij}} \frac{\partial \overline{u_i}}{\partial x_j} - c_2 \frac{\bar{\varepsilon}^2}{k}. \end{aligned} \tag{36}$$

The five empirical coefficients (C_μ, c_1, c_2, σ_k, σ_ε) are fixed by considering experimental results for simple-geometry flows.

B. The limit of the turbulent-viscosity concept

The concept of turbulent viscosity is properly applicable only when there is a significant separation between the length scale of inhomogeneity of the mean field and the mixing length of the agitation. This condition is not satisfied in turbulent flows where *turbulent motions display a continuous distribution of scale sizes; turbulent motions do not occur at some characteristic mixing length*. Furthermore, the largest (spatial) scales of the turbulent motion are as large as the scale of inhomogeneity of the mean flow. Consequently, the concept of a local transport has not much justification, except in a very crude qualitative sense.

IV. THE STATISTICAL MECHANICS OF TURBULENCE

This section is devoted to the statistical description of turbulent motions, also called *turbulent eddies*. By analogy with a molecule, an eddy may be seen as a "glob" of fluid of a given size, or (spatial) scale, that has a certain structure and life history of its own. The turbulent activity of the bulk is the net result of the interactions between the eddies.

It is characteristic of turbulence that fluctuations are unpredictable in details; however, statistically distinct properties can be identified and profitably examined.

A. The mean kinetics of the turbulent motion

From the Navier-Stokes equations, one can establish for each component of the turbulent velocity field $u_i'(\vec{x},t)$:

$$\frac{\partial u_i'}{\partial t} + \frac{\partial}{\partial x_j}\left(u_i'u_j' - \overline{u_i'u_j'}\right) + \left(u_j'\frac{\partial \overline{u_i}}{\partial x_j} + \overline{u_j}\frac{\partial u_i'}{\partial x_j}\right) = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu\frac{\partial^2 u_i'}{\partial x_j \partial x_j} \tag{37}$$

and for the mean kinetic energy $\overline{k'} = \frac{1}{2}\,\overline{u_i'^2}$:

$$\frac{\overline{D}}{\overline{D}t}\overline{k'} = \underbrace{-\frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_i'u_i'u_j'} + \frac{1}{\rho}\overline{u_j'p'} - 2\nu\overline{u_i'S_{ij}'}\right)}_{\text{turbulent mixing}} \underbrace{-\overline{u_i'u_j'}\;\overline{S_{ij}}}_{\text{production}} \underbrace{-2\nu\overline{|S'|^2}}_{\text{dissipation: }\bar{\varepsilon}}, \tag{38}$$

where S_{ij}' denotes the turbulent strain-rate tensor.

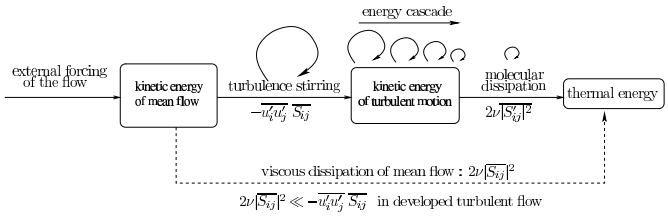

FIG. 6: Sketch of the overall kinetics of a turbulent flow.

The transport term is responsible for a turbulent mixing of kinetic energy between the various scales of motion (eddies). This mixing is peculiar: It achieves a conservative transfert of kinetic energy from the largest eddies (of size comparable to the scale of inhomogeneity of the mean flow) to smaller and smaller eddies. This process is usually referred to as the *energy cascade* (Richardson [1922]). The turbulent kinetic energy is ultimately dissipated into thermal energy through the action of the molecular viscosity.

When turbulence is developed,

$$-\frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_i'u_i'u_j'} + \frac{1}{\rho}\overline{u_j'p'} - 2\nu\overline{u_i'S_{ij}'}\right) \approx -\frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_i'u_i'u_j'} + \frac{1}{\rho}\overline{u_j'p'}\right) \tag{39}$$

The local energy budget then simplifies as

$$\frac{\overline{D}}{\overline{D}t}\overline{k'} = \underbrace{-\frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_i'u_i'u_j'} + \frac{1}{\rho}\overline{u_j'p'}\right)}_{\text{energy cascade}} - \overline{u_i'u_j'}\;\overline{S_{ij}} \quad -2\nu\overline{|S'|^2}. \tag{40}$$

In homogeneous and stationary turbulence, this budget reduces to

$$-\overline{u_i'u_j'}\;\overline{S_{ij}} = 2\nu\overline{|S'|^2}, \tag{41}$$

which simply states that the energy received from the mean flow is eventually dissipated under molecular viscosity. The kinetics of a turbulent flow are summarized in Fig. 6.

B. The characteristic (length) scales of turbulence; the turbulent Reynolds number

1. *The turbulent two-point velocity correlation*

The quantities of most theoretical interest in the statistical description of turbulence are averages of the form

$$R_{ij\ldots}(\vec{x},t;\vec{x'},t';\ldots) = \left\langle u_i(\vec{x},t)u_j(\vec{x'},t')\cdots\right\rangle \tag{42}$$

called *velocity correlation functions*. A complete specification of turbulence implies the knowledge of all correlation functions. The simplest (non-zero) and probably most important is the two-point correlation function.

In this context, Taylor's development of two-point velocity correlations laid the groundwork for the modern statistical approach of (stationary in time) turbulence (Taylor [1935]):

$$R_{ij}(\vec{x},\vec{r}) \equiv \langle u_i(\vec{x},t) u_j(\vec{x}+\vec{r},t)\rangle \tag{43}$$

where $\langle\ \rangle$ now stands for ensemble average. Hereafter, the primes will be often omitted (for simplicity) to denote turbulent fluctuations, and the mass density of the fluid will be taken equal to unity ($\rho = 1$).

Turbulence that occurs in nature is usually not homogeneous. There is frequently important variation of the local production and transport of turbulence, related to the inhomogeneity of the mean flow. Despite the importance of these effects, it is thought that the small-scale statistical properties of general turbulent flows should be *locally homogeneous and isotropic*, and therefore, the study of idealized homogeneous and isotropic turbulence is not entirely without interest. In such a case, the tensor R_{ij} depends only on the separation, or scale, $r \equiv |\vec{r}|$.

It is profitable to break the correlation tensor into its longitudinal and transverse components (Batchelor [1967]):

$$R_\parallel(r) = \frac{1}{3}\langle u_i^2\rangle f(r) \quad \text{and} \quad R_\perp(r) = \frac{1}{3}\langle u_i^2\rangle g(r). \tag{44}$$

Geometrical considerations lead to

$$R_{ij}(r) = \frac{1}{3}\langle u_i^2\rangle \left(\frac{f(r)-g(r)}{r^2} r_i r_j + g(r)\delta_{ij} \right) \tag{45}$$

and the incompressibility condition yields

$$g(r) = f(r) + \frac{r}{2}\frac{df(r)}{dr}. \tag{46}$$

All the components of the correlation tensor R_{ij} can be deduced from the longitudinal auto-correlation function $f(r)$.

Note the possibility of adding a term $h(r)\varepsilon_{ijk}r_k$ (ε_{ijk} is the permutation tensor) within the parentheses in Eq. (45). Such a term preserves isotropy but would not be invariant under reflection (parity inversion). Turbulence is normally expected to be mirror symmetric, and, therefore, this extra term is usually omitted. Turbulence in a rotating frame, however, may not be mirror symmetric. Dynamo action in plasma also requires turbulence not symmetric under reflection.

2. *The macroscale and the microscale of turbulence*

- for $r = 0$, the (longitudinal) correlation function $R_\parallel(0) = \frac{1}{3}\langle u_i^2\rangle > 0$.

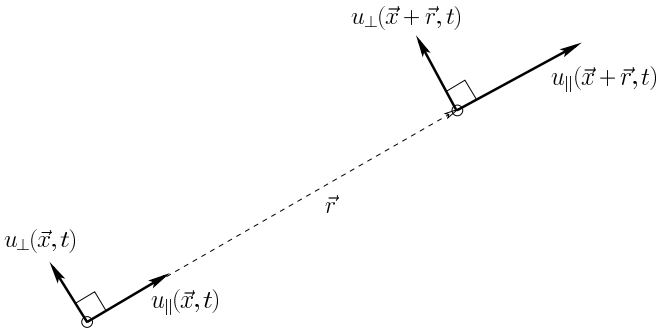

FIG. 7: The longitudinal and transverse components of the velocity along the direction $\vec{r}$. By definition, $R_\parallel(r) = \langle u_\parallel(\vec{x},t)u_\parallel(\vec{x}+\vec{r},t)\rangle$ and $R_\perp(r) = \langle u_\perp(\vec{x},t)u_\perp(\vec{x}+\vec{r},t)\rangle$.

- for large enough r, $u_\parallel(\vec{x},t)$ and $u_\parallel(\vec{x}+\vec{r},t)$ are uncorrelated so that $R_\parallel(r) = 0$.

The behavior of $R_\parallel(r)$ therefore exhibits a correlation length, which also characterizes the size of the largest eddies. This macroscale is usually called the *integral scale*; it is estimated by

$$L \equiv \int_0^\infty f(r)dr \quad : \text{integral scale of turbulence.} \tag{47}$$

Experimental studies indicate that the integral scale is of the order of the size of the (local) mean-flow inhomogeneity (see Fig. 8). In the context of turbulent viscosity, the integral scale may be viewed as the mixing length: $L \approx \ell_m$.

A microscale can be exhibited by describing the behavior of $f(r)$ for $r \approx 0$. The condition of homogeneity implies $f'(0) = 0$; the Taylor's development of $f(r)$ thus yields

$$f(r) = 1 - \frac{1}{2}\left(\frac{r}{\lambda}\right)^2 + \mathcal{O}(r^4) \text{ with } \frac{1}{\lambda^2} = -f''(0) > 0. \tag{48}$$

The microscale λ is called the *Taylor's microscale*. Taylor hypothesized that λ "may roughly be regarded as a measure of the diameters of the smallest eddies which are responsible for the dissipation of energy".

When considering the *longitudinal increment* $\delta u_\parallel(r) \equiv u_\parallel(\vec{x}+\vec{r},t) - u_\parallel(\vec{x},t)$, one obtains

- $\dfrac{\langle \delta u_\parallel(r)^2\rangle}{2\langle u_\parallel^2\rangle} \approx \dfrac{1}{2}\left(\dfrac{r}{\lambda}\right)^2$ for $r \approx 0$

and

- $\dfrac{\langle \delta u_\parallel(r)^2\rangle}{2\langle u_\parallel^2\rangle} \approx 1$ for $r \geq L$ (integral scale)

The representation of $\langle \delta u_\parallel(r)^2\rangle / 2\langle u_\parallel^2\rangle$ (as a function of the scale r) thus provides a direct visualization of the logarithmic separation between the macroscale and the microscale of turbulent velocity fluctuations, as shown in Fig. 9.

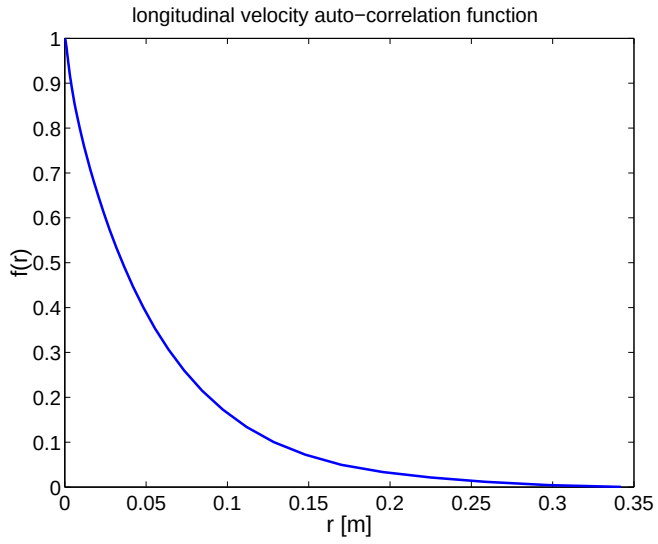

FIG. 8: The longitudinal velocity auto-correlation function $f(r)$, measured downstream from a vertical cylinder (on the centerline of the wake). The integral scale $\int_0^\infty f(r)dr$ is found of the order of the diameter of the cylinder ($D = 0.10$ m). Velocity fluctuations are recorded in time at a fixed position in the flow by using a hot-wire anemometer (see Fig. 5). The mean-flow velocity $\overline{u}$ is used to recast the temporal velocity signal in the space domain, according to $u(x, t+\tau) = u(x+r, t)$ with $r = -\overline{u}\tau$. This transformation is known as the *Taylor's hypothesis*; it states that turbulence may be considered as *frozen* for small time lags τ, provided that the mean-flow velocity is large compared to the turbulent velocity: $\sqrt{< (u-\overline{u})^2 >} \;/\; \overline{u} \ll 1$ (this ratio is called the *turbulent rate*).

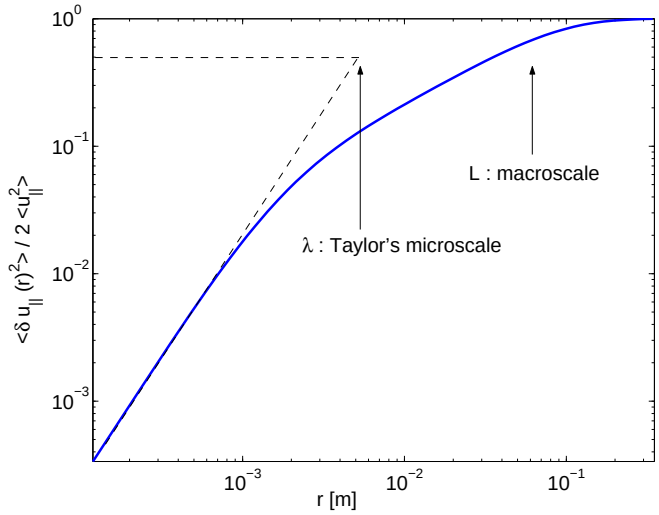

FIG. 9: The macroscale and microscale of turbulent fluctuations are well separated. From dimensional arguments, it can be established that $L/\lambda \sim R_\lambda$ (turbulent Reynolds number). Here, $R_\lambda \approx 180$ (turbulent flow behind a vertical cylinder).

3. *The turbulent Reynolds number*

Here, the Reynolds' decomposition is required to distinguish unambiguously the mean-flow velocity and the turbulent velocity.

Turbulence is primarily a dissipative process, and, with this respect, the behavior of the finest scales of motion is essential. The definition of the Taylor's microscale (48) yields

$$\bar{\varepsilon} = 2\nu\overline{|S'|^2} \sim \nu \left(\frac{u'}{\lambda}\right)^2, \tag{49}$$

where u' is the turbulent velocity scale ($u' \sim \overline{k'}^{1/2}$).

On the other hand, $\bar{\varepsilon} = -\overline{u_i'u_j'}\;\overline{S_{ij}}$ from Eq. (41). Typically, $-\overline{u_i'u_j'}\;\overline{S_{ij}} \sim u'^2|\overline{S}|$ and therefore $|\overline{S}| \sim \nu/\lambda^2$. The local supremacy of the turbulent stress over the mean-flow viscous stress then reads

$$Re_{\text{local}} \sim \frac{-\overline{u_i'u_j'}\;\overline{S_{ij}}}{\nu|\overline{S}|^2} \sim \left(\frac{u'\lambda}{\nu}\right)^2. \tag{50}$$

This naturally motivates the introduction of a local order parameter for the turbulent agitation:

$$R_\lambda \equiv \frac{u'\lambda}{\nu} \quad : \text{the turbulent Reynolds number.} \tag{51}$$

R_λ is also called the *Reynolds number based on the Taylor's microscale*.

The fully developed turbulent regime is obtained for high turbulent Reynolds numbers. One therefore obtains as a criterion of (local) developed turbulence:

$$R_\lambda \sim \frac{\sqrt{\overline{|S'|^2}}}{|\overline{S}|} \gg 1, \tag{52}$$

which indicates that the gradients of the fluctuating velocity are very large compared to the mean-flow gradients.

It is important to emphasize on the fact that turbulence properties are local in space (and time if the flow is not stationary). For the Poiseuille's flow in a pipe, the turbulence in the vicinity of the wall, where the shear (mean-flow gradient) is strong, is different in nature from the turbulence far away from the wall, where the shear is almost zero. In this flow, R_λ depends on the distance to the wall.

The local Reynolds number $Re_{\text{local}} \sim R_\lambda^2$ according to Eq. (50). It has been previously established that $\nu_{\text{turb.}}/\nu \sim Re_{\text{local}}$. The integral scale L may be viewed as the mixing length of turbulence, and consequently, $\nu_{\text{turb.}} \sim u'L$. It follows that

$$Re_{\text{local}} \sim \frac{u'L}{\nu} \sim R_\lambda^2. \tag{53}$$

By combining the Eqs. (51) and (53), one obtains

$$\left(\frac{L}{\lambda}\right) \sim R_\lambda. \tag{54}$$

The logarithmic separation between the integral scale and the Taylor's microscale is proportional to the logarithm of the turbulent Reynolds number.

4. From G. I. Taylor (1935) to A. N. Kolmogorov (1941)

In Taylor's approach, the turbulent-velocity-gradient scale is (rigorously) given by (Batchelor [1967])

$$\left\langle |S|^2 \right\rangle = \frac{15}{2}\frac{u_{\rm rms}^2}{\lambda^2}, \tag{55}$$

where $u_{\rm rms} \equiv \langle u_\parallel^2 \rangle^{1/2}$ denotes the root-mean-square of any velocity component (by assuming isotropy). Eq.(55) infers a turbulent-agitation process, which is uncorrelated at the microscale λ. This is definitively not the case in actual turbulence; the velocity field is (strongly) correlated at the spatial size of the smallest eddies. This feature rather motivates the introduction of a microscale η such that

$$\left\langle |S|^2 \right\rangle \sim \frac{\langle \delta u_\parallel(\eta)^2 \rangle}{\eta^2}. \tag{56}$$

This is the approach followed by Kolmogorov (1941). In this situation, it is required to determine explicitly the behavior of $\langle \delta u_\parallel(r)^2 \rangle$ as a function of the scale r. The microscale η is usually called the *Kolmogorov's scale*; it nails down the size of the smallest eddies.

C. The difficulty of an analytical treatment

As demonstrated previously, the (longitudinal) two-point velocity correlation function contains important information about the range of excited scales in turbulence; it also determines the distribution of the turbulent kinetic energy in wavenumber. So far, there is no analytical theory that predicts this correlation function by starting from the Navier-Stokes equations. In that sense, turbulence remains an unsolved problem.

The need for a statistical description of turbulence arises from both the intrinsic complexity of individual solutions of the Navier-Stokes equations and the instability of these solutions to infinitesimal (extrinsic) perturbations in the initial or boundary conditions. This makes natural to examine ensembles of realizations rather than each individual realization, and to look for simple robust statistical features insensitive to the details of perturbations.

The statistical problem is a priori well posed if at initial time t_0, the mean velocity $\overline{u}_i(\vec{x}, t_0)$ and the correlation function $R_{ij}(\vec{x}, t_0; \vec{x}', t_0')$ are given, and if it is assumed that the distribution of the turbulent velocity field $u_i(\vec{x}, t_0)$ is gaussian. At times $t > t_0$, the (multivariate) distribution of $u_i(\vec{x}, t)$ deviates from the gaussian distribution, due to the correlations induced by the (nonlinear) dynamics of the Navier-Stokes equations. This departure from the gaussian distribution is governed by the entire (unclosed) hierarchy of statistical equations for the correlation functions (42). The central problem is to find an appropriate closure condition to convert this infinite hierarchy into a closed subset, that is the "closure problem of turbulence". In this context, the direct-interaction-approximation proposed by Kraichnan (1959) is certainly the most realized attempt (see Frisch [1995], for a modern review).

The statistical features of turbulence are rather investigated from a phenomenological standpoint, that is, starting from hypotheses motivated by experimental and numerical observations. Interestingly, this is the approach adopted by the mathematician Andrei Kolmogorov. This line of study has yielded very fruitful results during the past half century and continues to expand nowadays.

D. The Kolmogorov's theory of homogeneous and isotropic turbulence

Kolmogorov's approach of the (out-of-equilibrium) statistical problem of turbulence is interesting; it yields a relevant prediction for the spectrum of kinetic energy from (very) simple arguments.

1. Kolmogorov's similarity hypotheses

Kolmogorov's theory focuses on the fluctuations of velocity increments

$$\delta u_i(\vec{x}, \vec{r}, t) \equiv u_i(\vec{x} + \vec{r}, t) - u_i(\vec{x}, t). \tag{57}$$

These fluctuations are assumed stationary (in time), homogeneous and isotropic (and mirror symmetric).

Kolmogorov's theory relies on two similarity hypotheses (Kolmogorov [1941]):

- at scales r small compared to the integral scale L, the distributions of the velocity increments $\delta u_i(r)$ are universal (independent of the stirring mechanism of turbulence) and fixed by the kinematic viscosity of the fluid ν and the mean energy-dissipation rate (per unit mass) $\bar{\varepsilon}$.

In particular, for the second-order (longitudinal) moment $B_\parallel(r) \equiv \langle \delta u_\parallel(r)^2 \rangle$:

$$B_\parallel(r) = \sqrt{\nu\bar{\varepsilon}}\ \Phi(\frac{r}{\eta}) \quad \text{for } r \ll L, \tag{58}$$

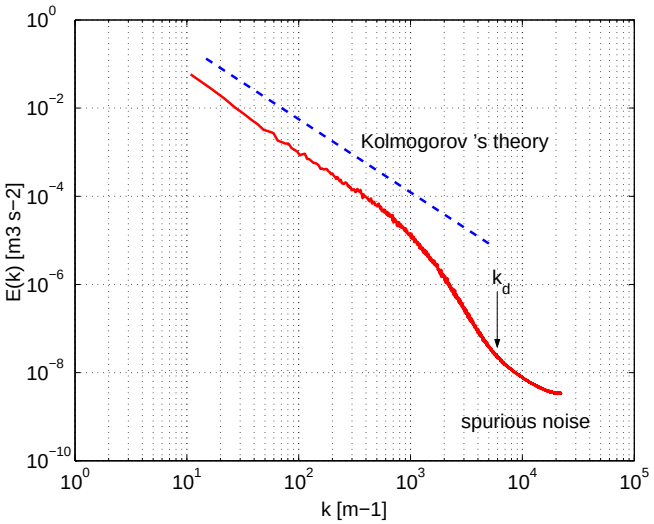

FIG. 10: The energy spectrum in wavenumber measured experimentally (C. Baudet et al.). The velocity signal is recorded in time at a fixed position in a turbulent jet. Turbulence is locally homogeneous and isotropic (at small scales) and the turbulent Reynolds number is 390. One observes a decreasing spectrum in good agreement with the prediction of Kolmogorov's theory: $E(k) \sim k^{-5/3}$. The hypothesis of "frozen turbulence" has been used: $\omega = \overline{u}k$, where $\overline{u}$ is the mean velocity component.

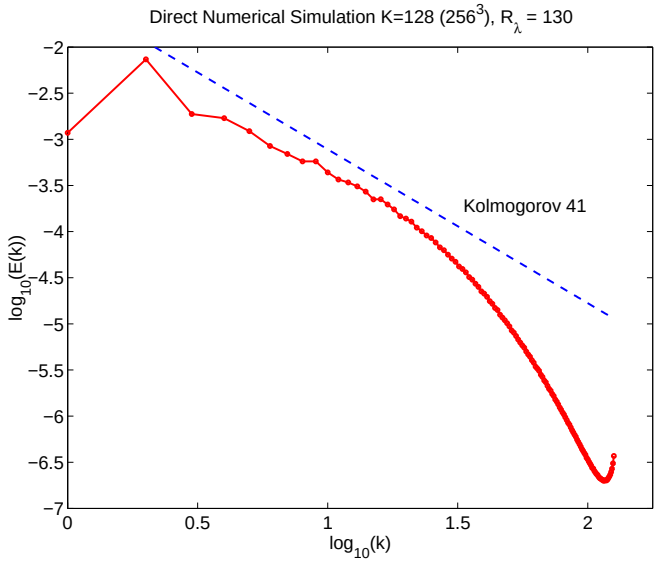

FIG. 11: The kinetic energy spectrum $E(k)$ in wavenumber, obtained from a direct numerical simulation of the (three-dimensional) Navier-Stokes equation (E. Lévêque). The energy is not equally distributed among the Fourier modes; $E(k) \sim k^2$ would be expected at thermodynamic equilibrium. At wavenumbers k between those, where energy is fed into turbulence, and the higher wavenumbers, where the energy is dissipated by viscosity, the spectrum is found close to the Kolmogorov's similarity law $E(k) \sim k^{-5/3}$.

where η denotes the *elementary scale of turbulent motions* and Φ is a universal function. A dimensional argument yields

$$\eta = \left(\frac{\nu^3}{\bar{\varepsilon}}\right)^{1/4}. \tag{59}$$

- at scales r large compared to the elementary scale η, the distributions of $\delta u_i(r)$ do not depend on ν.

Therefore,

$$\langle \delta u_{\|}(r)^2 \rangle = B\ (\bar{\varepsilon} r)^{2/3} \quad \text{for } \eta \ll r \ll L, \tag{60}$$

where B is a universal constant. The range of scales $\eta \ll r \ll L$ is called the *inertial range*.

It is interesting to note that the law (60) does not include any characteristic scale. This feature is related to the idea that the energy cascade is a self-similar process in scale. At a given scale r, all detailed statistical information about the source of energy is lost; the only parameter which controls the cascade is $\bar{\varepsilon}$, the rate of cascade. This parameter enters because the cascade conserves the energy.

2. *The spectrum law*

By assuming that the energy spectrum decreases sufficiently rapidly at high wavenumbers, one derives from the similarity law (60) the celebrated prediction

$$E(k) = C\bar{\varepsilon}^{2/3}k^{-5/3}, \tag{61}$$

where C is termed the *Kolmogorov's constant*. This law is corroborated by a large body of experimental (Fig. 10) and numerical (Fig. 11) measurements (with $C \approx 2$).

One can extend the prediction of Kolmogorov's theory to all wavenumbers

$$E(k) = C\bar{\varepsilon}^{2/3}k^{-5/3}f(\frac{k}{k_d}), \tag{62}$$

where f is a universal function and

$$k_d = \left(\frac{\bar{\varepsilon}}{\nu^3}\right)^{1/4} \sim \frac{1}{\eta} \tag{63}$$

is the *Kolmogorov's dissipative wavenumber*.

E. The dynamical mechanism of the energy cascade

Kolmogorov's theory envisages a cascade of kinetic energy from large scales to small scales which is local in scale size. It implies that an eddy of a given scale mainly interacts with eddies of similar scale. Indeed, it is plausible that motions on much larger scales should act to

transport this eddy without distorting it. On the opposite, the shears associated with excitations at much smaller scales should cancel out over the extend of the eddy. This suggests that distortions of the given eddy, and therefore the mechanism of transfer of energy to eddies of smaller scale, should be principally due to interaction with other eddies of similar scale. The mechanism responsible for this transfer is the process of *vortex stretching*.

1. A bit of kinematics

The vorticity field $\vec{\omega}(\vec{x},t)$ is defined as the curl of the velocity field

$$\vec{\omega}(\vec{x},t) = \vec{\nabla} \times \vec{u}(\vec{x},t). \tag{64}$$

By taking the curl of the Navier-Stokes equation, one obtains for the vorticity field

$$\frac{D\vec{\omega}}{Dt} \equiv \frac{\partial \vec{\omega}}{\partial t} + (\vec{u}.\vec{\nabla})\vec{\omega} = (\vec{\omega}.\vec{\nabla})\vec{u} + \nu\Delta\omega. \tag{65}$$

In Eq.(65), the vorticity-stretching term $(\vec{\omega}.\vec{\nabla})\vec{u}$ plays an important role. Let us consider the previous equation without the viscous term:

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega}.\vec{\nabla})\vec{u}. \tag{66}$$

Eq. (66) is identical to the equation for the evolution of an infinitesimal line element $\delta\vec{s}$ of fluid:

$$\frac{D\vec{\delta s}}{Dt} = (\vec{\delta s}.\vec{\nabla})\vec{u}. \tag{67}$$

The chaotic nature of turbulence tends to separate any two fluid elements initially near each other. Consequently, there is a tendency to stretch initial vorticity distributions into thin and elongated structure (until viscosity ultimately stops the thinning). Furthermore, if $\vec{\delta s}$ is taken along a vorticity line in Eq. (67), one gets

$$\frac{D\delta s}{Dt} = \overbrace{\frac{\omega_i\omega_j}{\omega^2}\frac{\partial u_i}{\partial x_j}\delta s = \frac{1}{\omega^2}\frac{D}{Dt}\left(\frac{\omega^2}{2}\right)\delta s}^{\text{from Eq. (66)}}$$
$$\text{or} \quad \frac{1}{2}\frac{D\omega^2}{\omega^2} = \frac{D\delta s}{\delta s}, \tag{68}$$

from which it is deduced that $\omega/\delta s$ is conserved during the stretching process. This means that the stretching of the vorticity line is accompanied by an intensification of the vorticity (the fluid in the vortex spins harder). An initial distribution of *vorticity tends to stretch and concentrate on thin and elongated structures* (see Fig. 12 for a numerical evidence).

Kolmogorov's theory relies on the picture of a uniform sea of random disordered whirls, with the cascade process consisting of the fission of eddies into smaller ones. This picture appears to be in conflict with what is inferred from the vortex stretching process.

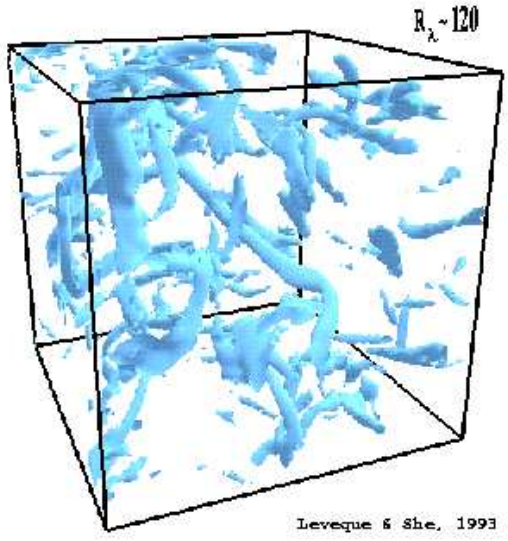

FIG. 12: Snapshot of high-enstrophy isosurfaces from a numerical simulation of three-dimensional turbulence; the local enstrophy (squared vorticity) is defined by $|\vec{\nabla}\times\vec{u}(\vec{x},t)|^2$. The swirling activity of the flow concentrates into very localized fluid structures: The *vortex filaments*.

Finally, note that in two-dimensional flows, the vortex-stretching term $(\vec{\omega}.\vec{\nabla})\vec{u}$ vanishes. Because of the absence of vortex stretching, two-dimensional turbulence is different from three-dimensional turbulence.

F. The phenomenon of intermittency

The Kolmogorov's law for the energy spectrum is well supported. However, higher-order statistics are not universal in the sense of Kolmogorov's 1941 hypotheses.

Kolmogorov's similarity hypotheses lead to the following form for the moments of $\delta u_\parallel(r)$:

$$\left\langle |\delta u_\parallel(r)|^p \right\rangle = B_p\ (\bar{\varepsilon} r)^{p/3} \quad \text{for } \eta \ll r \ll L, \tag{69}$$

where the B_p are universal constants. These moments are called the *velocity structure functions*.

From Eq. (69), the normalized moments

$$\frac{\langle |\delta u_\parallel(r)|^p \rangle}{\langle |\delta u_\parallel(r)|^2 \rangle^{p/2}} \tag{70}$$

are universal, independent of the mean dissipation rate $\bar{\varepsilon}$ and the scale r (in the inertial range). Instead, measured

value for the normalized moments increase dramatically with both the order p and $1/r$.

The power-law scaling of velocity structure functions is not denied, but the scaling exponents do not increase linearly with the order p:

$$\langle \delta u_\parallel(r)|^p \rangle \sim r^{\zeta_p} \quad \text{but} \quad \zeta_p - \frac{p}{2}\zeta_2 \neq 0. \tag{71}$$

This deviation from the Kolmogorov's linear scaling law (69) is an indication of a strong scale-dependent intermittency at small scales.

1. The build-up of intermittency along the cascade chain

At scales $r \approx L$, where L denotes the integral scale of turbulence, fluid motions are statistically independent and the probability density function of $\delta u_\parallel(L)$ is found nearly gaussian. At scales $r < L$, intrinsic nonlinear fluid dynamics operate and turbulent motions become intermittent; fluid activity comes in intense locally-organized motions embedded in a sea of relatively quiescent and disordered eddies (as a result of the vortex-stretching process).

As a consequence, the pdf of $\delta u_\parallel(r)$ develops long tails and becomes strongly non-gaussian. These deviations from the gaussian shape may be quantified by the flatness (fourth-order normalized moment), defined by

$$F(r) \equiv \frac{\langle \delta u_\parallel(r)^4 \rangle}{\langle \delta u_\parallel(r)^2 \rangle^2} \ . \tag{72}$$

For a centered gaussian distribution $F = 3$; as long tails develop F increases. The quantity $F(r)/3$ may therefore be roughly thought of as the ratio of intense to quiescent fluid motions at scale r. In that sense, $\log(F(r)/3)$ provides a quantitative measure of intermittency at the scale size r.

The normalized (to the gaussian value) flatness is plotted as a function of the scale ratio r/L for two turbulent flows in Fig. 13. We observe at scales $r \geq L$, $F(r) \simeq 3$, in agreement with the picture of disordered fluid motions: There is no intermittency, since the flatness $F(r)$ is independent of the scale r and (almost) equal to the gaussian value $F = 3$. At smaller scales, $F(r)$ displays a power-law dependence on r: Intermittency grows up linearly with $\log(1/r)$. This scaling behavior is inherent to the non-linear inertial fluid dynamics and means (in the context of vortex stretching) that the vorticity becomes concentrated in an increasingly sparse collection of intense filaments as the scale size r decreases.

2. The refined theory of Kolmogorov and Oboukhov

In 1962, Kolmogorov and Oboukhov (Kolmogorov & Oboukhov [1962]) suggested a refinement of the 1941 theory, in which the spatial fluctuations of the energy-dissipation rate (per unit mass) are taken into account.

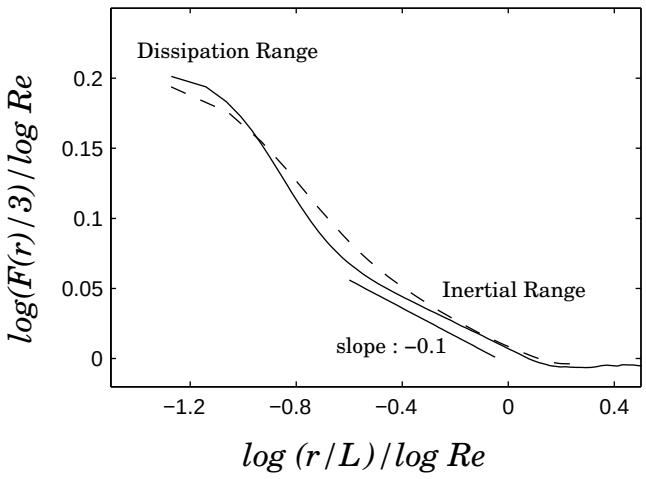

FIG. 13: Scale dependence of the flatness of longitudinal velocity increments for two different turbulent flows: (solid line) a turbulent jet ($R_\lambda = 380$) and (dashed line) a direct numerical simulation ($R_\lambda = 140$). The macroscopic parameter Re is the local Reynolds number. For a centered gaussian distribution, $F = 3$. We observe that $F(r)$ is not constant at scales $r \leq L$: Kolmogorov's similarity hypotheses are violated.

$$\begin{aligned} \varepsilon(\vec{x},t) &= 2\nu|S(\vec{x},t)|^2 \\ &= \frac{\nu}{2}\sum_{i,j}\left(\frac{\partial u_i(\vec{x},t)}{\partial x_j} + \frac{\partial u_j(\vec{x},t)}{\partial x_i}\right)^2. \end{aligned} \tag{73}$$

Locally, the dissipation rate at scale r is defined as the space average over a ball of radius r:

$$\varepsilon_r(\vec{x},t) = \frac{1}{\frac{4}{3}\pi r^3}\int_{|\vec{y}|<r} \varepsilon(\vec{x}+\vec{y},t)d\vec{y}. \tag{74}$$

The previous similarity hypotheses are refined by considering that turbulence is locally conditioned (at scale r) by the value of $\varepsilon_r(\vec{x},t)$:

$$B_\parallel(\vec{x},t,r|\varepsilon_r(\vec{x},t)) = B(\vec{x},t)r^{2/3}\varepsilon_r(\vec{x},t)^{2/3}. \tag{75}$$

By integrating over all possible values of $\varepsilon_r(\vec{x},t)$, one gets

$$B_\parallel(\vec{x},t,r) = B(\vec{x},t)r^{2/3}\left\langle \varepsilon_r(\vec{x},t)^{2/3}\right\rangle. \tag{76}$$

If turbulence is stationary, and locally homogeneous and isotropic:

$$B_\parallel(r) = B(\bar{\varepsilon}r)^{2/3}\left(\frac{L}{r}\right)^{-\mu} \tag{77}$$

by noting

$$\langle \varepsilon_r(\vec{x},t)^{2/3}\rangle = \bar{\varepsilon}^{2/3}\left(\frac{L}{r}\right)^{-\mu}. \tag{78}$$

The parameter μ is called the *intermittency parameter*. It characterizes the deviation from the 1941 theory for the second-order moment.

More generally, the velocity structure functions write

$$\langle |\delta u_{\parallel}(r)|^p \rangle = B_p(\bar{\varepsilon} r)^{p/3} \left(\frac{r}{L}\right)^{\tau_{p/3}}, \qquad (79)$$

where the corrections to the linear law $\zeta_p = p/3$ are contained in

$$\langle \varepsilon_r(\vec{x},t)^{p/3} \rangle = \bar{\varepsilon}^{p/3} \left(\frac{r}{L}\right)^{\tau_{p/3}} \quad : \quad \zeta_p = \frac{p}{3} + \tau_{p/3}. \qquad (80)$$

Heuristically, Kolmogorov and Oboukhov related the phenomenon of intermittency to the fluctuations of the local energy-dissipation rate. They also opened a Pandora box containing all a priori admissible distributions for ε_r (Kraichnan [1974]).

In the refined theory of Kolmogorov and Oboukhov (1962) the distributions of $\delta u_{\parallel}(r)$ are fixed by the kinematic viscosity ν, the mean dissipation rate $\bar{\varepsilon}$ and the integral scale L. The introduction of the integral scale L is not anodyne. It is related to the idea that the energy cascade results from the iteration of the same elementary process of fission of eddies into smaller eddies. The number of cascade steps required for an excitation to propagate from the integral scale L to the small scale r is measured by $\log(r/L)$. It is assumed that each step of the cascade is stochastic in nature and statistically independent of the previous steps. The result is a build-up of intermittency at each cascade step, which may be viewed as a *multiplicative process* (Castaing [1990]).

3. The log-normal model

Kolmogorov and Oboukhov proposed a plausible candidat for the distribution of ε_r. It is the log-normal model, for which the pdf of $\log \varepsilon_r$ is supposed gaussian:

$$\mathcal{P}(\log \varepsilon_r) = \frac{1}{\sqrt{2\pi\sigma_r^2}} \exp\left(\frac{-(\log \varepsilon_r - m_r)^2}{2\sigma_r^2}\right). \qquad (81)$$

The parameters m_r and σ_r^2 denote respectively the mean and the variance of the random variable $\log \varepsilon_r$. The log-normal model yields a quadratic law for the scaling exponents ζ_p:

$$\zeta_p = \frac{p}{3} + \frac{1}{2}\mu p(p-3) \quad \text{with} \quad \mu > 0. \qquad (82)$$

In Kolmogorov's 1941 theory, the scaling exponents are fixed by dimensional arguments. Here, all the possible values for μ are a priori permitted. This is the detailed nature of the non-linear dynamics of turbulence which should fix the value of μ. This relation has not been established so far (as mentioned in the introduction).

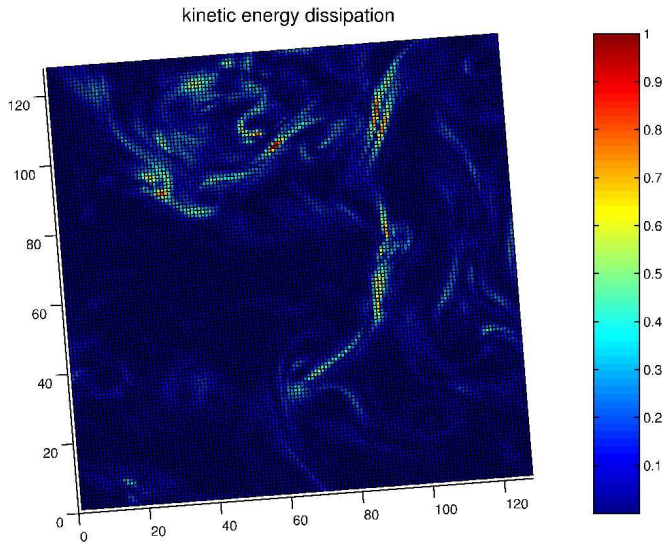

FIG. 14: Slice of a snapshot of the energy-dissipation rate $\varepsilon(\vec{x},t)$, obtained from a numerical simulation of the (three-dimensional) Navier-Stokes equation (E. Lévêque). The magnitude of $\varepsilon(\vec{x},t)$ is represented by a colorbar ranging from 0 to 1. Energy dissipation is concentrated on fine structures; it is not uniformly distributed.

4. The log-normality and the vortex-stretching process

In the context of vortex stretching, the build-up of intermittency along the cascade chain is related to a swirling activity of the flow which concentrates on a more and more little fraction of the volume.

A simple model assumes that the vorticity is statistically independent of the local velocity shear, and obeys the stochastic equation

$$\frac{D\omega(t)}{Dt} = b(t)\omega(t), \qquad (83)$$

where $b(t)$ denotes the effective velocity shear; $b(t)$ is random and independent of $\omega(t)$. The previous equation leads to

$$\log\left(\frac{\omega(t)}{\omega(0)}\right) = \int_0^t b(s)ds. \qquad (84)$$

For times t very large compared to the correlation time of $b(s)$, the statistics of $\omega(t)$ become log-normal (according to the central-limit theorem). Furthermore, if one assumes that $\varepsilon \sim \nu\omega^2$, this simple model gives a support to the log-normal hypothesis. However, a more realistic model should take into account the (strong) correlation between $b(t)$ and $\omega(t)$. In that case, an other (statistical) distribution for $\omega(t)$ would be obtained. What is the stochastic process selected by the vorticity-stretching mechanism of turbulence? It is an alternative approach to the problem of turbulence (see Saffman [1998], for a review).

Finally, let us mention that the importance of the fluid dynamics in the cascade process, naturally calls for a Lagrangian representation of turbulence. The Lagrangian coordinate system moves with the fluid and therefore gets ride of advection effects by the large-scale fluid structures (sweeping effects), but focuses on distortion effects responsible for the local energy-transfer mechanism. But there are severe problems in using a Lagrangian representation. First, the pressure and the viscous terms of the Navier-Stokes equation display intractable non-linearity in Lagrangian coordinates (in the three-dimensional case). Second, after a substantial time of evolution, the labelling of fluid particles has such a mixed-up relation to current positions that it becomes inappropriate.

year	grid size DNS	R_λ
1972	32^3 (Orszag & Patterson)	35
1985	128^3	
1991	256^3 (Vincent & Menneguzzi)	140
1995	512^3	
2002	1024^3 (Gotoh)	400
2005	2048^3 (Earth Simulator, Japan)	700

TABLE I: Resolution of the direct (without modelling) numerical simulation of the Navier-Stokes equation in a cubic domain with periodic boundary conditions (a prototype for homogeneous and isotropic turbulence). The grid size is roughly multiplied by a factor two (in each direction) every five years. This evolution is closely related to the increase in size and performance of computers (see Fig. 15).

V. THE LARGE-EDDY SIMULATION OF TURBULENCE

A. General introduction

Turbulence exhibits a wide range of spatial and temporal scales of motion, which needs to be resolved in a direct (without modelling) numerical simulation (DNS). The Kolmogorov's elementary scale η fixes the mesh resolution. On the other hand, the integration domain must encompass the largest scales of motion, comparable to the integral scale L. Therefore, the number of grid points required to suitably resolve all spatial scales, is given by

$$\left(\frac{L}{\eta}\right)^3 \sim Re^{9/4} \quad \text{with} \quad Re \equiv \frac{u_{\rm rms} L}{\nu}. \tag{85}$$

When the CFL time-step condition is factored in, i.e.,

$$\sigma = \frac{u_{rms}\delta t}{\eta} \leq 1, \tag{86}$$

one ends up with a computational cost which grows as the cube of the Reynolds number. This imposes dramatic constraints on the numerical simulation of turbulent flows (see Table I). Excessive computing costs motivate the development of "reduced models of turbulence", which usually requires much less computational ressources but remains relevant (in some degree) to describe the kinetics of the flow.

Roughly speaking, large-scale eddies carry most of the kinetic energy; their size and strength make them the most effective transporters of the conserved quantities (momentum, heat, mass, etc.), and small-scale eddies are mainly responsible for the dissipation; they are weaker and provide little transport. From a mechanical viewpoint, large-scale dynamics are therefore of primary importance, and the (costly) computation of small-scale motions should be avoided in a numerical simulation.

In this context, the RANS methods are the most popular. This is the approach followed by Osborne Reynolds, which consists in breaking the velocity field into a mean-flow component and a turbulent component, and modeling the effects of turbulent fluctuations on the mean flow. The RANS methods lead to the estimation of mean quantities but do not resolve the turbulent fluctuations; they are based on questionable closure conditions and often appeal to numerous empirical parameters (see section III). But their computational cost is unbeatable.

The large-eddy simulation (LES) offers a compromise between the DNS and the RANS methods. In a LES, the large-scale dynamics of turbulence are explicitly integrated in time, and the interaction with the unsolved small-scale motions is modelled. In order to separate the small-scale component and the large-scale component of the turbulent velocity field, an explicit filtering procedure is used (Lesieur & Métais [1996], for a review on LES).

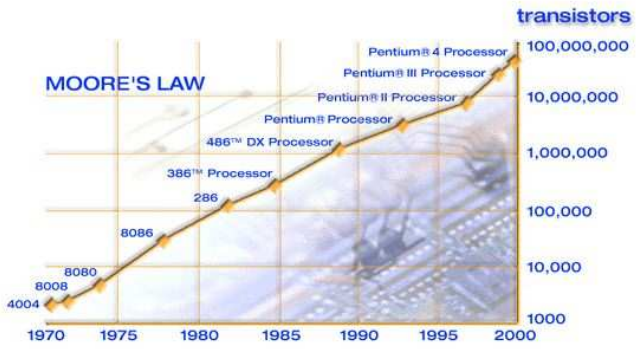

FIG. 15: The number of transistors in a chipset follows an exponential law; the Moore's law (Moore [1965]).

B. The equation for the large-scale (filtered) velocity field

The instantaneous velocity field $u_i(\vec{x},t)$ is decomposed into a large-scale component

$$u_i^<(\vec{x},t) = \int H_\Delta(\vec{x'}) u_i(\vec{x}-\vec{x'},t) d\vec{x'}, \qquad (87)$$

obtained by spatially filtering the velocity field, and a small-scale component $u_i^>(\vec{x},t)$ such that

$$u_i(\vec{x},t) = u_i^<(\vec{x},t) + u_i^>(\vec{x},t). \qquad (88)$$

The convolution kernel $H_\Delta(\vec{x})$ eliminates the fluctuations of the velocity field at scales smaller than Δ.

The equation for the large-scale (filtered) velocity field is obtained by filtering the Navier-Stokes equation and reads

$$\frac{\partial u_i^<}{\partial t} + u_j^< \frac{\partial u_i^<}{\partial x_j} = -\frac{\partial}{\partial x_j}\left(p^<\delta_{ij} + \tau_{ij}^<\right) + \nu \frac{\partial^2 u_i^<}{\partial x_j \partial x_j}, \qquad (89)$$

where

$$\tau_{ij}^< \equiv (u_i u_j)^< - u_i^< u_j^< \qquad (90)$$

denotes the *subgrid-scale stress tensor*. $\tau_{ij}^<$ arises from the nonlinearity of the Navier-Stokes equation and encompasses all interactions between the resolved and the unresolved scales of motion.

Eq.(89) is amenable to numerical discretization at the mesh resolution Δ (Ferziger & Perić [2002], a book on *Computational Methods for Fluid Dynamics*), which is typically much more affordable than the DNS if $\Delta \gg \eta$. However, in order to close Eq.(89), it is necessary to express the subgrid-scale stress tensor $\tau_{ij}^<$ in terms of the large-scale (filtered) velocity field.

C. The Smagorinsky's model

The first, and still very widely used, proposal for the subgrid-scale stress tensor, is the Smagorinsky's model (Smagorinsky [1963]).

1. *The hypothesis of turbulent viscosity*

The idea is to consider the deviatoric part of $\tau_{ij}^<$ by analogy with the viscous stress in a Newtonian fluid:

$$\tau_{ij}^< \underbrace{- \frac{1}{3}\tau_{kk}^<}_{\text{included in the pressure}} \approx -2\nu_T S_{ij}^<, \qquad (91)$$

where ν_T is the (scalar) *eddy viscosity* (at scale Δ) and $S_{ij}^<$ denotes the resolved (filtered) strain-rate tensor.

Following the mixing-length model introduced by Prandtl ([1925]), Smagorinsky proposed

$$\nu_T = (C_s\Delta)^2 |S^<| \quad \text{with} \quad |S^<| \equiv \sqrt{2 S_{ij}^< S_{ij}^<}. \qquad (92)$$

C_s is an empirical constant (the Smagorinsky's constant $C_s \approx 0.2$ for homogeneous and isotropic turbulence). The Smagorinsky's length scale $\ell_s = C_s\Delta$ is analogous to the mixing length, and proportional to the filter width Δ.

The dynamical equation for the filtered velocity field then reads

$$\left(\frac{\partial}{\partial t} + u_j^< \frac{\partial}{\partial x_j}\right) u_i^< = -\frac{\partial p^<}{\partial x_i} + (\nu + \nu_T)\frac{\partial^2 u_i^<}{\partial x_j \partial x_j} \qquad (93)$$

with the eddy viscosity $\nu_T = (C_s\Delta)^2\sqrt{2 S_{ij}^< S_{ij}^<}$.

Eq.(93) is identical to the usual Navier-Stokes equation except for the effective viscosity $\nu+\nu_T$. In the framework of Kolmogorov's 1941 theory, the elementary scale of the (turbulent) solution of Eq.(93) expresses as

$$\eta^< = \left(\frac{(\nu+\nu_T)^3}{\varepsilon}\right)^{1/4}. \qquad (94)$$

The energy-dissipation rate is given by $\varepsilon = (\nu+\nu_T)|S^<|^2$, and therefore

$$\eta^< = C_s\Delta\left(1 + \frac{\nu}{\nu_T}\right)^{1/2}. \qquad (95)$$

If turbulence is developed, the eddy-viscosity is much larger than the molecular viscosity, and consequently $\eta^< \approx C_s\Delta$. In the Smagorinsky's model, the eddy-viscosity is thought so that the Smagorinsky's length scale ℓ_s coincides with the elementary scale $\eta^<$ of the filtered velocity field.

Despite its popularity, the Smagorinsky's model possesses a number of shortcomings. For inhomogeneous flows, it is too dissipative, that is, it transfers to much energy to the unresolved subgrid motions. The Smagorinsky's model also performs poorly in the vicinity of a wall; the Smagorinsky's length scale becomes large compared to the energy-containing scales. This suggests to adjust dynamically the coefficient C_s, in order to ensure the constitutive relation $\ell_s \approx \eta^<$ everywhere in the flow. This issue is actually addressed in the dynamic model proposed by Germano ([1991]). The behavior of the Smagorinsky's model is greatly improved but the computational cost is heavy.

[1971] Ruelle, D. & Takens, F. 1971, Comm. Math. Phys., 20, 167-192

[1975] Monin, A. S. & A. M. Yaglom, A. M. 1975, *Statis-*

tical Fluid Mechanics, ed. Mit Press, Cambridge, Massachusetts (USA)
[2004] Kritsuk, A. G. & Norman, M. L. 2004, ApJ, 601, L55
[1883] Reynolds, O. 1883, Philos. Trans. R. Soc. London Ser. A, 174, 935-982
[1985] Eckmann, J.-P. & Ruelle, D. 1985, Rev. Modern Phys., 57, 617-656
[1922] Richardson, L. F. 1922, *Weather Prediction by Numerical Process*, ed. Cambridge University Press, Cambridge, United Kingdom
[1941] Kolmogorov, A. N. 1941, Dokl. Akad. Nauk SSSR, 32, 19-21 — Kolmogorov, A. N. 1941, Dokl. Akad. Nauk SSSR, 30, 299-303
[1995] Frisch, U. 1995, *Turbulence : The Legacy of A. N. Kolmogorov*, ed. Cambridge University Press, Cambridge, United Kingdom
[1994] She, Z.-S. & Lévêque, E. 1994, Phys. Rev. Lett., 72, 334
[1967] G. K. Batchelor, G.K. 1967, *An Introduction to Fluid Dynamics*, ed. Cambridge University Press, Cambridge, United Kingdom
[1841] Poiseuille, J. L. M. 1841, C. R. Acad. Sci. Paris, 12, 112
[1895] Reynolds, O. 1895, Phil. Trans. Roy. Soc. London Ser. A, 186, 123-164
[1823] Navier, C. L. M. H. 1823, Mem. Acad. R. Sci. Paris, 6, 389-416.
[1843] Stokes, G. G. 1843, Trans. Camb. Phil. Soc., 8, 105
[1976] Comte-Bellot, G. 1976, Annu. Rev. Fluid Mech., 8, 209
[1993] Wilcox, D. C. 1993, *Turbulence Modeling for CFD*, ed. DWC Industries Inc., La Canada, USA
[1925] Prandtl, L. 1925, Zeitschrift Angew. Math. und Mech., 3, 136-139
[1942] Kolmogorov, A. N. 1942, Izvestia Acad. Sci. USSR, Phys. 6, 56-58
[1945] Prandtl, L. 1945, Nachr. Akad. Wiss. Götingen Math-Phys. K1, 6-19
[1972] Jones, W. P. & Launder B. E. 1972, Int. J. of Heat Mass Tran., 15, 301-304
[1935] Taylor, G. I. 1935, Proc. R. Soc. London Ser. A, 151, 421-464
[1962] Kolmogorov, A. N. 1962, J. Fluid Mech., 13, 82 — Oboukhov, A. M. 1962, J. Fluid Mech., 13, 77
[1974] Kraichnan, R. H. 1974, J. Fluid Mech., 62 (2), 305-330
[1998] Pullin, D. I. & Saffman, P. G. 1998, Annu. Rev. Fluid Mech., 30, 31
[1990] Castaing, B. , Gagne, Y. & Hopfinger, E. 1990, Physica D, 46, 177
[1965] Moore, G. 1965, Electronics, 38 (8).
[2002] Ferziger, J. H. & Perić, M. 2002, *Computational Methods for Fluid Dynamics*, ed. Springer-Verlag (3rd edition), Berlin Heidelberg, Germany
[1996] Lesieur, M. & Métais, O. 1994, Annu. Rev. Fluid Mech., 28, 45-82
[1963] Smagorinsky, J. 1963, Mon. Wea. Rev., 91, 99-164
[1991] Germano, M. , Piomelli, U. , Moin, P. & Cabot, W. H. 1991, Phys. Fluids A, 3, 1760-1765